Stochastic Processes

Final Exam, Solutions

1. (15 marks)
   
   (a) (6 marks) We have $M_0 = 0$. If $B$ is initially at position 1, then the time for $A$ to find $B$ is Geometric with parameter $1 - \alpha$. Thus, $M_1 = \frac{1}{1-\alpha}$. For $k > 1$, conditioning on $B$’s first move, we obtain by the law of total expectation that $M_k = \alpha(1 + M_k) + (1 - \alpha)(1 + M_{k-1})$, or $M_k = \frac{k}{1-\alpha} + M_{k-1}$. Recursing this gives $M_k = \frac{k-1}{1-\alpha} + M_1 = \frac{k}{1-\alpha}$.

   (b) (9 marks)
   
   (i) For a given $k$, the state space is $S = \{0, 1, \ldots, k\}$.

   (ii) The transition matrix is given by

   $$
   P = \begin{bmatrix}
   1 & 0 & \cdots & 0 \\
   1 - \alpha & \alpha & 0 & \cdots & 0 \\
   0 & 1 - \alpha & \alpha & 0 & \cdots & 0 \\
   \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\
   0 & \cdots & 0 & 1 - \alpha & \alpha & 0 \\
   0 & \cdots & 0 & 1 - \alpha & \alpha & 0 
   \end{bmatrix}
   $$

   where the rows going down (and columns going right) correspond to the states $0, 1, \ldots, k$, in that order.

   (iii) All states have period 1. State 0 is recurrent and states $1, \ldots, k$ are transient.

   (iv) The chain is not irreducible because state 0 does not communicate with any other state.

2. (15 marks)

   (a) (4 marks) False. The Galton-Watson branching process or the simple random walk as discussed in class are both time-homogeneous, but neither is stationary.
(b) (4 marks) True. We have for \( m < n \) and \( i, j \in S \), and any \( k \) such that \( n + k \) and \( m + k \) are both nonnegative,
\[
P(X_n = j \mid X_m = i) = \frac{P(X_n = j, X_m = i)}{P(X_m = i)}
\]
\[
= \frac{P(X_{n+k} = j, X_{m+k} = i)}{P(X_{m+k} = i)} \quad \text{(by stationarity)}
\]
\[
= P(X_{n+k} = j \mid X_{m+k} = i).
\]
Hence \( X \) is time-homogeneous.

(c) (4 marks) True. If \( X \) is time reversible then the stationary distribution \( \pi \) (which under the given conditions exists, is unique, and has all positive components) satisfies the local balance equations \( \pi_i p_{ij} = \pi_j p_{ji} \) for all \( i, j \in S \). For these to be satisfied it is necessary that if \( p_{ij} > 0 \) then \( p_{ji} > 0 \) and if \( p_{ij} = 0 \) then \( p_{ji} = 0 \).

(d) (3 marks) False. A counterexample is given by the transition matrix in Problem 4.

3. (15 marks)

(a) (7 marks) Following the hint, we follow just \( \{X_n : n \geq 0\} \), which keeps track of just the unmutated individuals. This process is itself a branching process in which each individual has 0 offspring with probability \( \frac{1}{3} \) and 2 offspring with probability \( \frac{2}{3} \) (“offspring” only count here if they are unmutated). We wish to find the probability of ultimate extinction for this unmutated population. The generating function of the family size distribution is \( G(s) = \frac{1}{3} + \frac{2}{3}s^2 \). The solution to \( s = G(s) \) is the smallest root in \([0, 1]\) of the quadratic equation \( \frac{2}{3}s^2 - s + \frac{1}{3} = 0 \).
The roots are
\[
\frac{1 \pm \sqrt{1 - \frac{8}{9}}}{\frac{4}{3}} = \frac{1 \pm 1/3}{\frac{4}{3}} = \frac{1}{2} \text{ or } 1.
\]
Thus, the probability of ultimate extinction of the unmutated population is \( \frac{1}{2} \).

(b) (8 marks) Since \( \{X_n : n \geq 0\} \) is a branching process we have that \( E[X_n] = \mu^n \), where \( \mu_X \) is the mean family size. From part(a), the mean family size (of unmutated offspring) is \( 2 \times \frac{2}{3} = \frac{4}{3} \). Thus, \( E[X_n] = (\frac{4}{3})^n \). For \( E[Y_n] \) we follow the hint and condition on \( X_{n-1} \) and \( Y_{n-1} \) (note that \( \{Y_n : n \geq 0\} \) is not a branching process of the type discussed in class). Suppose we are given that \( X_n = j \) and \( Y_n = k \). Then each of the \( j \) unmutated individuals will have on average \( 2 \times \frac{1}{12} = \frac{1}{6} \) mutated offspring, and each of the \( k \) mutated individuals will have on average \( 2 \times \frac{1}{6} = \frac{1}{3} \) mutated offspring.
\[ \frac{3}{4} = \frac{3}{2} \text{ mutated offspring. That is, } E[Y_n \mid X_{n-1} = j, Y_{n-1} = k] = \frac{j}{6} + \frac{3k}{2}. \] 

Then by the law of total expectation, 
\[ E[Y_n] = \frac{1}{6} E[X_{n-1}] + \frac{3}{2} E[Y_{n-1}] = \frac{1}{6}(\frac{4}{3})^{n-1} + \frac{3}{2} E[Y_{n-1}]. \] 

Recursing, and using the fact that 
\[ E[Y_0] = 0, \] 

we have 
\[ E[Y_n] = \frac{1}{6}(\frac{4}{3})^{n-1} + \frac{3}{2} E[Y_{n-1}] = \frac{1}{6}(\frac{4}{3})^{n-1} + \frac{3}{2} \left( \frac{1}{6}(\frac{4}{3})^{n-2} + \frac{3}{2} E[Y_{n-2}] \right) = \frac{1}{6}(\frac{4}{3})^{n-1} + \frac{3}{2} \left( \frac{1}{6}(\frac{4}{3})^{n-2} + \frac{3}{2} \left( \frac{1}{6}(\frac{4}{3})^{n-3} + \frac{3}{2} E[Y_{n-3}] \right) \right) \]

\[ \vdots \]

\[ = \frac{1}{6} \sum_{k=0}^{n-1} \left( \frac{3}{2} \right)^k \left( \frac{4}{3} \right)^{n-k-1} \]

\[ = \frac{1}{6} \left( \frac{4}{3} \right)^{n-1} \sum_{k=0}^{n-1} \left( \frac{9}{8} \right)^k \]

\[ = \frac{1}{6} \left( \frac{4}{3} \right)^{n-1} \frac{(9/8)^n - 1}{9/8 - 1} \]

\[ = \left( \frac{4}{3} \right)^n \left[ \left( \frac{9}{8} \right)^n - 1 \right] = \left( \frac{3}{2} \right)^n - \left( \frac{4}{3} \right)^n. \]

4. (15 marks)

(a) (9 marks) Note that local balance is not satisfied because the local balance equations are \( \pi_1 = \pi_2, \pi_1 = \pi_3, \) and \( 2\pi_2 = \pi_3, \) but the first two equations imply \( \pi_2 = \pi_3, \) which contradicts the third equation. The first two global balance equations are

\[ \pi_1 = \frac{1}{3} \pi_2 + \frac{2}{3} \pi_3 \quad (1) \]

\[ \pi_2 = \frac{1}{3} \pi_1 + \frac{1}{3} \pi_3, \quad (2) \]

which together with the normalization constraint \( \pi_1 + \pi_2 + \pi_3 = 1 \) are enough to determine \( \pi_1, \pi_2, \) and \( \pi_3. \) Plugging (1) into (2) gives

\[ \pi_2 = \frac{1}{3} \left( \frac{1}{3} \pi_2 + \frac{2}{3} \pi_3 \right) + \frac{1}{3} \pi_3, \]
or $\pi_2 = \frac{5}{8} \pi_3$. Plugging this back into (1) gives $\pi_1 = \frac{1}{3} \left( \frac{5}{8} \right) \pi_3 + \frac{2}{3} \pi_3 = \frac{7}{8} \pi_3$. The normalization constraint gives $\frac{7}{8} \pi_3 + \frac{5}{8} \pi_3 + \pi_3 = 1$, or $\pi_3 = \frac{2}{3}$. Then $\pi_1 = \frac{7}{20}$ and $\pi_2 = \frac{1}{4}$.

(b) (6 marks) A continuous time Markov chain for which \( \{X_n : n \geq 0\} \) is the embedded discrete time chain has infinitesimal rates $q_{ij} = v_i p_{ij}$ for some positive values $v_1, v_2, v_3$, and the transition probabilities $p_{ij}$ are the entries in the transition matrix $P$ from part(a). If this continuous time chain were time reversible then the local balance equations $\pi_i q_{ij} = \pi_j q_{ji}$ would be satisfied by the stationary distribution $\pi$ (note that this $\pi$ is not the same as the one in part(a)). Writing these out gives

$$
\pi_1 v_1 = \pi_2 v_2; \quad \pi_1 v_1 = \pi_3 v_3; \quad \pi_2 v_2 \frac{2}{3} = \pi_3 v_3 \frac{1}{3}.
$$

But the first two equations imply $\pi_2 v_2 = \pi_3 v_3$, which contradicts the third equation. Thus the local balance equations cannot be satisfied and so the continuous time chain cannot be time reversible.

5. (15 marks)

(a) (9 marks) Let the transition probabilities of the embedded jump chain be denoted by $p_{ij}$. Conditioning on the first jump we have that

$$
\eta_i = P(T_A < \infty \mid X(0) = i) = \sum_{j \in S} P(T_A < \infty \mid X(0) = i, \text{first jump is to } j) p_{ij} = \sum_{j \in A} p_{ij} + \sum_{j \in A^c} P(T_A < \infty \mid X(0) = i, \text{first jump is to } j) p_{ij} = \sum_{j \in A} \eta_j p_{ij} + \sum_{j \notin A} \eta_j p_{ij} = \sum_{j \in S} \eta_j p_{ij},
$$

noting that $\eta_j = 1$ for $j \in A$. But since $p_{ij} = q_{ij} / v_i$ for $i \neq j$ and $p_{ii} = 0$, we see that

$$
\eta_i = \sum_{j \neq i} \eta_j q_{ij} / v_i \quad \text{or} \quad v_i \eta_i = \sum_{j \neq i} \eta_j q_{ij}.
$$

(b) (6 marks) Letting $T_i$ denote the holding time in state $i$ and conditioning on the
first jump of the chain, we have that

$$\mu_i = E[T_A \mid X(0) = i] = \sum_{j \in S} E[T_A \mid X(0) = i, \text{first jump is to } j] p_{ij}$$

$$= \sum_{j \in A} E[T_i] p_{ij} + \sum_{j \notin A} (E[T_i] + \mu_j) p_{ij}$$

$$= E[T_i] \sum_{j \in S} p_{ij} + \sum_{j \notin A} \mu_j p_{ij}$$

$$= E[T_i] + \sum_{j \notin A} \mu_j p_{ij}$$

Since $E[T_i] = 1/v_i$, $p_{ij} = q_{ij}/v_i$, and $p_{ii} = 0$ we obtain

$$\mu_i = \frac{1}{v_i} + \sum_{j \notin A} \mu_j \frac{q_{ij}}{v_i} \quad \text{or} \quad v_i \mu_i = 1 + \sum_{j \notin A} q_{ij} \mu_j.$$  

Finally, since $\mu_j = 0$ for $j \in A$, we can also write this as

$$v_i \mu_i = 1 + \sum_{j \neq i} q_{ij} \mu_j.$$