

REVIEW OF ORDINARY DIFFERENTIAL EQUATION MODELS

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FIRST ORDER EQUATIONS

Basic important fact: A first order differential equation with an initial condition has exactly one solution, if the function in the differential equation is decent.

Finding solution analytically is hard, except for very simple equations.

Example : Exponential growth or decay,

$$y' = ay, \quad y(0) = y_0.$$

Solution : Separate variables

$$\frac{dy}{y} = a dt.$$

[This equation doesn't mean anything because dy and dt by themselves don't mean anything, but pretend it does, because when we integrate the result will be valid.]

Now integrate to get

$$\ln y = at + c$$

(don't forget the constant of integration).

Substitute initial condition $t = 0, y = y_0$ to solve for c , obtaining $c = \ln y_0$. Then take exponential of both sides to give solution,

$$y = y_0 e^{at}.$$

The solution is also valid if $y_0 = 0$, where the solution for c doesn't work.

This is the solution both if $a < 0$ (exponential decay) and if $a > 0$ (exponential growth).

Very few differential equations can be integrated explicitly. Look for qualitative information (how does the solution behave as $t \rightarrow \infty$?).

An *equilibrium* of an autonomous (not depending on the independent variable) differential equation

$$y' = g(y)$$

is a solution y_∞ of the equation

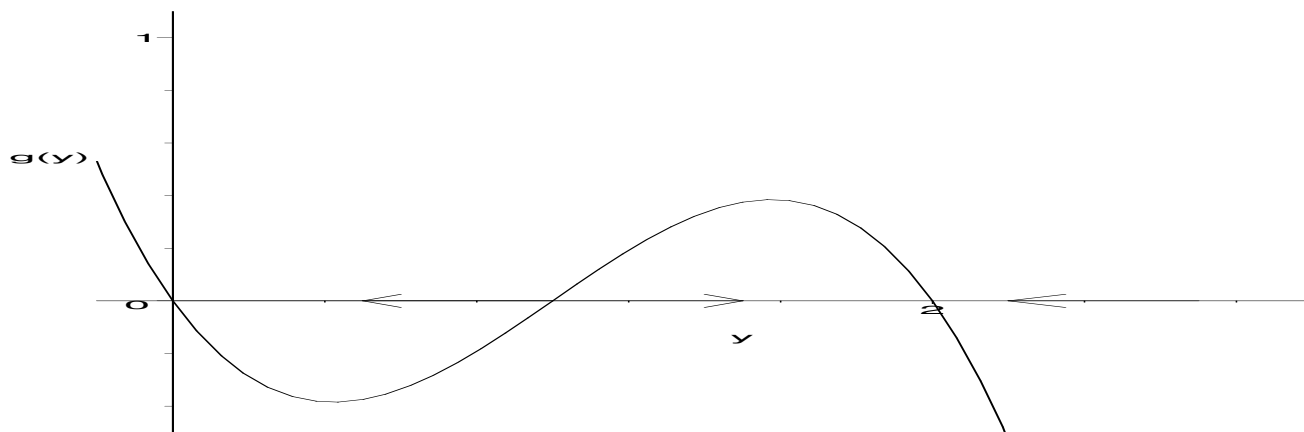
$$g(y) = 0.$$

An equilibrium y_∞ corresponds to a constant solution $y = y_\infty$ of the differential equation.

Important facts about equilibria :

- If y_∞ is an equilibrium with $g'(y_\infty) < 0$ solutions with initial value near y_∞ approach y_∞ as $t \rightarrow \infty$. The equilibrium is *asymptotically stable*.
- If y_∞ is an equilibrium with $g'(y_\infty) > 0$, then solutions with initial value near y_∞ move away from y_∞ as $t \rightarrow \infty$. The equilibrium is *unstable*.
- If y_∞ is an equilibrium with $g'(y_\infty) = 0$, no conclusion about stability of the equilibrium can be drawn.

The procedure for analyzing the qualitative behaviour of an autonomous differential equation is to draw the *phase line*, the t -axis of the (t, y) plane in which the graph of the function $g(y)$ is drawn. Where $g(y) > 0$, draw an arrow on the axis to the right, indicating that for the solution of the differential equation $y' > 0$ and the solution of the differential equation is increasing. Similarly, where $g(y) < 0$, draw an arrow on the axis to the left, indicating that for the solution of the differential equation $y' < 0$ and the solution of the differential equation is decreasing. It is clear that these arrows point towards equilibria y_∞ with $g'(y_\infty) < 0$ (asymptotically stable equilibrium) and away from equilibria y_∞ with $g'(y_\infty) > 0$ (unstable equilibrium).



EXAMPLES

Find all equilibria and analyze their stability for each of the following differential equations representing population models.

1. $y' = ry \left(1 - \frac{y}{K}\right)$
2. $y' = ry \ln \frac{K}{y}$
3. $y' = ry \frac{K-y}{K+ay}$
4. $y' = ry \left[1 - \left(\frac{y}{K}\right)^\theta\right]$
5. $y' = ry e^{-\frac{y}{K}} - dy$
6. $y' = -ry \left(1 - \frac{y}{a}\right) \left(1 - \frac{y}{K}\right)$.

SYSTEMS OF TWO DIFFERENTIAL EQUATIONS

Models for two interacting populations require a system of two first order differential equations, of the form

$$\begin{aligned}y' &= g(y, z) \\z' &= h(y, z).\end{aligned}$$

An *equilibrium* is a point (y_∞, z_∞) in the (y, z) plane, called the *phase plane* such that

$$g(y_\infty, z_\infty) = h(y_\infty, z_\infty) = 0.$$

An equilibrium corresponds to a constant solution of the system. It is an intersection of the two *null clines*

$$g(y_\infty, z_\infty) = 0, \quad h(y_\infty, z_\infty) = 0.$$

Important facts about equilibria :

- The *linearization* of the system at an equilibrium is the linear system

$$\begin{aligned}u' &= g_y(y_\infty, z_\infty)u + g_z(y_\infty, z_\infty)v \\v' &= h_y(y_\infty, z_\infty)u + h_z(y_\infty, z_\infty)v\end{aligned}$$

- If every solution of the linearization at an equilibrium approaches zero, the equilibrium is asymptotically stable, and if the linearization has unbounded solutions, the equilibrium is unstable.
- Every solution of the linearization at an equilibrium approaches zero if and only if

$$\begin{aligned}g_y(y_\infty, z_\infty) + h_z(y_\infty, z_\infty) &< 0 \\g_y(y_\infty, z_\infty)h_z(y_\infty, z_\infty) - g_z(y_\infty, z_\infty)h_y(y_\infty, z_\infty) &> 0.\end{aligned}$$

- A bounded solution approaches an equilibrium, or is a periodic solution, or approaches a periodic solution (Poincaré-Bendixson theorem).

The first two of these facts extend directly to systems of arbitrary dimension, the third has a more complicated extension, but the last is restricted to two dimensions.

EXAMPLES

Find all equilibria and analyze their stability for each of the following systems representing population models.

1. (Species in competition)

$$\begin{aligned}y' &= y(\lambda - ay - bz) \\z' &= z(\mu - cy - dz),\end{aligned}$$

where λ, μ, a, b, c, d are positive constants.

2. (Predator-prey system)

$$\begin{aligned}y' &= ry \left(1 - \frac{y}{K}\right) - \frac{ayz}{y + A} \\z' &= sz \left(\frac{y}{y + A} - \frac{J}{J + A}\right).\end{aligned}$$