1. (15 marks)

(a) (7 marks) Define $M_1, M_2$ and $M_3$, where $M_i$ is the expected number of steps to get to the target vertex starting at a vertex that is $i$ edges away from the target vertex. By symmetry, the expected number of steps to get to the target vertex only depends on how many edges away the starting vertex is, so the $M_i$ are well defined. We want $M_3$. Starting from the opposite vertex the first move will take us to a vertex 2 edges away from the target, so $M_3 = 1 + M_2$. For $M_2$, condition on the first move, giving

$$M_2 = \frac{1}{2}(1 + M_1) + \frac{1}{2}(1 + M_3) = 1 + \frac{1}{2}M_1 + \frac{1}{2}M_3 \quad (1)$$

For $M_1$, condition on the first move to give

$$M_1 = \frac{1}{3}(1) + \frac{2}{3}(1 + M_2) = 1 + \frac{2}{3}M_2 \quad (2)$$

Plugging (2) into (1) gives $M_2 = 1 + \frac{2}{3}(1 + \frac{1}{2}M_2) + \frac{1}{2}M_3$, or $M_2 = 2 + \frac{3}{5}M_3$. Thus, we get $M_3 = 1 + 2 + \frac{2}{3}M_3$, or $M_3 = 9$.

(b) (8 marks) Define $M_1, M_2, M_3$ as in part (a). Once again, we have $M_3 = 1 + M_2$. For $M_2$, two of the possible first moves will take the particle one edge closer to the target and one of the possible first moves will take the particle one edge further from the target. So conditioning on the first move gives

$$M_2 = \frac{2}{3}(1 + M_1) + \frac{1}{3}(1 + M_3) = 1 + \frac{2}{3}M_1 + \frac{1}{3}M_3 \quad (3)$$

For $M_1$, one of the possible first moves will take the particle to the target vertex and two of the possible first moves will take the particle one edge further from the target vertex. So conditioning on the first move, we have

$$M_1 = \frac{1}{3}(1) + \frac{2}{3}(1 + M_2) = 1 + \frac{2}{3}M_2 \quad (4)$$

Plugging (4) into (3) gives $M_2 = 1 + \frac{2}{3}(1 + \frac{2}{3}M_2) + \frac{1}{3}M_3$, or $M_2 = 3 + \frac{3}{5}M_3$. So we get $M_3 = 1 + 3 + \frac{3}{5}M_3$, or $M_3 = 10$. 
2. (15 marks)

(a) (9 marks) For $\{Y_n^{(2)} : n \geq 0\}$ the period is 2 and there are 2 communicating classes.

For $\{Y_n^{(3)} : n \geq 0\}$ the period is 4 and there is 1 communicating class.

For $\{Y_n^{(4)} : n \geq 0\}$ the period is 1 and there are 4 communicating classes.

(b) (4 marks) Let $C_0, C_1, C_2, C_3$ be the four cyclic classes that the $X$ chain cycles through. The chain $\{(X_n, Y_n) : n \geq 0\}$ cannot be irreducible since, for example, suppose $i \in C_0$ and $j \in C_1$ then from state $(i, j)$ the chain could never reach state $(i, k)$, where $k \in C_2$.

(c) (2 marks) Let $i$ be a given state. Since the period of the $X$ chain is 4 there is some $m$ that is a multiple of 4 such that $p_{ii}(m) > 0$ (if $p_{ii}(n) = 0$ for all $n$ then the period would not be 4). With this $m$, and conditioning on $T_1$, we have

$$P(W_1 = i \mid W_0 = i) = P(W_1 = i \mid X_0 = i) \quad \text{(since } W_0 = X_0)$$

$$= \sum_{k=1}^{\infty} P(W_1 = i \mid T_1 = k, X_0 = i)P(T_1 = k \mid X_0 = i)$$

$$\geq P(W_1 = i \mid T_1 = m, X_0 = i)P(T_1 = m)$$

$$= P(X_m = i \mid X_0 = i)P(T_1 = m)$$

$$= p_{ii}(m)P(T_1 = m) > 0,$$

since given $T_1 = m$ we have that $W_1 = X_m$ (hence the 4th line), and the inequality in the 5th line follows since $p_{ii}(m) > 0$ by choice of $m$ and $P(T_1 = m) = P(Z_1 = m) > 0$ by assumption. Hence the period of state $i$ must be 1 (since the period must divide 1). Since the chain is irreducible (given) or since state $i$ was arbitrary in the above argument, we can conclude that the chain $\{W_n : n \geq 0\}$ has period 1.
3. (15 marks)

(a) (5 marks) Let \( T_j \) be the first time, after time 0, that the chain visits state \( j \). Conditioning on \( T_j \) we have

\[
p_{ij}(n) = P(X_n = j \mid X_0 = i) = \sum_{r=1}^{n} P(X_n = j \mid T_j = r, X_0 = i)P(T_j = r \mid X_0 = i)
\]

(for \( r > n \) the conditional probability will be 0)

\[
= \sum_{r=1}^{n} P(X_n = j \mid X_r = j, X_{r-1} \neq j, \ldots, X_1 \neq j, X_0 = i) \times P(X_r = j, X_{r-1} \neq j, \ldots, X_1 \neq j \mid X_0 = i)
\]

(by the Markov property)

\[
= \sum_{r=1}^{n} p_{jj}(n-r)f_{ij}(r).
\]

(b) (5 marks) Multiply the first and last expressions above by \( s^n \) and sum from \( n = 1 \) to \( \infty \). Noting that \( p_{ij}(0) = 0 \) on the LHS we get \( P_{ij}(s) \) and noting that the last expression is the \( n \)th term in the convolution of the two sequences \( \{p_{jj}(n)\} \) and \( \{f_{ij}(n)\} \) we get \( P_{jj}(s)F_{ij}(s) \) on the right.

(c) (5 marks) Set \( s = 1 \) in part(b), giving \( P_{ij}(1) = F_{ij}(1)P_{jj}(1) \). But \( P_{jj}(1) = \sum_{n=0}^{\infty} p_{jj}(n) \), which is finite if state \( j \) is transient (from proposition in class). Since \( F_{ij}(1) \leq 1 \) (it is the probability that state \( j \) is ever visited starting in state \( i \)) we have that \( P_{ij}(1) < \infty \) for all \( i \). But \( P_{ij}(1) = \sum_{n=0}^{\infty} p_{ij}(n) \).