1. (15 marks)

(a) (3 marks) If there is $k = 1$ coin, the number of flips required for a given person is a Geometric($\frac{1}{2}$) random variable, with mean 2. Thus, it will take on average 6 flips for all 3 people to leave the room.

(b) (6 marks) Now suppose there are $k = 2$ coins. Let $m_i$ be the expected number of time periods required for everyone to leave if we start with $i$ people in the room. Then $m_1 = 2$. By conditioning on the first set of flips we get

$$m_2 = \frac{1}{4}(1 + m_2) + \frac{1}{2}(1 + m_1) + \frac{1}{4}(1),$$

where $1 + m_2$, $1 + m_1$ and 1 are the expected number of time periods required given that the first set of flips gave 0, 1, or 2 heads, respectively. Solving for $m_2$, and substituting $m_1 = 2$, we obtain

$$m_2 = \frac{4}{3} \left( \frac{1}{4} + \frac{3}{2} + \frac{1}{4} \right) = \frac{8}{3}.$$

In a similar manner we compute $m_3$ as

$$m_3 = \frac{1}{4}(1 + m_3) + \frac{1}{2}(1 + m_2) + \frac{1}{4}(1 + m_1).$$

Solving for $m_3$ and substituting in $m_2 = 8/3$ and $m_1 = 2$, we obtain

$$m_3 = \frac{4}{3} \left( \frac{1}{4} + \frac{11}{6} + \frac{3}{4} \right) = \frac{34}{9}.$$

(c) (6 marks) If there are $k = 3$ coins, we still have $m_2 = 8/3$ and $m_1 = 2$, since a third coin is ignored if we start with 1 or 2 people. Starting with 3 people, all 3 coins are flipped in the first time period. Conditioning on the possible numbers of heads, we have

$$m_3 = \frac{1}{8}(1 + m_3) + \frac{3}{8}(1 + m_2) + \frac{3}{8}(1 + m_1) + \frac{1}{8}(1).$$

Solving for $m_3$ and substituting in $m_2 = 8/3$ and $m_1 = 2$, we obtain

$$m_3 = \frac{8}{7} \left( \frac{1}{8} + \frac{11}{8} + \frac{9}{8} + \frac{1}{8} \right) = \frac{22}{7}.$$
2. (15 marks)

(a) (1 mark) The state space is

\[ S = \{(1, 2), (1, 3), (2, 1), (2, 3), (3, 1), (3, 2)\}. \]

That is, \( S \) is all pairs \((i, j)\) such that \( i \) and \( j \) are integers, \( 1 \leq i, j \leq 3 \) and \( i \neq j \), where \( i \) is the vertex of the cat and \( j \) is the vertex of the mouse.

(b) (4 marks) The transition matrix is

\[
P = \begin{pmatrix}
0 & 0 & 0 & p & 0 & 1-p \\
0 & 0 & 0 & 1-p & 0 & p \\
p & 0 & 0 & 0 & 1-p & 0 \\
0 & 1-p & 0 & 0 & p & 0 \\
p & 0 & 1-p & 0 & 0 & 0 \\
1-p & 0 & p & 0 & 0 & 0
\end{pmatrix}.
\]

(c) (2 marks) The sequence of transitions \((1, 2) \rightarrow (3, 2) \rightarrow (1, 2)\) has probability \((1-p)^2 > 0\) and so state \((1, 2)\) can be returned to in 2 steps. Also, the sequence of transitions \((1, 2) \rightarrow (2, 3) \rightarrow (3, 1) \rightarrow (1, 2)\) has probability \(p^3 > 0\) and so state \((1, 2)\) can be returned to in 3 steps. Since the period of state \((1, 2)\) must divide both 2 and 3, the period of \((1, 2)\) must be 1. Finally, it’s easy to see that the chain is irreducible, and so the period of every state is 1.

(d) (5 marks) Since the columns of \(P\) sum to 1 it’s easy to see that the stationary distribution is

\[
\pi = \left(\frac{1}{6}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6}\right).
\]

Then the long-run proportion of time the mouse spends at vertex \(i\) is

\[
i = 1 : \pi_{(2,1)} + \pi_{(3,1)} = \frac{1}{6} + \frac{1}{6} = \frac{1}{3}
i = 2 : \pi_{(1,2)} + \pi_{(3,2)} = \frac{1}{6} + \frac{1}{6} = \frac{1}{3}
i = 3 : \pi_{(1,3)} + \pi_{(2,3)} = \frac{1}{6} + \frac{1}{6} = \frac{1}{3}.
\]

(e) (3 marks) Knowing where the mouse was previous to the immediate past can tell us where the cat is. For example, consider

\[
P(Y_n = 1 \mid Y_{n-1} = 2, Y_{n-2} = 3) \quad (1)
\]

and

\[
P(Y_n = 1 \mid Y_{n-1} = 2, Y_{n-2} = 1). \quad (2)
\]
In (1), the mouse jumped from 3 to 2 so we know the cat is at vertex 3. Then the probability that the mouse jumps to vertex 1 is $p$ (the cat jumps from 3 to 2). In (2), the mouse jumped from 1 to 2 so we know the cat is at vertex 1. In this case the probability the mouse jumps to vertex 1 is 0. So conditioning further in the past changes the conditional probability and so the process $\{Y_n : n \geq 0\}$ does not possess the Markov property.

3. (15 marks)

(a) (5 marks) Let $A$ be the event that there were at least 3 calls and let $B$ be the event that there was at least 1 emergency call. We wish to compute $P(A \cap B)$, which can be computed as $1 - P((A \cap B)^c) = 1 - P(A^c \cup B^c)$. We have

$$P(A^c) = P(\text{“0, 1, or 2 calls”}) = e^{-24\lambda}(1 + 24\lambda + (24\lambda)^2/2)$$

$$P(B^c) = P(\text{“0 emergency calls”}) = e^{-24\lambda p}$$

$$P(A^c \cap B^c) = P(\text{“0 emergency calls” and “0, 1 or 2 non-emergency calls”})$$

$$= P(\text{“0 emergency calls”})P(\text{“0, 1 or 2 non-emergency calls”})$$

$$= e^{-24\lambda p}e^{-24\lambda (1-p)}(1 + 24\lambda(1-p) + (24\lambda(1-p))^2/2)$$

$$= e^{-24\lambda}(1 + 24\lambda(1-p) + (24\lambda(1-p))^2/2).$$

Then

$$P(A^c \cup B^c) = P(A^c) + P(B^c) - P(A^c \cap B^c)$$

$$= e^{-24\lambda}(1 + 24\lambda + (24\lambda)^2/2) + e^{-24\lambda p}$$

$$- e^{-24\lambda}(1 + 24\lambda(1-p) + (24\lambda(1-p))^2/2)$$

$$= e^{-24\lambda p} + e^{-24\lambda}(24\lambda p + (24\lambda)^2(2p - p^2)/2)$$

$$= e^{-24\lambda p} + 24\lambda pe^{-24\lambda}(1 + 12\lambda(2-p)).$$

and

$$P(A \cap B) = 1 - e^{-24\lambda p} - 24\lambda pe^{-24\lambda}(1 + 12\lambda(2-p)).$$

An (arithmetically) simpler solution is to compute $P(A \cap B)$ as $P(A \cap B) = P(A) - P(A \cap B^c)$, and noting that

$$P(A \cap B^c) = P(\text{“at least 3 calls” \cap “0 emergency calls”})$$

$$= P(\text{“at least 3 non-emergency calls” \cap “0 emergency calls”})$$

$$= P(\text{at least 3 non-emergency calls})P(0 \text{ emergency calls}),$$
where the last equality follows because the number of non-emergency and emergency calls in a given 24 hour time period are independent. Then

\[ P(A \cap B^c) = e^{-24\lambda p} \left[ 1 - e^{-24\lambda(1-p)} \left( 1 + 24\lambda(1-p) + \frac{(24\lambda(1-p))^2}{2} \right) \right] \]

and

\[ P(A \cap B) = 1 - e^{-24\lambda} \left( 1 + 24\lambda + \frac{(24\lambda)^2}{2} \right) - e^{-24\lambda p} \left[ 1 - e^{-24\lambda(1-p)} \left( 1 + 24\lambda(1-p) + \frac{(24\lambda(1-p))^2}{2} \right) \right] \]

\[ = 1 - e^{-24\lambda p} - e^{-24\lambda} \left( 24\lambda + \frac{(24\lambda)^2}{2} - 24\lambda(1-p) - \frac{(24\lambda(1-p))^2}{2} \right) \]

\[ = 1 - e^{-24\lambda p} - e^{-24\lambda} \left( 24\lambda p + \frac{(24\lambda)^2(2p-p^2)}{2} \right) \]

\[ = 1 - e^{-24\lambda p} - 24\lambda p e^{-24\lambda(1 + 12\lambda(2 - p))}, \]

as before.

(b) (5 marks) If \( N_1(t) \) and \( N_2(t) \) denote the number of emergency and non-emergency calls, respectively, by time \( t \), then conditioned on \( N_1(10) = m \) the (unordered) event times, say \( U_1, \ldots, U_m \), are independent Uniform(0,10) random variables, and conditioned on \( N_2(10) = n \), the (unordered) event times, say \( V_1, \ldots, V_n \), are also independent Uniform(0,10) random variables, and are independent of \( U_1, \ldots, U_m \) (because the \( N_1 \) process is independent of the \( N_2 \) process). That is, all \( m + n \) random variables \( U_1, \ldots, U_m, V_1, \ldots, V_n \) are i.i.d. Uniform(0,10) under the given conditioning. As such, every ordering of these \( m + n \) random variables is equally likely. There are \( (m + n)! \) possible orderings and \( m(m + n - 1)! \) of these have an emergency call as the first event. Therefore,

\[ P(1\text{st call is emergency } \mid N_1(10) = m, N_2(10) = n) = \frac{m(m + n - 1)!}{(m + n)!} = \frac{m}{m + n}. \]

Similarly, there are \( m!n! \) orderings such that the first \( m \) calls are emergency calls. Therefore,

\[ P(1\text{st } m \text{ calls are emergency } \mid N_1(10) = m, N_2(10) = n) = \frac{m!n!}{(m + n)!} = \frac{1}{\binom{m+n}{m}}. \]
(c) (5 marks) Let $p_c(s)$ be the probability that an emergency call arriving at time $s$ arrives during congested traffic. Then

$$p_c(s) = \begin{cases} 
0 & 0 \leq s \leq 7 \\
1 & 7 < s < 9 \\
0 & 9 \leq s \leq 16 \\
1 & 16 < s < 18 \\
0 & 18 \leq s < 24 
\end{cases}$$

and $p_c(s)$ is periodic with period 24. Then $N_c(t)$ has a Poisson distribution with parameter $\lambda(t)$, where

$$\lambda(t) = \begin{cases} 
0 & 0 \leq t \leq 7 \\
(t - 7)p\lambda & 7 < s < 9 \\
2p\lambda & 9 \leq s \leq 16 \\
(2 + (t - 16))p\lambda & 16 < s < 18 \\
4p\lambda & 18 \leq s < 24 
\end{cases}$$

4. (15 marks)

(a) (3 marks) The infinitesimal generator matrix is

$$G = \begin{bmatrix}
1 & -(p_1 + q_1) & p_1 & 0 & \cdots & 0 & q_1 \\
2 & q_2 & -(p_2 + q_2) & p_2 & 0 & \cdots & 0 \\
& & \ddots & \ddots & \ddots & \ddots & \vdots \\
& & & 0 & q_{n-1} & -(p_{n-1} + q_{n-1}) & p_{n-1} & 0 \\
& & & & & & & 0 & q_n & -(p_n + q_n)
\end{bmatrix}$$

(b) (6 marks) We solve the global balance equations

$$\begin{align*}
\pi_1 &= 4\pi_4 + \pi_2 \\
3\pi_2 &= \pi_1 + 2\pi_3 \\
5\pi_3 &= 2\pi_2 + 3\pi_4 \\
7\pi_4 &= 3\pi_3
\end{align*}$$

From (6) we have $\pi_4 = \frac{3}{7}\pi_3$. Plugging this into (5) gives $5\pi_3 - \frac{9}{7}\pi_3 = 2\pi_2$, or $\pi_3 = \frac{7}{13}\pi_2$. Plugging this into (4) gives $3\pi_2 = \pi_1 + \frac{14}{13}\pi_2$, or $\pi_2 = \frac{13}{25}\pi_1$. Solving in terms of $\pi_1$ we have $\pi_3 = \frac{7}{25}\pi_1$ and $\pi_4 = \frac{3}{25}\pi_1$. The normalizing constraint then gives $\pi_1(1 + \frac{13}{25} + \frac{7}{25} + \frac{3}{25}) = 1$, or $\pi_1 = \frac{25}{48}$. Then we have $\pi_2 = \frac{13}{48}$, $\pi_3 = \frac{7}{48}$, and $\pi_4 = \frac{3}{48}$ (one can check that these satisfy (3) as well).
(c) (6 marks) In this case we check the local balance equations

\[ i\pi_i = (i + 1)\pi_{i+1} \quad \text{for} \ i = 1, \ldots, n - 1 \quad (7) \]
\[ n\pi_n = \pi_1 \quad (8) \]

From the equations (7) we obtain recursively

\[ \pi_{i+1} = \frac{i}{i+1}\pi_i = \frac{i-1}{i+1}\pi_{i-1} = \ldots = \frac{1}{i+1}\pi_1, \]

for \( i = 1, \ldots, n - 1 \). Note that when \( i = n - 1 \) the above relationship is exactly (8), which is a redundant equation. The normalization constraint gives

\[ \pi_1 \left(1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n}\right) = 1, \]

or

\[ \pi_1 = \frac{1}{\sum_{k=1}^{n} 1/k} \quad \text{and} \quad \pi_i = \frac{1/i}{\sum_{k=1}^{n} 1/k} \quad \text{for} \ i = 2, \ldots, n. \]

5. (15 marks)

(a) (3 marks) There are a couple of ways to do this. We will need the generating function of \( X_2 \), the number of individuals in generation 2, for part(c), so we will compute this here and it can be used to compute \( E[X_2] \). Let \( G_2(s) \) denote the generating function of \( X_2 \). By the composition property of generating functions, \( G_2(s) = G\left(H(s)\right) \). For the given even and odd generation family size distributions, we have

\[ G(s) = \sum_{k=0}^{\infty} s^k \left(\frac{1}{2}\right)^{k+1} = \frac{1}{2^{1 - s/2}} = \frac{1}{2 - s} \]
\[ H(s) = \sum_{k=0}^{\infty} s^k pq^k = p \cdot \frac{1}{1 - sq} = \frac{p}{1 - sq}. \]

Then

\[ G_2(s) = G(H(s)) = \frac{1}{2 - \frac{p}{1 - sq}} = \frac{1 - sq}{2 - 2sq - p}. \]

The mean number of individuals in generation 2 is \( G_2'(1) \). We have

\[ G_2'(s) = \frac{(2 - 2sq - p)(-q) - (1 - sq)(-2q)}{(2 - 2sq - p)^2} = \frac{pq}{(2 - 2sq - p)^2} \]

and

\[ G_2'(1) = \frac{pq}{(2(1 - q) - p)^2} = \frac{pq}{p^2} = \frac{q}{p}. \]
(b) (3 marks) We can consider just the even generations and note that ultimate extinction occurs if and only if the even generations ultimately go extinct. Considering just the even generations this is just a regular Galton-Watson branching process with family size generating function \( G_2(s) \) (as computed in part(a); that is, the “family” of any individual is the set of grandchildren of that individual). The probability of ultimate extinction is equal to 1 if and only if the mean family size is less than or equal to 1, which is to say \( q \leq 1/2 \).

(c) (9 marks) The probability of ultimate extinction is less than 1 if \( q > 1/2 \). Let \( \eta \) denote the probability of ultimate extinction. From the observations in part(b) and the computation of \( G_2(s) \) in part(a), \( \eta \) is the smallest solution in \([0, 1]\) to the equation \( \eta = G_2(\eta) \), which is \( \eta = \frac{1-\eta q}{2-2\eta q - p} \), or \( 2\eta - 2\eta^2 q - p\eta = 1 - q\eta \), or

\[
2q\eta^2 - (2 - p + q)\eta + 1 = 0 \quad \text{or} \quad 2q\eta^2 - (1 + 2q)\eta + 1 = 0.
\]

The 2 roots are

\[
\eta = \frac{1 + 2q \pm \sqrt{(1 + 2q)^2 - 8q}}{4q} = \frac{1 + 2q \pm \sqrt{1 - 4q + 4q^2}}{4q} = \frac{1 + 2q \pm \sqrt{(1 - 2q)^2}}{4q} = \frac{1 + 2q \pm (2q - 1)}{4q} \quad \text{for} \ q > 1/2
\]

\[
= 1 \quad \text{or} \quad \frac{1}{2q}.
\]

Therefore, for \( q > 1/2 \), the probability of ultimate extinction is \( \eta = \frac{1}{2q} \).