Applied Stochastic Processes

Midterm, Brief Solutions

1. (15 marks)

(a) (2 marks) When $N = 1$, each move from the line $x = 0$ will go to one of the lines $x = -1$ or $x = 1$ with probability $1/2$. Thus, the number of moves to go from the line $x = 0$ to one of the lines $x = -1$ or $x = 1$ has a Geometric($1/2$) distribution, with mean $M_{0}^{(1)} = 1/(1/2) = 2$.

(b) (2 marks) Conditioning on the first move from $(0, 0)$ we obtain

$$M_{0}^{(N)} = (1 + M_{0}^{(N)}) \frac{1}{2} + (1 + M_{1}^{(N)}) \frac{1}{4} + (1 + M_{-1}^{(N)}) \frac{1}{4}$$

$$= (1 + M_{0}^{(N)}) \frac{1}{2} + (1 + M_{1}^{(N)}) \frac{1}{2},$$

using observation (ii). Solving for $M_{0}^{(N)}$, we get $M_{0}^{(N)} = 2 + M_{1}^{(N)}$.

(c) (7 marks) For $k = 1, \ldots, N - 1$, we can condition on the particle’s first move starting from the line $x = k$ to obtain

$$M_{k}^{(N)} = (1 + M_{k}^{(N)}) \frac{1}{2} + (1 + M_{k-1}^{(N)}) \frac{1}{4} + (1 + M_{k+1}^{(N)}) \frac{1}{4},$$

noting that $M_{N}^{(N)} = 0$. This is equivalent to

$$2M_{k}^{(N)} = 4 + M_{k-1}^{(N)} + M_{k+1}^{(N)} \quad \text{or} \quad M_{k}^{(N)} - M_{k-1}^{(N)} = 4 + M_{k+1}^{(N)} - M_{k}^{(N)}.$$

Starting from $k = 1$, we get

$$M_{1}^{(N)} - M_{0}^{(N)} = 4 + M_{2}^{(N)} - M_{1}^{(N)}$$

$$= (4)(2) + M_{3}^{(N)} - M_{2}^{(N)}$$

$$\vdots$$

$$= (4)(N - 1) + M_{N}^{(N)} - M_{N-1}^{(N)} = 4(N - 1) - M_{N-1}^{(N)},$$

using $M_{N}^{(N)} = 0$. But, from part(b), $M_{1}^{(N)} - M_{0}^{(N)} = -2$. Therefore, solving for $M_{N-1}^{(N)}$, we obtain $M_{N-1}^{(N)} = 2 + 4(N - 1)$.

(d) (4 marks) Starting from the line $x = 0$, in order to get to one of the lines $x = -N$ or $x = N$ we must first get to one of the lines $x = -1$ or $x = 1$. The mean number of moves to do this is $M_{0}^{(1)}$. Once we get to one of the lines $x = -1$ or $x = 1$ we must then get to one of the lines $x = -2$ or $x = 2$. From observation (ii), the mean number of moves to do this is $M_{1}^{(2)}$. Continuing this reasoning, we see that we can break up the number of moves to get from the line $x = 0$ to one of the lines $x = -N$ or $x = N$ into the number of moves to first get to $x = -1$ or $x = 1$, plus the number of moves after that to get to one of the lines $x = -2$ or $x = 2$, and so on, with the final block of moves being those
to get from one of the lines \( x = -(N - 1) \) or \( x = N - 1 \) to one of the lines \( x = -N \) or \( x = N \). The total expected number of moves is then the expression given in the problem statement:

\[
M_0^{(N)} = M_0^{(1)} + M_1^{(2)} + M_2^{(3)} + \ldots + M_{N-1}^{(N)}.
\]

From parts (a), (c) and the above expression, we then get that

\[
M_0^{(N)} = 2 + (2 + 4(2 - 1)) + (2 + 4(3 - 1)) + \ldots + (2 + 4(N - 1))
\]

\[
= 2N + 4 \sum_{i=1}^{N-1} = 2N + 4 \frac{N(N - 1)}{2} = 2N^2.
\]

2. (15 marks)

(a) (5 marks) The state space is \( S = \{0, 1, \ldots, M\} \) and the transition probability matrix is

\[
P = \begin{bmatrix}
0 & \frac{1}{M} & \frac{M}{M+1} & 0 & 0 & \ldots & 0 \\
1 & 0 & \frac{1}{M} & \frac{M-1}{M+1} & 0 & \ldots & 0 \\
2 & \frac{1}{M} & 0 & \frac{M-2}{M+1} & 0 & \ldots & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
M & 0 & 0 & 0 & 0 & \ldots & \frac{1}{M} \\
\end{bmatrix}
\]

(b) (2 marks) The chain is both irreducible and aperiodic.

(c) (6 marks) We use the global balance equations

\[
\pi_i = \frac{M + 1 - i}{M + 2 - i} \pi_{i-1}
\]

for \( i = 1, \ldots, M \) together with the normalization constraint \( \pi_0 + \pi_1 + \ldots + \pi_M = 1 \) to determine the stationary probabilities. The \( i \)th global balance equation can be solved recursively to give \( \pi_i \) in terms of \( \pi_0 \):

\[
\pi_i = \frac{M + 1 - i}{M + 2 - i} \pi_{i-1} = \frac{M + 1 - i}{M + 2 - i} \left( \frac{M + 1 - (i - 1)}{M + 2 - (i - 1)} \pi_{i-2} \right)
\]

\[
= \frac{M + 1 - i}{M + 2 - i} \left( \frac{M + 1 - (i - 1)}{M + 2 - (i - 1)} \right) \times \ldots \times \left( \frac{M}{M + 1} \right) \pi_0
\]

\[
= \frac{M!/(M - i)!}{(M + 1)!/(M - i + 1)!} \pi_0
\]

\[
= \frac{M + 1 - i}{M + 1} \pi_0.
\]

The normalization constraint then gives

\[
1 = \pi_0 \left( 1 + \frac{M}{M + 1} + \frac{M - 1}{M + 1} + \ldots + \frac{1}{M + 1} \right)
\]

\[
= \frac{\pi_0}{M + 1} \left( 1 + 2 + 3 + \ldots + (M + 1) \right) = \frac{\pi_0}{M + 1} \frac{(M + 1)(M + 2)}{2} = \pi_0 \frac{M + 2}{2}.
\]
Thus, we obtain
\[ \pi_0 = \frac{2}{M+2} \quad \text{and} \quad \pi_i = \frac{2(M+1-i)}{(M+1)(M+2)} \quad \text{for } i = 1, \ldots, M. \]

(d) (2 marks) Starting in state 0, the mean time to return to state 0 is
\[ \mu_0 = \frac{1}{\pi_0} = \frac{M+2}{2}. \]
To find the mean time, starting in state 0, until the first time urn 2 has all the red balls, we note that (i) starting in state \( M \) (urn 2 has all the red balls), the mean time to return to state \( M \) is \( \frac{1}{\pi_M} \), and (ii) from state \( M \), the first step must be to state 0, and so the mean time to return to state \( M \) is 1 plus the mean time, starting in state 0, until the first time urn 2 has all the red balls. That is, letting \( \mu_{0M} \) denote the mean time, starting in state 0, until the first time urn 2 has all the red balls, we have \( \mu_M = 1 + \mu_{0M} \), or \( 1/\pi_M = 1 + \mu_{0M} \), or \( \mu_{0M} = \frac{1}{\pi_M} - 1 \). So
\[ \mu_{0M} = \frac{(M+1)(M+2)}{2} - 1 = \frac{M+3}{2}. \]

3. (15 marks)

(a) (2 marks) The chain is irreducible because state 0 communicates with every other state (since \( p_{0i} = p_{i0} = a_i > 0 \) for every \( i \geq 1 \)), and, for any states \( i, j \geq 1 \) with \( i \neq j \), state \( j \) is accessible from state \( i \) through state 0 (e.g., \( p_{ij}(2) = p_{i0} p_{0j} = a_i a_j > 0 \)). The chain is aperiodic because \( p_{ii} = 1 - a_i > 0 \) for any \( i \geq 1 \) (though, by irreducibility, we just need to show aperiodicity for a single state). To be complete, we should note that \( 1 - a_i > 0 \) for every \( i \) because \( a_i > 0 \) and \( \sum_{i=1}^{\infty} a_i = 1 \) by assumption. If \( a_i = 1 \) for some \( i \), the normalization condition would imply all the other \( a_j \)'s were 0, which would be a contradiction.

(b) (6 marks) Starting in state 0, the chain must first go to some state \( i \), where \( i \geq 1 \). From state \( i \), the chain must either go to state 0 or stay in state \( i \). Therefore, for \( n \geq 2 \), starting in state 0, the only ways for the chain to make a first return to state 0 at time \( n \) are, for \( i \geq 1 \), to go to state \( i \) at the first step, stay in state \( i \) for \( n-2 \) steps, then go back to state 0 at the \( n \)th step. Thus,
\[ f_{00}(n) = \sum_{i=1}^{\infty} (p_{0i})(p_{ii}^{n-2})(p_{i0}) = \sum_{i=1}^{\infty} a_i^2 (1 - a_i)^{n-2}. \]

To relate the \( f_{00}(n) \) to recurrence of state 0, we note that, starting in state 0, the event that the chain eventually returns to state 0 is equivalent to the event that the chain returns to state 0 for the first time at step \( n \), for some \( n \geq 1 \). That is,
\[ P(\text{chain ever returns to state 0} \mid \text{start in state 0}) = \sum_{n=1}^{\infty} f_{00}(n). \]

State 0 and so, by irreducibility, every state is recurrent if and only if the sum above is equal to 1. Computing this sum (noting that \( f_{00}(1) = 0 \)), we have
\[
\sum_{n=2}^{\infty} f_{00}(n) = \sum_{n=2}^{\infty} \sum_{i=1}^{\infty} a_i^2 (1 - a_i)^{n-2} = \sum_{i=1}^{\infty} a_i^2 \sum_{n=2}^{\infty} (1 - a_i)^{n-2} \\
= \sum_{i=1}^{\infty} a_i^2 \frac{1}{1 - (1 - a_i)} = \sum_{i=1}^{\infty} a_i = 1.
\]
(c) (4 marks) To show that a stationary distribution does not exist, one can show that the mean return time to state 0 is equal to infinity. A more direct way would be to show that the global balance equations do not have a normalizable solution. The $i$th global balance equation, for $i \geq 1$, is
\[
\pi_i = a_i \pi_0 + (1 - a_i) \pi_i \quad \text{or} \quad \pi_i = \pi_0.
\]
In other words, any solution to the global balance equations must satisfy $\pi_i = c$, for all $i \geq 0$ and for some constant $c$. Since there are infinitely many states, the sum of all components of such a solution is equal to infinity, for any $c > 0$. Thus, a stationary distribution does not exist.

(d) (3 marks) If $p_{0i} = a_i$, $p_{i0} = 1 - a_i$ and $p_{ii} = a_i$ for $i \geq 1$, then the $i$th global balance equation becomes
\[
\pi_i = a_i \pi_0 + a_i \pi_i \quad \text{or} \quad \pi_i = \frac{a_i}{1 - a_i} \pi_0.
\]
A stationary distribution exists if
\[
\sum_{i=1}^{\infty} \frac{a_i}{1 - a_i} < \infty.
\]
To show that this is so, we can use the fact that $a_i \to 0$ as $i \to \infty$ since $\sum_{i=1}^{\infty} a_i < \infty$. Therefore, there is some $N > 0$ such that $a_i < 1/2$ for all $i \geq N$. This implies $1 - a_i > 1/2$ for all $i \geq N$. Therefore,
\[
\sum_{i=1}^{\infty} \frac{a_i}{1 - a_i} = \sum_{i=1}^{N} \frac{a_i}{1 - a_i} + \sum_{i=N+1}^{\infty} \frac{a_i}{1 - a_i} < \sum_{i=1}^{N} \frac{a_i}{1 - a_i} + 2 \sum_{i=N+1}^{\infty} a_i < \infty,
\]
since in the last expression the first sum is a finite sum and the second sum is less than or equal to 2.