MTHE/STAT455, STAT855, Stochastic Processes

Midterm Exam

Instructions:

(a) The exam is closed book. No books are allowed. You may use one 8.5 × 11 inch sheet of notes and a calculator.

(b) There are 3 questions. Stat 855 students must do all of the problems. Mthe/Stat 455 students must do 2 of questions 1, 2 and 3, and if you answer all three questions, you must specify which two you want graded (the default if you do not specify is questions 1 and 2).

(c) Each question is worth 15 marks for a total of 30 marks (Mthe/Stat455) or 45 marks (Stat855)

(d) Show all your work. Partial credit is given.

(e) Read the hints!

(f) You have 120 minutes. Good luck!
1. (15 marks) A Markov chain has \( n \) states, labelled 1, \ldots, \( n \). Let \( P \) denote the transition matrix. On each step of the chain a fair die is rolled once, and if a 6 is rolled the chain is stopped in the state it happens to be in. The die is also rolled in the initial state and if a 6 is rolled the chain is stopped in the initial state without making any steps. If the initial state is state 1, find the probability that the chain is in state 1 when it is stopped,

(a) (8 marks) if each entry of \( P \) is \( \frac{1}{n} \). \textit{Hint:} Define appropriate unknowns, you only need 2. Condition on the outcome of the roll in the initial state and on the first step of the chain if the outcome of the roll is not a 6.

(b) (7 marks) if \( n = 3 \), and

\[
P = \begin{bmatrix}
0 & \frac{1}{2} & \frac{1}{2} \\
\frac{1}{2} & 0 & \frac{1}{2} \\
\frac{1}{2} & \frac{1}{2} & 0
\end{bmatrix}.
\]

2 (15 marks) In the following parts, let \( P \) be the transition matrix of a Markov chain with entries \( p_{ij} \) and let the state space be \( \{0, 1, \ldots, n\} \).

(a) (7 marks) If \( p_{i,i+1} = \frac{n-i}{n} \) for \( i = 0, \ldots, n-1 \) and \( p_{i,i-1} = \frac{i}{n} \) for \( i = 1, \ldots, n \), and all other \( p_{ij} = 0 \), find the stationary distribution of this Markov chain.

(b) (8 marks) Let \( n = 3 \) and suppose

\[
P = \begin{bmatrix}
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
1 & 0 & 0 & 0
\end{bmatrix}.
\]

(i) (4 marks) Find a (row) vector \( a = (a_0, a_1, a_2, a_3) \) that satisfies \( a_i > 0 \) for \( i = 0, 1, 2, 3 \), \( a_0 + a_1 + a_2 + a_3 = 1 \), \( a = aP \), \( a = aP^2 \), and \( a = aP^4 \).

(ii) (4 marks) Find 3 (row) vectors \( a = (a_0, a_1, a_2, a_3) \), \( b = (b_0, b_1, b_2, b_3) \), and \( c = (c_0, c_1, c_2, c_3) \) that satisfy \( a_i, b_i, c_i > 0 \) for \( i = 0, 1, 2, 3 \), \( a_0 + a_1 + a_2 + a_3 = 1 \), \( b_0 + b_1 + b_2 + b_3 = 1 \), \( c_0 + c_1 + c_2 + c_3 = 1 \), \( a = aP \), \( b = bP^2 \), and \( c = cP^4 \), and \( a \neq b \neq c \).
3. (15 marks) Let $P$ be the $n \times n$ transition matrix of an irreducible, positive recurrent Markov chain with stationary distribution $\pi$, and state space \( \{1, \ldots, n\} \).

(a) (6 marks) Let $Q$ be an $n \times n$ transition matrix. Consider the Markov chain with transition matrix \( \frac{1}{2}(P + Q) \). Explain why this Markov chain is irreducible and give an example of a $Q \neq P$ such that this Markov chain also has stationary distribution $\pi$.

(b) (9 marks) Suppose we wish to add a state, say state 0. Let $R$ be the \((n+1) \times (n+1)\) transition matrix of a Markov chain with state space \( \{0, 1, \ldots, n\} \) such that $R = \begin{bmatrix} a & b^T \\ c & dP \end{bmatrix}$, such that $a$ and $d$ are scalars, $b$ and $c$ are both $n \times 1$ vectors. Find $a$, $b$, $c$, and $d$ such that the Markov chain has stationary distribution $(p, (1-p)\pi)$, where $p \in (0, 1)$ is given.