1. (15 marks)

(a) (8 marks) For $m_1$, by conditioning on the next roll, we get

$$m_1 = (1 + m_2)\frac{1}{6} + (1 + m_1)\frac{5}{6}, \quad (1)$$

because in the next roll we will roll the same number as was previously rolled with probability $1/6$, and will roll a different number with probability $5/6$ (which puts us back in the situation of $m_1$). For $m_2$, again by conditioning on the next roll, we get

$$m_2 = (1)\frac{1}{6} + (1 + m_1)\frac{5}{6}. \quad (2)$$

Equations (1) and (2) are 2 equations in 2 unknowns, which we can solve to get $m_1$ and $m_2$. Plugging (2) into (1), we get

$$m_1 = 1 + \frac{1}{6} \left(1 + \frac{5}{6}m_1\right) + \frac{5}{6}m_1$$

or

$$m_1 = 36 \times \frac{7}{6} = 42.$$

Then $E[X_3] = 1 + m_1 = 1 + 42 = 43$.

(b) (7 marks) If $X_{k-1} = j$ then the first time the same number appeared on $k - 1$ consecutive rolls was on roll $j$. Now condition further on the next roll. With probability $1/6$ the next roll will be the same as the previous roll and in this case $X_k = j + 1$. If the next roll is different then we have to start over, and the expected further number of rolls to get $k$ consecutive same rolls is $E[X_k]$. That is,

$$E[X_k \mid X_{k-1} = j, (j + 1)st \text{ roll is different from previous roll}] = j + E[X_k].$$

So using the law of total probability to get $E[X_k \mid X_{k-1} = j]$ we have

$$E[X_k \mid X_{k-1} = j] = (j + 1)\frac{1}{6} + (j + E[X_k])\frac{5}{6}.$$  

or

$$E[X_k \mid X_{k-1}] = X_{k-1} + \frac{1}{6} + \frac{5}{6}E[X_k].$$
Unconditioning, we get

\[ E[X_k] = E[X_{k-1}] + \frac{1}{6} + \frac{5}{6}E[X_k] \]

or

\[ E[X_k] = 6E[X_{k-1}] + 1. \]

2. (15 marks) In the following solutions we let \( p_{ij}(n) \) denote the \( n \)-step transition probabilities in the \( X \) chain, and \( q_{ij}(n) \) denote the \( n \)-step transition probabilities in the \( Y \) chain. Then \( q_{ij}(n) = p_{ij}(2n) \) for all \( i, j, n \).

(a) (7 marks)

(i) (3 marks) Suppose state \( i \) had period \( d \) in the \( Y \) chain, with \( d > 1 \). Then \( q_{ii}(n) > 0 \) only if \( d \) divides \( n \). So \( p_{ii}(2n) > 0 \) only if \( d \) divides \( n \). But then \( 2d \) divides all these times in the \( X \) chain, which implies that the period of state \( i \) in the \( X \) chain is at least \( 2d \). If \( d > 1 \) this implies the period of state \( i \) in the \( X \) chain is at least 4, which is a contradiction.

(ii) (4 marks) Since state \( i \) is recurrent in the \( X \) chain, by a proposition from class, we have \( \sum_{n=0}^{\infty} p_{ii}(n) = \infty \). But since the period of state \( i \) is 2, \( p_{ii}(n) = 0 \) for all odd \( n \). Therefore,

\[ \sum_{n=0}^{\infty} p_{ii}(n) = \sum_{n=0}^{\infty} p_{ii}(2n) = \sum_{n=0}^{\infty} q_{ii}(n). \]

Therefore, \( \sum_{n=0}^{\infty} q_{ii}(n) \) is also equal to \( \infty \), which implies that state \( i \) is recurrent in the \( Y \) chain, again by the proposition from class.

(b) (8 marks)

(i) (4 marks) Removed from exam.

(ii) (4 marks) Let \( i \) and \( j \) be any two states. We will show that \( j \) is accessible from \( i \) in the \( Y \) chain. The argument that \( i \) is accessible from \( j \) is the same.

Since the \( X \) chain is irreducible, let \( n \) be such that \( p_{ij}(n) > 0 \). Following the hint we have that

\[ p_{ij}(m + n) \geq p_{ii}(m)p_{ij}(n) \quad (3) \]

for any \( m \geq 0 \) (this follows from the Chapman-Kolmogorov equations). Since the \( X \) chain is irreducible, all states have period 1. Therefore, for state \( i \), it cannot be that \( p_{ii}(m) > 0 \) only for even \( m \); otherwise the period of state \( i \) would be at least 2 in the \( X \) chain. So there is an odd \( m \) such that \( p_{ii}(m) > 0 \). But then \( p_{ii}(2m) \geq p_{ii}(m)p_{ii}(m) > 0 \) also, and \( 2m \) is even. So in (3), if \( n \) is
odd we can choose \( m \) odd with \( p_{ii}(m) > 0 \), and if \( n \) is even we can choose \( m \) even with \( p_{ii}(m) > 0 \). In either case, we can choose \( n \) and \( m \) so that \( m + n \) is even, so that we can write \( n + m = 2r \) for some positive integer \( r \). So from (3), with this choice of \( n \) and \( m \), we have

\[ q_{ij}(r) = p_{ij}(2r) = p_{ij}(m + n) \geq p_{ii}(m)p_{ij}(n) > 0, \]

so that \( j \) is accessible from \( i \) in the \( Y \) chain.

3. (15 marks)

(a) (3 marks) The mean family size is \( G'(1) \). We have

\[ G'(s) = p\alpha(1-s)^{p-1} = \frac{p\alpha}{(1-s)^{1-p}}. \]

As \( s \to 1 \), \( G'(s) \to \infty \), so the mean family size is \( \infty \).

(b) (12 marks) For \( G_n(s) \) we need to start computing the compositions of \( G \) with itself. Starting with \( n = 2 \), and noting that \( 1 - G(s) = \alpha(1 - s)^p \), we have

\[ G_2(s) = G(G(s)) = 1 - \alpha(1 - G(s))^p = 1 - \alpha^{1+p}(1 - s)^{p^2}. \]

Continuing for \( n = 3 \) we have

\[ G_3(s) = G_2(G(s)) = 1 - \alpha^{1+p}(1 - G(s))^p = 1 - \alpha^{1+p^2}(1 - s)^{p^3}. \]

It's fairly clear I hope that for general \( n \) we will get

\[ G_n(s) = 1 - \alpha^{1+p+p^2+...+p^{n-1}}(1 - s)^{p^n} \]

(this can be proven by induction on \( n \)). Then

\[ P(X_n = 0) = G_n(0) = 1 - \alpha^{1+p+p^2+...+p^{n-1}} = 1 - \alpha^{(1-p^n)/(1-p)} \]

and if \( \eta \) denotes the probability of ultimate extinction, then

\[ \eta = \lim_{n \to \infty} P(X_n = 0) = 1 - \alpha^{1/(1-p)}. \]