Applied Stochastic Processes

Midterm Exam

Instructions:

(a) The exam is closed book. No books are allowed. You may use one 8.5 × 11 inch sheet of notes and a calculator.

(b) There are 3 questions. Stat 855 students must do all of the problems. Stat 455 students must do 2 of questions 1, 2 and 3, and if you answer all three of questions 1, 2 and 3, you must specify which two you want graded (the default if you do not specify is questions 1 and 2).

(c) Each question is worth 15 marks for a total of 30 marks (Stat455) or 45 marks (Stat855).

(d) Show all your work. Partial credit is given.

(e) You have 120 minutes. Good luck!
1. (15 marks) Let \( \{X_n : n \geq 0\} \) be a Markov chain with state space \( S = \{0, 1, 2, 3, 4, 5, 6\} \) and transition probability matrix

\[
P = \begin{bmatrix}
0 & 1 & 2 & 3 & 4 & 5 & 6 \\
0 & \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 \\
2 & 0 & 0 & 0 & 0 & 0 & 0 \\
3 & 0 & 0 & 0 & 0 & 0 & 0 \\
4 & 0 & 0 & 0 & 0 & 0 & 1 \\
5 & 0 & 0 & 0 & 0 & 0 & 0 \\
6 & 1 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
\]

(a) (6 marks) Identify the equivalence classes and for each class, state whether it is transient or recurrent, and give the period of the class. For any class with period greater than 1 give the cyclic classes.

(b) (4 marks) Starting in state 0 what is the mean number of steps to get to state 1?

(c) (5 marks) If a stationary distribution for this Markov chain exists find one.

2. (15 marks) Let \( \{X_n : n \geq 0\} \) be an irreducible, positive recurrent Markov chain. Let \( p_{ij}(n) \) denote the \( n \)-step transition probability from \( i \) to \( j \) and let \( \ell_{ij}(n) = P(X_n = j, X_{n-1} \neq i, \ldots, X_1 \neq i | X_0 = i) \) for \( n \geq 2 \), and \( \ell_{ij}(1) = p_{ij} \) (i.e., \( \ell_{ij}(n) \) is the probability that the chain visits \( j \) at time \( n \) during a sojourn from \( i \) back to \( i \)). Let

\[
P_{ij}(s) = \sum_{n=0}^{\infty} p_{ij}(n) s^n \quad \text{and} \quad L_{ij}(s) = \sum_{n=0}^{\infty} \ell_{ij}(n) s^n
\]

be the generating functions of the sequences \( \{p_{ij}(n)\}_{n=0}^{\infty} \) and \( \{\ell_{ij}(n)\}_{n=0}^{\infty} \), respectively.

(a) (7 marks) Suppose the transition probability matrix of \( \{X_n : n \geq 0\} \) is given by

\[
P = \begin{bmatrix}
0 & p & q \\
1 & 0 & 0 \\
1 & 0 & 0
\end{bmatrix},
\]

where \( 0 < p < 1 \) and \( q = 1 - p \) (say the rows of \( P \) correspond to the states 0, 1 and 2, respectively). Compute \( P_{0j}(s) \) and \( L_{0j}(s) \) explicitly for \( j = 0, 1, 2 \), and show that

\[
L_{0j}(s) = \frac{P_{0j}(s)}{P_{00}(s)}.
\]

(b) (8 marks) More generally, show that

\[
P_{ij}(s) = P_u(s) L_{ij}(s)
\]

for \( i \neq j \). Hint: Compute \( p_{ij}(n) \) by conditioning on the last time before time \( n \) that the chain was in state \( i \).
3. (15 marks) Consider an irreducible, positive recurrent Markov chain \( \{X_n : n \geq 0\} \). We say that two states, \( i \) and \( j \) \( (i \neq j) \), are symmetric if the probability that \( j \) is visited during a sojourn from \( i \) back to \( i \) is equal to the probability that \( i \) is visited during a sojourn from \( j \) back to \( j \). Show that if states \( i \) and \( j \) are symmetric, then the expected number of visits to \( j \) during a sojourn from \( i \) back to \( i \) is equal to one. \textit{Hint}: Let \( N_j \) denote the number of visits to state \( j \) during a sojourn from \( i \) back to \( i \). Write \( E[N_j] = \sum_{k=1}^\infty P(N_j \geq k) \) then show that \( P(N_j \geq k) = \theta(1 - \theta)^{k-1} \) for some \( \theta \).