Stochastic Processes

Final Exam, Solutions

1. (15 marks)

(a) (8 marks) Let $S_m$ be the number of steps until the walk first reaches state $m$ starting in state 0, and let $S_{i,i+1}$ be the number of steps until the walk first reaches state $i+1$ starting in state $i$, for $i \in \mathbb{Z}$. Note that by spatial homogeneity, $E[S_{i,i+1}] = E[S_{0,1}] = E[S_1]$ for all $i \in \mathbb{Z}$. By the Markov property and temporal homogeneity, and since the walk can move up only one step at a time we can write $S_m = S_{0,1} + S_{1,2} + \ldots + S_{m-1,m}$ so that $E[S_m] = mE[S_{0,1}] = mE[S_1]$. To compute $E[S_1]$ we condition on the first step to get

$$E[S_1] = p(1) + (1 - p)(1 + E[S_{-1,0}] + E[S_{0,1}]) = 1 + (1 - p)2E[S_1].$$

Then solving for $E[S_1]$ we get

$$E[S_1] = \frac{1}{1 - 2(1 - p)}$$

and finally

$$E[S_m] = \frac{m}{1 - 2(1 - p)}.$$

(b) (7 marks) Let $\mu_k$ be as defined in the hint. Conditioning on the first step we have

$$\mu_0 = 1 + \frac{1}{2}(\mu_{-1} + \mu_1) = 1 + \frac{1}{2}(\mu_1 + \mu_1) = 1 + \mu_1$$

$$\mu_k = 1 + \frac{1}{2}(\mu_{k-1} + \mu_{k+1}) \quad \text{for } k = 1, \ldots, m-2$$

$$\mu_{m-1} = 1 + \frac{1}{2}(\mu_{m-2} + \mu_m) = 1 + \frac{1}{2}\mu_{m-2}$$

and we have the boundary condition $\mu_m = 0$. To solve these equations let $d_k = \mu_k - \mu_{k+1}$ for $k = 0, \ldots, m-1$. Then we have

$$d_0 = 1 \quad \text{and} \quad d_k = 2 + d_{k-1} \quad \text{for } k = 1, \ldots, m-1$$

which gives $d_k = 2k + 1$, for $k = 0, \ldots, m-1$. Noting that $\mu_0 = \mu_0 - \mu_m = d_0 + \ldots + d_{m-1}$ we have

$$\mu_0 = 1 + 3 + \ldots + (2(m-1) + 1) = (1 + \ldots + 2m) - 2(1 + \ldots + m) = (2m)(2m+1) - 2m(m+1) = \frac{2m(m+1)}{2} = m^2.$$
2. (15 marks)

(a) (7 marks) Letting $Y$ denote a random variable with the same distribution as the $Y_i$'s, then the transition probabilities for the $Z$ chain can be computed by conditioning on $Y$:

$$P(Z_{n+1} = j \mid Z_n = i) = P(X_{S_{n+1}} = j \mid X_{S_n} = i) = P(X_Y = j \mid X_0 = i) = \sum_{n=1}^{\infty} P(X_Y = j \mid X_0 = i, Y = n) P(Y = n \mid X_0 = i)$$

$$= \sum_{n=1}^{\infty} P(X_n = j \mid X_0 = i) P(Y = n) = \sum_{n=1}^{\infty} p_{ij}(n) P(Y = n),$$

where $p_{ij}(n)$ is the $n$-step transition probability from $i$ to $j$ in the $X$ chain, and the fourth equality above follows because $Y$ is independent of $X$. To show that $Z$ has stationary distribution $\pi$ it suffices to check that $\pi$ satisfies the global balance equations for the $Z$ chain. That is, we need to check that $\pi$ satisfies

$$\pi_j = \sum_{i \in S} \pi_i P(Z_1 = j \mid Z_0 = i)$$

Computing the right hand side above we get

$$\sum_{i \in S} \pi_i P(Z_1 = j \mid Z_0 = i) = \sum_{i \in S} \pi_i \sum_{n=1}^{\infty} p_{ij}(n) P(Y = n)$$

$$= \sum_{n=1}^{\infty} P(Y = n) \sum_{i \in S} \pi_i p_{ij}(n)$$

$$= \sum_{n=1}^{\infty} P(Y = n) \pi_j$$

$$= \pi_j \sum_{n=1}^{\infty} P(Y = n) = \pi_j.$$

(b) (8 marks) Let $q_{ij}$, $i, j \in S$, denote the transition probabilities of the modified chain. From the description we have that

$$q_{ij} = \begin{cases} 
    b_i p_{ij} & \text{if } i \neq j \\
    1 - b_j + b_j p_{jj} & \text{if } i = j.
\end{cases}$$
We note that the modified chain spends a Geometric($b_i$) number of steps in state $i$ before making a transition according to $P$ (which may be back to state $i$). From the hint, intuitively the long run proportion of time the modified chain spends in state $i$ should be proportional to $\pi_i$ times the mean of a Geometric($b_i$) distribution; that is, $\pi_i$ times $1/b_i$. To verify this we can check the global balance equations for the modified chain. Checking the $j$th global balance equation we have

$$\sum_{i \in S} \pi_i b_{ij} q_{ij} = \frac{\pi_j}{b_j} (1 - b_j + b_j p_{jj}) + \sum_{i \neq j} \frac{\pi_i}{b_i} b_i p_{ij}$$

$$= \frac{\pi_j}{b_j} - \pi_j + \sum_{i \in S} \pi_i p_{ij}$$

$$= \frac{\pi_j}{b_j} - \pi_j + \pi_j = \frac{\pi_j}{b_j}.$$

Since $b_i \geq b > 0$ for all $i \in S$, we also have that

$$\sum_{i \in S} \frac{\pi_i}{b_i} \leq \sum_{i \in S} \frac{\pi_i}{b} = \frac{1}{b} < \infty$$

Therefore, the vector $\psi = (\psi_i)_{i \in S}$, where

$$\psi_i = \frac{\pi_i / b_i}{\sum_{k \in S} \pi_k / b_k}$$

is the stationary distribution for the modified chain.

3. (15 marks)

(a) (7 marks) Fix $t > 0$ and define an arrival at time $s$ to be of type 1 if $s < t$ and the arrival has departed by time $t$, and to be of type 2 otherwise. Let $N_1(t)$ be the number of type 1 arrivals by time $t$. Then $Y(t) = N_1(t)$ and $P(Y(t) = k) = P(N_1(t) = k)$. But we know that $N_1(t)$ has a Poisson distribution with parameter $\lambda \int_0^t p_1(s) ds$, where $p_1(s)$ is the probability that an arrival at time $s$ is of type 1 and, for $s < t$, is given by

$$p_1(s) = P(\text{service time is less than } t - s) = G(t - s).$$

Thus,

$$P(Y(t) = k) = \frac{[\lambda \int_0^t G(t - s) ds]^k}{k!} e^{-\lambda \int_0^t G(t - s) ds},$$

for $k \geq 0$. 
(b) (8 marks) Let $S_i$ and $Y_i$ denote the $i$th arrival time and service time, respectively, so the $i$th departure time is $S_i + Y_i$. The event $\{R(t) > x\}$ occurs if and only if none of the arrivals in the interval $[0, t + x]$ depart in the interval $(t, t + x]$, i.e.,

$$P(R(t) > x) = P\left( \bigcap_{i=1}^{N(t+x)} \{S_i + Y_i \notin (t, t + x]\}\right),$$

where $\{N(t) : t \geq 0\}$ is the Poisson arrival process. Given $n$ arrivals in $[0, t + x]$, $n \geq 1$, the $n$ arrival times are distributed as the order statistics, $U_{(1)}, \ldots, U_{(n)}$, of $n$ independent Uniform$(0, t + x)$ random variables, $U_1, \ldots, U_n$, so

$$P\left( \bigcap_{i=1}^{N(t+x)} \{S_i + Y_i \notin (t, t + x]\} \mid N(t + x) = n \right) = P\left( \bigcap_{i=1}^{n} \{U_i + Y_i \notin (t, t + x]\} \right) = P\left( \bigcap_{i=1}^{n} \{U_i + Y_i \notin (t, t + x]\} \right) = \prod_{i=1}^{n} P(U_i + Y_i \notin (t, t + x]) = P(U_1 + Y_1 \notin (t, t + x)]^n.$$

For notational convenience let $d(t, x) = P(U_1 + Y_1 \notin (t, t + x])$. Conditioning on $U_1$ we get

$$d(t, x) = \frac{1}{t + x} \int_{0}^{t+x} P(U_1 + Y_1 \notin (t, t + x] \mid U_1 = u) du$$

$$= \frac{1}{t + x} \left[ \int_{0}^{t} P(U_1 + Y_1 \notin (t, t + x] \mid U_1 = u) du + \int_{t}^{t+x} P(U_1 + Y_1 \notin (t, t + x] \mid U_1 = u) du \right]$$

$$= \frac{1}{t + x} \left[ \int_{0}^{t} \left( 1 - G(t - u) + G(t + x - u) \right) du + \int_{t}^{t+x} G(t + x - u) du \right],$$

where $G = 1 - G$. Then by the law of total probability

$$P(R(t) > x) = E[d(t, x)^{N(t+x)}]$$

$$= \exp\{\lambda(t + x)(d(t, x) - 1)\}$$

$$= \exp\left\{ -\lambda(t + x) + \lambda\left( \int_{0}^{t} G(t - u) du + \int_{0}^{t+x} G(t + x - u) du \right) \right\}.$$
4. (15 marks)

(a) (7 marks) We have \( \pi_i g_{i,j} = \pi_j g_{j,i} \) for all \( i \neq j \) from the local balance equations and we are given that \( g_{i,j} = g_{-i,-j} \) for all \( i \neq j \). In particular, \( g_{0,i} = g_{0,-i} \) and \( g_{i,0} = g_{-i,0} \) for all \( i \neq 0 \). Then

\[
\pi_i g_{i,0} = \pi_0 g_{0,i} = \pi_0 g_{0,-i} = \pi_{-i} g_{-i,0} = \pi_{-i} g_{i,0}
\]

and so \( \pi_i = \pi_{-i} \) for all \( i \) such that \( g_{i,0} > 0 \). Suppose now that \( g_{i,0} = 0 \). Then there is a sequence of states, say \( j_1, \ldots, j_n \), such that \( g_{i,j_1}, g_{j_1,j_2}, \ldots, g_{j_{n-1},j_n}, g_{j_n,0} \) are all positive (in which case \( g_{j_1,i}, g_{j_2,j_1}, \ldots, g_{j_{n-1},j_n}, g_{j_n,0} \) are also all positive), and repeatedly applying the local balance equations, then the relation \( g_{i,j} = g_{-i,-j} \), then the local balance equations again, then the relation \( g_{i,j} = g_{-i,-j} \) again yields

\[
\pi_i = \pi_{-i}.
\]

(b) (8 marks) If we choose the \( \lambda_i \) to be positive and such that we can find a solution to the local balance equations then the conditions in part (a) will be satisfied and it then suffices to find \( \pi_i \) for \( i \geq 0 \) and set \( \pi_{-i} = \pi_i \). For \( i \geq 0 \) the local balance equations are \( \pi_i \lambda_i = \pi_{i+1} \lambda_{i+1} \), so that

\[
\frac{\pi_{i+1}}{\lambda_{i+1}} = \frac{\pi_i}{\lambda_i} \Rightarrow \frac{\pi_{i+1}}{\lambda_{i+1}} = \frac{\pi_{i-1}}{\lambda_{i-1}} = \ldots = \frac{\pi_0}{\lambda_0}.
\]

So we can set \( \lambda_i, i > 0 \), to be anything that satisfies \( \sum_{i=1}^{\infty} \frac{1}{\lambda_i} < \infty \). For example, set \( \lambda_i = \frac{\lambda}{\lambda_i} \) for some \( \lambda > 0 \), and set \( \lambda_0 = \frac{1}{2} \). Then we get

\[
\pi_0 = \left[ 1 + \sum_{i=1}^{\infty} \frac{\lambda^i}{i!} \right]^{-1} = e^{-\lambda}
\]

and

\[
\pi_i = \frac{\lambda^i}{2^i} e^{-\lambda}; \quad \pi_{-i} = \pi_i \quad \text{for } i \geq 0.
\]

The embedded discrete time Markov chain is just the symmetric simple random walk, so it has no stationary distribution.
5. (15 marks)

(a) (8 marks) We can write the local balance equations as

\[
\begin{align*}
\pi_i \frac{1}{i+2} &= \pi_{i+1} \frac{i+2}{i+3} \\
\pi_{-i} \frac{1}{|i|+2} &= \pi_{-(i+1)} \frac{|i|+2}{|i|+3}
\end{align*}
\]

for \( i \geq 0 \). From the equations above we can see that \( \pi_{-i} = \pi_i \) for \( i > 0 \) (assuming the local balance equations are satisfied). For \( i > 0 \) we get

\[
\begin{align*}
\pi_{i+1} &= \frac{i+3}{(i+2)^2} \pi_i \\
&= \frac{i+3}{(i+2)^2} \left( \frac{i+2}{(i+1)^2} \right) \pi_{i-1} \\
&= \frac{i+3}{(i+2)(i+1) \ldots (3)(2^2)} \pi_0 \\
&= \frac{i+3}{2(i+2)!} \pi_0 \\
&= \frac{i+2}{2(i+2)!} \pi_0 + \frac{1}{2(i+2)!} \pi_0 \\
&= \frac{1}{2(i+1)!} \pi_0 + \frac{1}{2(i+2)!} \pi_0.
\end{align*}
\]

\( \pi_0 \) is obtained from the normalization constraint

\[
1 = \sum_{i=-\infty}^{\infty} \pi_i = \pi_0 \left( 1 + 2 \sum_{i=1}^{\infty} \frac{1}{2(i+1)!} + 2 \sum_{i=1}^{\infty} \frac{1}{2(i+2)!} \right)
\]

\[
= \pi_0 \left( 1 + e - \frac{1}{0!} - \frac{1}{1!} + e - \frac{1}{0!} - \frac{1}{1!} - \frac{1}{2!} \right)
\]

\[
= \pi_0 (2e - 3.5),
\]

which gives \( \pi_0 = \frac{1}{2e-3.5} = 0.51638 \). Then

\[
\pi_i = \frac{0.51638}{2} \left( \frac{1}{(i+1)!} + \frac{1}{(i+2)!} \right) = 0.25819 \left( \frac{1}{(i+1)!} + \frac{1}{(i+2)!} \right)
\]

for \( i > 0 \), and \( \pi_{-i} = \pi_i \).
(b) (7 marks) Let $M_{i,j}$ be the mean time to reach state $j$ starting in state $i$. The thing to note about this chain is that it is probabilistically symmetric about state 0; that is, any sample path segment has the same probability as the negative of that sample path segment (the one obtained by flipping it about the $x$-axis when graphing it). By this symmetry we have $M_{1,0} = M_{-1,0}$. We first compute the mean return time to state 0 (which is $1/\pi_0$) by conditioning on the first step. This gives
\[
\frac{1}{\pi_0} = \frac{1}{2}(1 + M_{1,0}) + \frac{1}{2}(1 + M_{-1,0}) = 1 + M_{-1,0}.
\]
Therefore, $M_{-1,0} = \frac{1}{\pi_0} - 1$. Next, we compute $M_{0,1}$ by conditioning on the first step. This gives
\[
M_{0,1} = \frac{1}{2}(1) + \frac{1}{2}(1 + M_{-1,0} + M_{0,1}) = 1 + \frac{1}{2} \left( \frac{1}{\pi_0} - 1 \right) + \frac{1}{2} M_{0,1}
\]
or
\[
M_{0,1} = 2 + \frac{1}{\pi_0} - 1 = 1 + \frac{1}{\pi_0}.
\]
From part(a) we have
\[
M_{0,1} = 1 + 2e - 3.5 = 2e - 2.5 = 2.9366.
\]