1. (15 marks)

(a) (4 marks) Each individual in the initial population is the root of a Galton-Watson branching process starting at a single individual. If the initial population is of size \( k \) then we have \( k \) independent and identically distributed branching processes, each of which will go extinct with probability \( \eta \), and the entire population will go extinct with probability \( \eta^k \). Letting \( X_0 \) denote the size of the initial population we have that

\[
P(\text{pop'n goes extinct}) = \sum_{k=0}^{\infty} P(\text{pop'n goes extinct} \mid X_0 = k) P(X_0 = k) = \sum_{k=0}^{\infty} \eta^k P(X_0 = k) = E[\eta^{X_0}] = G(\eta) = \eta,
\]

since \( \eta = G(\eta) \).

(b) (3 marks) As in part (a), we still have \( P(\text{pop'n goes extinct}) = E[\eta^{X_0}] \), but now with the distribution of \( X_0 \) being discrete uniform on \( \{0, 1, \ldots, 9\} \), we have

\[
P(\text{pop'n goes extinct}) = E[\eta^{X_0}] = \frac{1}{10} \sum_{k=0}^{9} \eta^k = \begin{cases} \frac{1-\eta^{10}}{10(1-\eta)} & \text{if } 0 \leq \eta < 1 \\ 1 & \text{if } \eta = 1 \end{cases}
\]

(c) (8 marks) First we compute \( \eta \), the extinction probability starting from a single individual, when the family size distribution is Geometric\((p)\). The mean family size is \( \mu = \frac{1-p}{p} \), and for \( p \geq \frac{1}{2} \) we have \( \mu \leq 1 \), and so \( \eta = 1 \) in this case. Assume now that \( p < \frac{1}{2} \). The generating function of the family size distribution is

\[
G(s) = \sum_{k=0}^{\infty} s^k p(1-p)^k = p \sum_{k=0}^{\infty} (s(1-p))^k = \frac{p}{1-s(1-p)}.
\]

Then the equation \( s = G(s) \) is equivalent to the quadratic equation \((1-p)s^2 - s + p = 0\), with roots

\[
s = \frac{1 \pm \sqrt{1 - 4p(1-p)}}{2(1-p)} = \frac{1 \pm |1 - 2p|}{2(1-p)} = \frac{1 \pm (1 - 2p)}{2(1-p)}
\]
since \( p < \frac{1}{2} \). The two roots are 1 and \( \frac{p}{1-p} \), and so \( \eta = \frac{p}{1-p} \), the smaller of the two. From part (a) this is also the extinction probability when the initial population size has distribution with generating function \( G \). Therefore, in this case we have

\[
P(\text{pop'n goes extinct}) = \begin{cases} \frac{p}{1-p} & \text{for } p < \frac{1}{2} \\ 1 & \text{for } p \geq \frac{1}{2} \end{cases}.
\]

From part (b), the extinction probability in terms of \( p \) is given by

\[
P(\text{Pop'n goes extinct}) = \begin{cases} 1 - (\frac{p}{1-p})^{10} & \text{for } p < \frac{1}{2} \\ 1 & \text{for } p \geq \frac{1}{2} \end{cases}.
\]

2. (15 marks)

(a) (5 marks) The transition matrix \( P \) has \( f(0) \) on the diagonal, and for \( k \geq 1 \) it has \( f(k) \) on the \( k \)th upper off diagonal and \( f(-k) \) on the \( k \)th lower off diagonal. The thing to note about \( P \) is that it is symmetric; as such the entries of each column sum to 1. Therefore, the vector \( \pi = (\ldots, 1, 1, 1, \ldots) \) of all ones satisfies the balance equations \( \pi = \pi P \), and this \( \pi \) is not normalizable. Therefore, there is no invariant distribution because if there were it the unique solution to this equation. Therefore, the chain is not positive recurrent.

(b) (5 marks) The \( j \)th balance equation, \( j \geq 0 \), is given by

\[
\pi_j = \alpha(j) \sum_{k \text{ even}} \pi_k + \beta(j) \sum_{k \text{ odd}} \pi_k.
\]

For notational convenience, let

\[
\alpha_{\text{even}} = \sum_{j \text{ even}} \alpha_j, \quad \beta_{\text{even}} = \sum_{j \text{ even}} \beta_j, \quad \pi_{\text{even}} = \sum_{j \text{ even}} \pi_j.
\]

Then summing (1) over \( j \) even gives \( \pi_{\text{even}} = \alpha_{\text{even}} \pi_{\text{even}} + \beta_{\text{even}} (1 - \pi_{\text{even}}) \), which gives

\[
\pi_{\text{even}} = \frac{\beta_{\text{even}}}{\beta_{\text{even}} + 1 - \alpha_{\text{even}}},
\]

so from (1) we get

\[
\pi_j = \frac{\alpha(j) \beta_{\text{even}} + \beta(j) (1 - \alpha_{\text{even}})}{\beta_{\text{even}} + 1 - \alpha_{\text{even}}}
\]

Since an invariance distribution exists the chain is positive recurrent.
(c) (5 marks) For \( j \geq 1 \), the \( j \)th balance equation is \( \pi_j = \frac{j}{j+1} \pi_{j-1} \) and recursing this down gives

\[
\pi_j = \frac{j}{j+1} \pi_{j-1} = \left( \frac{j}{j+1} \right) \left( \frac{j-1}{j} \right) \pi_{j-2} = \ldots = \frac{j!}{(j+1)!} \pi_0 = \frac{1}{j+1} \pi_0.
\]

Thus,

\[
\sum_{j=0}^{\infty} \pi_j = \pi_0 \sum_{j=1}^{\infty} \frac{1}{j} = \infty
\]

for any \( \pi_0 > 0 \). Therefore, any solution to the balance equations is not normalizable, and hence the chain is not positive recurrent.

3. (15 marks) To argue that \( P(T = k) = P(T > k - r - 1)qp^r \) for \( k > r \) we can condition to get

\[
P(T = k) = P(T = k \mid T > k - r - 1)P(T > k - r - 1)
\]

\[
+ P(T = k \mid T \leq k - r - 1)P(T \leq k - r - 1).
\]

The second summand is obviously 0 because if \( T \leq k - r - 1 \) then \( T \) cannot equal \( k \). So it remains to argue that \( P(T = k \mid T > k - r - 1) = qp^r \). But if \( T > k - r - 1 \) the only way for it to equal \( k \) is for moves \( k, k-1, \ldots, k-r \) to have been up and for move \( k-r-1 \) to have been down, and the probability of this is \( qp^r \). Clearly, \( P(T = r) = p^r \) and \( P(T = k) = 0 \) for \( k < r \). Now we can compute \( G(s) \) as follows:

\[
G(s) - p^r s^r = qp^r \sum_{k=r+1}^{\infty} s^k P(T > k - r - 1)
\]

\[
= qp^r \sum_{k=r+1}^{\infty} s^k \sum_{j=k-r}^{\infty} P(T = j)
\]

\[
= qp^r \sum_{j=1}^{\infty} P(T = j) \sum_{k=r+1}^{r+j} s^k
\]

\[
= \frac{qp^r s^{r+1}}{1-s} \sum_{j=1}^{\infty} P(T = j)(1-s^j) = \frac{qp^r s^{r+1}}{1-s} (1 - G(s)).
\]

Solving for \( G(s) \) we obtain

\[
G(s) = \frac{p^n s^n - p^{n+1} s^{n+1}}{1 - s + qp^n s^{n+1}}.
\]