1. (15 marks)

(a) (6 marks) Let $p_1$ be the probability that player 1 wins when he rolls first and let $\bar{p}_1$ be the probability that player 1 wins if he doesn’t roll first. By symmetry, $\bar{p}_1$ doesn’t depend on which of players 2, \ldots, 6 rolls first. Conditioning on the outcome of the first roll, we obtain

$$p_1 = (1) \frac{1}{6} + \bar{p}_1 \frac{5}{6} \quad \text{and} \quad \bar{p}_1 = p_1 \frac{1}{6} + \bar{p}_1 \frac{4}{6} + (0) \frac{1}{6}.$$ 

From the second equation we obtain $\bar{p}_1 = \frac{1}{2} p_1$. Plugging this into the first equation, we obtain $p_1 = \frac{1}{6} + \frac{5}{12} p_1$, or $p_1 = \frac{2}{7}$.

(b) (3 marks) The number of times the die is rolled until a winner is declared is a Geometric random variable with parameter $p = \frac{1}{6}$. Therefore, the expected number of times the die is rolled is 6.

(c) (6 marks) Now we condition on 3 possible outcomes for the first roll: a 1 is rolled, a 2 is rolled, or neither a 1 or 2 is rolled. By symmetry, given that neither a 1 or 2 was rolled first, the conditional probability that player 1 wins is the same whether player 3, 4, 5 or 6 rolls second. Let $p_1$ be as in part(a), $p_2$ be the probability that player 1 wins if player 2 rolls first, and $\bar{p}_{12}$ be the probability that player 1 wins if neither player 1 or 2 rolls first. Conditioning on the outcome of the first roll, we have

$$p_1 = (1) \frac{1}{4} + p_2 \frac{1}{4} + \bar{p}_{12} \frac{1}{2}, \quad p_2 = p_1 \frac{1}{4} + (0) \frac{1}{4} + \bar{p}_{12} \frac{1}{2}, \quad \bar{p}_{12} = p_1 \frac{1}{4} + p_2 \frac{1}{4} + \bar{p}_{12} \frac{3}{8} + (0) \frac{1}{8}.$$ 

The third equation gives $\bar{p}_{12} = \frac{2}{5} p_1 + \frac{2}{5} p_2$. Plugging this into the first and second equations gives

$$p_1 = \frac{1}{4} + \frac{1}{5} p_1 + \frac{9}{20} p_2 \quad \text{and} \quad p_2 = \frac{9}{20} p_1 + \frac{1}{5} p_2.$$ 

The second of these equations gives $p_2 = \frac{9}{16} p_1$. Plugging this into the first equation gives $p_1 = \frac{1}{4} + \frac{29}{60} p_1$, or $p_1 = \frac{16}{35}$. 
2. (15 marks)

(a) The state space is \( S = \{000, 001, 010, 011, 100, 101, 110, 111\} \). The following table summarizes all the possible transitions, each of which has probability 0.5 (the transitions are from the states in the first column to the states in the final column):

<table>
<thead>
<tr>
<th>((X_n, Y_n, Z_n))</th>
<th>((X_n, X_{n-1}, X_{n-2}))</th>
<th>(X_{n+1})</th>
<th>((X_{n+1}, X_n, X_{n-1}))</th>
<th>((X_{n+1}, Y_{n+1}, Z_{n+1}))</th>
</tr>
</thead>
<tbody>
<tr>
<td>000</td>
<td>000</td>
<td>0</td>
<td>000</td>
<td>000</td>
</tr>
<tr>
<td>000</td>
<td>000</td>
<td>1</td>
<td>001</td>
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<tr>
<td>001</td>
<td>011</td>
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<td>010</td>
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<tr>
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<tr>
<td>111</td>
<td>100</td>
<td>1</td>
<td>110</td>
<td>101</td>
</tr>
</tbody>
</table>

The transition matrix is therefore

\[
P = \begin{bmatrix}
0.5 & 0 & 0 & 0 & 0 & 0 & 0.5 \\
0 & 0 & 0 & 0.5 & 0.5 & 0 & 0 \\
0 & 0 & 0 & 0.5 & 0.5 & 0 & 0 \\
0.5 & 0 & 0 & 0 & 0 & 0 & 0.5 \\
0 & 0.5 & 0 & 0 & 0.5 & 0 & 0 \\
0 & 0.5 & 0 & 0 & 0 & 0.5 & 0 \\
0 & 0 & 0.5 & 0 & 0.5 & 0 & 0
\end{bmatrix},
\]

where the rows and columns correspond to the states listed in the order 000, 001, 010, 011, 100, 101, 110, 111.

(b) The chain is both irreducible and aperiodic.
(c) The stationary distribution is uniform on the state space $S$, since $P$ is doubly stochastic.

3. (15 marks)

(a) In the $X$ chain the transition from state $i$ to state $j$ occurs if and only if the $Y$ chain makes a transition directly from state $i$ to state $j$ or makes a transition from state $i$ to state 0, stays in state 0 for $n$ transitions, then makes a transition from state 0 to state $j$, for some $n \geq 1$. That is,

$$p_{ij} = P(X_1 = j \mid X_0 = i)$$

$$= P(Y_1 = j \mid Y_0 = i) + \sum_{n=1}^{\infty} P(Y_{n+1} = j, Y_n = 0, \ldots, Y_1 = 0 \mid Y_0 = i)$$

$$= q_{ij} + \sum_{n=1}^{\infty} q_{i0} q_{00}^{n-1} q_{0j}$$

$$= q_{ij} + q_{i0} q_{0j} \sum_{n=0}^{\infty} q_{00}^{n}$$

$$= q_{ij} + q_{i0} q_{0j} \frac{1}{1 - q_{00}},$$

as required.

(b) It suffices to show that

$$\psi_j = \sum_{i=1}^{M} \psi_i p_{ij} \quad \text{for } j = 1, \ldots, M.$$

We have

$$\sum_{i=1}^{M} \psi_i p_{ij} = \sum_{i=1}^{M} \psi_i \left( q_{ij} + \frac{q_{i0} q_{0j}}{1 - q_{00}} \right)$$

$$= \sum_{i=1}^{M} \psi_i q_{ij} + \frac{q_{0j}}{1 - q_{00}} \sum_{i=1}^{M} \psi_i q_{i0}$$

$$= \sum_{i=0}^{M} \psi_i q_{ij} + \frac{q_{0j}}{1 - q_{00}} \sum_{i=0}^{M} \psi_i q_{i0} - \psi_0 q_{0j} - \frac{q_{0j}}{1 - q_{00}} \psi_0 q_{00}$$

$$= \psi_j + \frac{q_{0j}}{1 - q_{00}} \psi_0 - \psi_0 q_{0j} - \frac{q_{0j}}{1 - q_{00}} \psi_0 q_{00}$$

$$= \psi_j + \psi_0 \left( \frac{q_{0j} - q_{0j} (1 - q_{00}) - q_{0j} q_{00}}{1 - q_{00}} \right) = \psi_j$$

as required.