1. (15 marks)

(a) (5 marks) Following the hint let \( m_k \) denote the expected additional number of rolls required if the first roll is \( k \). Let \( X \) denote the number of rolls required until two consecutive rolls are the same. We wish to find \( E[X] \). Conditioning on the outcome of the first roll we have

\[
E[X] = \sum_{k=1}^{6} E[X \mid \text{first roll is } k] \frac{1}{6} = \sum_{k=1}^{6} (1 + m_k) \frac{1}{6} = 1 + m_1
\]

where we use the fact given in the hint that all the \( m_k \) are the same. Condition on the second roll to find \( m_1 \). If the second roll is a one then \( m_1 = 1 \). If the second roll is \( k > 1 \), then \( m_1 = 1 + m_k \). Thus,

\[
m_1 = (1) \frac{1}{6} + \frac{1}{6} \sum_{k=2}^{6} (1 + m_k) = \frac{1}{6} + \frac{5}{6} (1 + m_1) = 1 + \frac{5}{6} m_1,
\]

where we again use the fact that all the \( m_k \) are the same. Solving for \( m_1 \) we get \( m_1 = 6 \). Plugging this back into (1) we get \( E[X] = 7 \).

(b) (5 marks) The simplest way to do this problem is to note that the number of rolls required is just the number of rolls until something other than the first roll is obtained. Regardless of what is rolled first, the number of rolls required until something other than the first roll is obtained is Geometrically distributed with probability of success \( \frac{5}{6} \), and mean \( \frac{6}{5} \). So the expected number of rolls until two consecutive rolls are different is one (for the first roll) plus the expected number of rolls to get something different than the first roll. The answer is \( 1 + \frac{6}{5} = 2.2 \).

(c) (5 marks) Let \( M_1 \) denote the expected number of rolls until two consecutive ones are rolled and let \( m_1 \) denote the expected additional number of rolls required if the first roll is a one. If the first roll is not a one, then we are probabilistically back to the start, so we have

\[
M_1 = (1 + M_1) \frac{5}{6} + (1 + m_1) \frac{1}{6} = 1 + \frac{5}{6} M_1 + \frac{1}{6} m_1,
\]
or
\[ M_1 = 6 + m_1. \]  
(2)

For \( m_1 \), if the second roll is a one we are done and if the second roll is not a one then we are back to the start:

\[ m_1 = (1) \frac{1}{6} + (1 + M_1) \frac{5}{6} = 1 + \frac{5}{6} M_1. \]

Plugging this into (2) we get \( M_1 = 6 + 1 + \frac{5}{6} M_1 \), or \( M_1 = 42 \).

2. (15 marks)

(a) (5 marks) \( S = \{0, 1, 2, \ldots \} \),

\[ P = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 0 & \cdots \\ \frac{1}{2} & 0 & \frac{1}{2} & 0 & \cdots \\ 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 & \cdots \\ 0 & 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 & \cdots \\ \vdots & \vdots & \ddots & \ddots & \ddots \end{bmatrix} \]

and the period is 1.

(b) (5 marks) \( S = \{0, 1, 2, \ldots \} \),

\[ P = \begin{bmatrix} 0 & 1 & 0 & \cdots \\ \frac{1}{2} & 0 & \frac{1}{2} & 0 & \cdots \\ 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 & \cdots \\ 0 & 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 & \cdots \\ \vdots & \vdots & \ddots & \ddots & \ddots \end{bmatrix} \]

and the period is 2.

(c) (5 marks) Starting in state 0, if the first step is to state 0 then the chain has returned to state 0, and if the first step is to state 1, then the probability that the chain will return to state 0 is the same as the probability that the simple symmetric random walk, starting in state 0, will ever reach state -1, and this probability is 1. So regardless of the first step, the probability that the chain returns to 0 is 1. Therefore, state 0 is recurrent. To see that state 0 is null recurrent we can condition on the first step again. If the first step is to state 0, then the expected time to return to state 0 is 1. If the first step is to state 1, then the expected time to return to state 0 is the same as one plus the expected
time for the simple symmetric random walk, starting in state 0, to reach state -1. This is actually finite and so state 0 is in fact positive recurrent. The exam asked you show something that wasn’t true. Therefore, everyone gets a free 5 marks for this part (unless you are in 455 and your total based on problems 1 and 3 is still better than any other combination). Exercise: Show that state 0 is positive recurrent.

3. (15 marks)

(a) (4 marks) If \(i\) and \(j\) are both transient and \(i\) communicates with \(j\), then \(f_{ij}\) and \(f_{ji}\) cannot both be equal to 1. For suppose they both equal to 1. Then starting in \(i\), with probability 1 the chain will visit \(j\) and once there with probability 1 the chain will go back to \(i\). So with probability 1 the chain, starting in \(i\), will return to \(i\), contradicting the assumption that \(i\) is transient.

(b) (4 marks) If \(i\) and \(j\) are both recurrent and \(i\) communicates with \(j\) then \(f_{ij}\) and \(f_{ji}\) must both be equal to 1. We will just argue that \(f_{ij}\) must equal 1. The argument that \(f_{ji}\) must equal 1 is similar. Since \(i\) and \(j\) communicate there is an \(m\) such that \(p_{ij}(m) > 0\). That is, starting in state \(i\) there is a positive probability that the chain will visit state \(j\). Each time the chain returns to state \(i\) (which it does so infinitely often with probability 1 since \(i\) is recurrent) there is the same positive chance that the chain will visit \(j\). This is equivalent to flipping a coin with a positive probability of heads until a heads is flipped (the coin is flipped each time the chain returns to \(i\) and flipping heads corresponds to the chain visiting \(j\) before the next return to \(i\)). The probability that a heads is eventually flipped is 1 (i.e., \(f_{ij} = 1\)) since this is the same as the probability that \(X < \infty\), where \(X\) has a Geometric distribution with positive parameter.

(c) (4 marks) No. If \(i\) and \(j\) are both recurrent and they do not communicate then they are in different classes, and recurrent classes are closed, so neither could be accessible from the other.

(d) (3 marks) Yes. As a trivial example suppose the state space is \(S = \{0, 1, 2, \ldots\}\) and \(p_{i,i+1} = 1\) for all \(i \geq 0\). Then all states are clearly transient, state \(i + 1\) is accessible from state \(i\), but state \(i\) is not accessible from state \(i + 1\).