Statistics 150
Instructor: David Steinsaltz
Spring 2001

F I N A L   E X A M

Wednesday, May 16 2001

1) (6 points) Shocks occur to a system according to a Poisson process of rate 2. Suppose that the system survives each shock with probability 0.9, independent of all other shocks. Compute the probability that the system survives until time 4.

2) (4 points) Let B_t be a standard Brownian motion. Compute the probability that there exists a time T at which $B_T \geq 2T + 1$. You may wish to use the fact that for any number σ, the process $e^{\sigma B_t - \sigma^2 t/2}$ is a martingale.

3) (9 points) A small barbershop, operated by a single barber, has room for at most two customers. Potential customers arrive at a Poisson rate of three per hour, and the successive service times are independent exponential random variables, with mean $\frac{1}{4}$ hour. If a customer arrives and finds two others already in the shop, he leaves.

a) What is the average number of customers in the shop over a very long time?

b) What fraction of the potential customers gets turned away?

c) By what factor would the barber's business increase, if she worked twice as fast?

4) (4 points) $B(t)$ is a standard Brownian motion. Compute $P\{B(1) + B(2) > 2\}$.

5) (9 points) $X(t)$ is a Yule process, that is, a pure birth process with birth rate $\lambda_n = n\lambda$ when the population is n, and $X(0) = 1$. Let $P_n(t) = P\{X(t) = n - 1\}$.

a) Show that $P_n(t) = e^{-\lambda t}(1 - e^{-\lambda t})^n$.

b) Let T_n be the time when the n-th individual is born (that is, when $X(t)$ reaches n.) Find the cumulative distribution function of T_n, and its expectation. (Hint: No integration is required.)

c) Find the density of T_2, conditioned on $X(1) = 2$.

6) (6 points) A circular silicon wafer of radius 4 cm has defects scattered over its surface according to a Poisson process, with the expected number of flaws in a wafer being 3. Each flaw has a cost, which is inversely proportional to its distance from the center of the wafer. That is, a flaw at distance r cm from the center costs \$$10r$.

A certain wafer is examined, and found to contain exactly two flaws. Without any information about where they are located, what is the expected total cost of the flaws?

7) (7 points) We repeatedly flip a coin, which comes up heads with probability 0.6, tails with probability 0.4, until we get the pattern heads-tails-heads. Let T be the number of the flip when we stop. Compute $E[T]$. (Hint: Use a Markov chain.)