1. (15 marks)

(a) (6 marks) If $f_r$ denotes the probability mass function of the family size in the modified branching process with parameter $r$ then

$$f_r(0) = r + (1 - r)f(0) \quad \text{and} \quad f_r(k) = (1 - r)f(k) \quad \text{for } k \geq 1$$

The generating function of this distribution is

$$G_r(s) = rs + \sum_{k=0}^{\infty} s^k(1 - r)f(k) = r + (1 - r)G(s).$$

If $\eta_r$ denotes the probability of ultimate extinction for the modified process, then $\eta_r$ is the smallest nonnegative solution to

$$\eta_r = G_r(\eta_r) \quad \text{or} \quad \eta_r = r + (1 - r)G(\eta_r).$$

Since $\eta$ satisfies $\eta = G(\eta)$ and $\eta < 1$, we have that the RHS above at $\eta$ satisfies

$$r + (1 - r)G(\eta) = r + (1 - r)\eta > r\eta + (1 - r)\eta = \eta.\quad \text{The generating function } G_r \quad \text{is convex on } [0, 1] \quad \text{so if } G_r(\eta) > \eta \quad \text{the equation } \eta_r = G_r(\eta_r) \quad \text{can have no solution smaller than } \eta \quad \text{(draw a picture). Since } \eta \text{ is also not a solution the solution must be larger than } \eta.$$

(b) (9 marks) With $f(k) = \left(\frac{1}{4}\right)\left(\frac{3}{4}\right)^k$ for $k = 0, 1, \ldots$ we have

$$G(s) = \sum_{k=0}^{\infty} s^k \left(\frac{1}{4}\right) \left(\frac{3}{4}\right)^k = \frac{1/4}{1 - 3s/4} = \frac{1}{4 - 3s}.$$ 

Then

$$G_r(s) = r + \frac{1 - r}{4 - 3s}$$

and the probability of ultimate extinction, $\eta_r$, is the smallest nonnegative solution to

$$\eta_r = r + \frac{1 - r}{4 - 3\eta_r} \quad \text{or} \quad \eta_r(4 - 3\eta_r) = r(4 - 3\eta_r) + 1 - r$$

or

$$3\eta_r^2 - (3r + 4)\eta_r + 3r + 1 = 0.$$
The solution to the above quadratic equation is

\[ \eta_r = \frac{3r + 4 \pm \sqrt{(3r + 4)^2 - 12(3r + 1)}}{6} \]

\[ = \frac{3r + 4 \pm \sqrt{9r^2 + 24r + 16 - 36r - 12}}{6} \]

\[ = \frac{3r + 4 \pm \sqrt{9r^2 - 12r + 4}}{6} \]

\[ = \frac{3r + 4 \pm \sqrt{(3r - 2)^2}}{6} \]

\[ = \frac{3r + 4 \pm |3r - 2|}{6}. \]

For \( r \geq \frac{2}{3} \) the smallest solution is \( \eta_r = 1 \). For \( r < \frac{2}{3} \) the smallest solution is

\[ \eta_r = \frac{3r + 4 - (2 - 3r)}{6} = \frac{6r + 2}{6} = r + \frac{1}{3}. \]

2. (15 marks)

(a) (4 marks) Note that the sum over all entries of \( P \) is \( M \) (since all the row sums are equal to 1). Therefore, if all the column sums are equal to \( c \) we must have \( Mc = M \), which implies \( c = 1 \).

(b) (4 marks) The vector \( \pi = (\frac{1}{M}, \ldots, \frac{1}{M}) \) (of dimension \( M \)) will satisfy \( \pi = \pi P \) (since all the column sums are equal to 1, by part(a)). Since this is a proper distribution it is a stationary distribution.

(c) (5 marks) Let \( \pi_1 \) and \( \pi_2 \) denote the unique stationary distributions corresponding to the transition matrices \( A \) and \( B \), respectively. Then check that

\[ \pi_\alpha = (\alpha \pi_1, (1 - \alpha) \pi_2), \]

where \( \alpha \in [0,1] \) satisfies \( \pi_\alpha = \pi_\alpha P \) and \( \pi_\alpha \) is a probability vector for each \( \alpha \).

(d) (2 marks) The Markov chain with transition matrix

\[
P = \begin{bmatrix}
0 & 1 \\
1 & 0
\end{bmatrix}
\]

is irreducible and positive recurrent and has stationary distribution \( \pi = (\frac{1}{2}, \frac{1}{2}) \), but has period 2.
3. (15 marks) The basic equation to set up is (2) below, with the conditional pdf of the most recent event time given by (1). Let $A_t$ denote the event that the last event before time $t$ was a type 1 event and let $L(t)$ denote the time of the last event before time $t$. As the hint indicates we want to condition on $N(t)$, which has a Poisson($\lambda t$) distribution, and then conditioned on $N(t) = n$ further condition on $L(t)$, which conditioned on $N(t) = n$ is distributed as max($U_1, \ldots, U_n$), where $U_1, \ldots, U_n$ are independent Uniform(0, $t$) random variables. Before proceeding, let us obtain the probability density function of $L(t)$ conditioned on $N(t) = n$. The conditional cdf of $L(t)$ satisfies

$$P(L(t) \leq s \mid N(t) = n) = P(\max(U_1, \ldots, U_n) \leq s) = P(U_1 \leq s, \ldots, U_n \leq s) = P(U_1 \leq s)^n = \left(\frac{s}{t}\right)^n,$$

for $s \in [0, t]$ (and the conditional cdf is 0 for $s < 0$ and 1 for $s > t$). Differentiating this we get the conditional pdf of $L(t)$ to be

$$f_{L(t)}(s) = \begin{cases} \frac{ns^{n-1}}{t^n} & \text{for } s \in [0, t] \\ 0 & \text{otherwise.} \end{cases} \quad (1)$$

Now conditioned on $L(t) = s$ and $N(t) = n$ the probability that the last event before time $t$ was a type 1 event is $p_1(s)$. That is, $P(A_t \mid L(t) = s, N(t) = n) = e^{-as}$. Putting this all together using the law of total probability, we have

$$P(A_t) = \sum_{n=0}^{\infty} \int_0^t P(A_t \mid L(t) = s, N(t) = n) f_{L(t)}(s) ds \times P(N(t) = n) \quad (2)$$

$$= \sum_{n=1}^{\infty} \int_0^t e^{-as} \frac{ns^{n-1}}{t^n} (\lambda t)^n e^{-\lambda t} \frac{1}{n!} e^{-\lambda t} \frac{s^n}{t^n} ds$$

$$= \lambda e^{-\lambda t} \int_0^t e^{-as} \sum_{n=1}^{\infty} \frac{(\lambda s)^n}{(n-1)!} ds$$

$$= \lambda e^{-\lambda t} \int_0^t e^{-as} e^{\lambda s} ds$$

$$= \lambda e^{-\lambda t} e^{t(\lambda - a)} - 1$$

$$= \frac{\lambda}{\lambda - a} (e^{-at} - e^{-\lambda t}) \quad (3)$$

The solution (4) holds if $a \neq \lambda$. If $a = \lambda$ then the solution above remains valid until (3). From there, if $a = \lambda$, it is easy to see that we get $P(A_t) = \lambda te^{-\lambda t}$. 

$$= \frac{\lambda}{\lambda - a} (e^{-at} - e^{-\lambda t}) \quad (4)$$
4. (15 marks)

(a) (7 marks) The transition matrix for the embedded chain is

\[
P = \begin{pmatrix}
0 & p_1 & p_2 & p_3 \\
1 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 \\
\end{pmatrix}
\]

and the stationary distribution is \( \psi = (\frac{1}{2}, \frac{p_1}{2}, \frac{p_2}{2}, \frac{p_3}{2}) \) (this is easily checked, and is intuitively arrived at since the embedded chain spends half its time in state 0 and for the rest of the time a proportion \( p_i \) is spent in state \( i \)).

(b) (8 marks) The generator matrix for the continuous time chain is

\[
G = \begin{pmatrix}
-\lambda_0 & \lambda_0 p_1 & \lambda_0 p_2 & \lambda_0 p_3 \\
\lambda_1 & -\lambda_1 & 0 & 0 \\
\lambda_2 & 0 & -\lambda_2 & 0 \\
\lambda_3 & 0 & 0 & -\lambda_3 \\
\end{pmatrix}
\]

and the stationary distribution satisfies \( \pi_i = C\psi_i / \lambda_i \), for \( i = 0, 1, 2, 3 \), where \( C \) is a normalizing constant. Thus,

\[
\pi_0 = \frac{C}{2\lambda_0} \quad \text{and} \quad \pi_i = \frac{Cp_i}{2\lambda_i} \quad \text{for} \quad i = 1, 2, 3
\]

where

\[
C = \left[ \frac{1}{2\lambda_0} + \frac{p_1}{2\lambda_1} + \frac{p_2}{2\lambda_2} + \frac{p_3}{2\lambda_3} \right]^{-1}.
\]

5. (15 marks)

(a) (10 marks) The number of subintervals of length \( h \) in the interval \([0,t)\) is \( \lfloor t/h \rfloor \) (\( \lfloor \cdot \rfloor \) is the floor function). So the probability that there are \( k \) events is

\[
P(N_h(t) = k) = \binom{\lfloor t/h \rfloor}{k} (\lambda h)^k (1 - \lambda h)^{\lfloor t/h \rfloor - k}
\]

We wish to see what happens to this as \( h \to 0 \). Write this as

\[
\frac{\lambda^k}{k!} \left( \lfloor t/h \rfloor \right) \left( \lfloor t/h \rfloor - 1 \right) \ldots \left( \lfloor t/h \rfloor - k + 1 \right) h^k \left( 1 - \lambda h \right)^{\lfloor t/h \rfloor} \left( 1 - \lambda h \right)^{t/h - k},
\]
where $\epsilon_h \in [0, 1)$ for any $h$. The product $(\lfloor t/h \rfloor)(\lfloor t/h \rfloor - 1)\ldots(\lfloor t/h \rfloor - k + 1)$ has $k$ terms so multiplying this by $h^k$ gives $t(t-h)(t-2h)\ldots(t-(k-1)h)$. Letting $h \to 0$ this goes to $t^k$. Also, clearly $(1 - \lambda h)^{\epsilon_h - k} \to 1$ as $h \to 0$. So now we have
\[
\lim_{h \to 0} P(N_h(t) = k) = \frac{(\lambda t)^k}{k!} \lim_{h \to 0} (1 - \lambda h)^{t/h}.
\]
For the last part write
\[
(1 - \lambda h)^{t/h} = \left(1 - \frac{\lambda t}{t/h}\right)^{t/h}
\]
from which it can be seen that $(1 - \lambda h)^{t/h} \to e^{-\lambda t}$ as $h \to 0$. So, finally, we see that
\[
\lim_{h \to 0} P(N_h(t) = k) = \frac{(\lambda t)^k}{k!} e^{-\lambda t}
\]
for $k \geq 0$, which shows that the limiting distribution of $N_h(t)$ is Poisson($\lambda t$).

(b) (5 marks) First, note that $T_h > t$ if and only if $N_h(t) = 0$. Therefore,
\[
\lim_{h \to 0} P(T_h > t) = \lim_{h \to 0} P(N_h(t) = 0) = e^{-\lambda t}
\]
from part(a) with $k = 0$. From this we can see that the limiting distribution of $T_h$ is Exponential($\lambda$).