Degree bounds for separating invariants

Martin Kohls
Technische Universität München

(joint work with Hanspeter Kraft)
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Our setup

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- $K[V] := S(V^*)$ i.e. the action of $G$ on $K[V]$ is given by
  \[ \sigma(f) := f \circ \sigma^{-1} \quad \text{for } f \in K[V], \quad \sigma \in G. \]
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The local and global beta number

- Local degree bound:
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The local and global beta number

- Local degree bound:
  \[ \beta(G, V) \] := least number \( d \) such that the invariants of degree \( \leq d \) generate \( K[V]^G \).

- Global degree bound, or Noether number:
  \[ \beta(G) := \sup \{ \beta(G, V) : \text{\( V \) a \( G \) -- module} \}. \]

- Noether’s bound: \( G \) finite, \( p \) \( \nmid \) \(|G| \) \( \Rightarrow \beta(G) \leq |G| \)
  (Fleischmann, Fogarty 2000/2001, Noether 1916 for \( p = 0 \) or \(|G| < p\))
Results of Barbara Schmid (1989) for characteristic 0, $|G| < \infty$

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- $H \leq G \implies \beta(H) \leq \beta(G) \leq \beta(H)(G : H)$. 

Definitions and Noether's bound
Schmid's results in characteristic 0
Sharpenings and infinite groups

Notation
Degree bounds for generating sets
Separating invariants
New Results
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- $N \trianglelefteq G \Rightarrow \beta(G) \leq \beta(N)\beta(G/N)$. 

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- $\beta(D_{2n}) = n + 1$.
- Determination of $\beta(G)$ for some small groups.
Sharpenings and some more results

- Domokos, Hegedus, Sezer: $G$ finite and $p \nmid |G| \Rightarrow$
  \[
  \beta(G) \leq \begin{cases} 
  \frac{3}{4} |G| & \text{for } |G| \text{ even and } G \text{ non cyclic} \\
  \frac{5}{8} |G| & \text{for } |G| \text{ odd and } G \text{ non cyclic}
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- Richman: $G$ finite and $p \not| |G| \Rightarrow \beta(G) = \infty$. 
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  \end{cases}
  \]

- Richman: $G$ finite and $p \mid |G| \Rightarrow \beta(G) = \infty$.

- Bryant, Derksen, Kemper: $G$ a linear algebraic group. Then
  \[
  \beta(G) < \infty \iff |G| < \infty \text{ and } p \nmid |G|.
  \]
Separate instead of generate!

Separating sets (Derksen, Kemper)

\[ S \subseteq K[V]^G \text{ separating} \quad \iff \quad \text{For all } x, y \in V \text{ we have:} \\
\text{if } f(x) = f(y) \text{ for all } f \in S \Rightarrow f(x) = f(y) \text{ for all } f \in K[V]^G. \]

Theorem: There always exists a finite separating set.
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Now define similarly:

- Local degree bound:
  \[ \beta_{\text{sep}}(G, V) := \text{ least number } d \text{ such that the invariants of degree } \leq d \text{ form a separating set.} \]
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Now define similarly:

- **Local degree bound:**
  \[ \beta_{sep}(G, V) := \text{least number } d \text{ such that the invariants of degree } \leq d \text{ form a separating set.} \]

- **Global degree bound:**
  \[ \beta_{sep}(G) := \sup \{ \beta_{sep}(G, V) : V \text{ a } G - \text{module} \}. \]
Known Results

- Derksen, Kemper (Noether bound): $G$ finite $\Rightarrow \beta_{\text{sep}}(G) \leq |G|$
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- Kemper: $H \leq G$ closed $\Rightarrow$
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Known Results

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- Derksen, Kemper: $G$ reductive, $U \leq V$ (submodule) $\Rightarrow$
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- Draisma, Kemper, Wehlau: $G$ finite $\Rightarrow$
  \[ \beta_{\text{sep}}(G) = \beta_{\text{sep}}(G, V_{\text{reg}}). \]
Things to do:

Problems/Questions:

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Problems/Questions:

- When is $\beta_{sep}(G)$ finite?
- Determine $\beta_{sep}(G)$ for as many groups as possible, or
- find (improved) upper and lower bounds!
Relative Degree bounds

**Theorem**

If $H \leq G$ is a subgroup of finite index $(G : H) < \infty$, and $V$ is an $H$-module, then

\[ \beta_{\text{sep}}(H, V) \leq \beta_{\text{sep}}(G, KG \otimes_{KH} V), \]

i.e.

\[ \beta_{\text{sep}}(H) \leq \beta_{\text{sep}}(G). \]
Relative Degree bounds

**Theorem**

\( H \leq G \) closed of finite index \((G : H) < \infty\), \(V\) an \(H\)-module \(\Rightarrow\)

\[ \beta_{\text{sep}}(H, V) \leq \beta_{\text{sep}}(G, KG \otimes_{KH} V), \quad \text{i.e. } \beta_{\text{sep}}(H) \leq \beta_{\text{sep}}(G) \]

Remark: \( N \trianglelefteq G \) closed \(\Rightarrow\) \( \beta_{\text{sep}}(G/N) \leq \beta_{\text{sep}}(G) \).
Relative Degree bounds

Theorem

\[ H \leq G \text{ closed of finite index } (G : H) < \infty, \, V \text{ an } H\text{-module} \Rightarrow \]

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Remark: \( N \trianglelefteq G \text{ closed } \Rightarrow \beta_{\text{sep}}(G/N) \leq \beta_{\text{sep}}(G). \)

Theorem

\[ N \trianglelefteq G \text{ closed of finite index } (G : N) < \infty \Rightarrow \]

\[ \beta_{\text{sep}}(G) \leq \beta_{\text{sep}}(G/N) \beta_{\text{sep}}(N). \]

E.g. for finite groups \( G, H: \)

\[ \beta_{\text{sep}}(G \rtimes H) \leq \beta_{\text{sep}}(G) \beta_{\text{sep}}(H). \]
$p$-groups and cyclic groups

\textbf{MAGMA} and $\beta_{\text{sep}}(G) = \beta_{\text{sep}}(G, V_{\text{reg}})$

$p = 2 : \quad \beta_{\text{sep}}(S_3) = \beta_{\text{sep}}(S_3, V_{\text{reg}}) = 4.$
**p-groups and cyclic groups**

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**Theorem (p-groups)**

\[ \text{char } K = p > 0, \quad G \text{ a } p\text{-group} \Rightarrow \quad \beta_{\text{sep}}(G) = |G|. \]
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\textbf{Theorem ($p$-groups)}

$\text{char } K = p > 0, \quad G \text{ a } p\text{-group} \Rightarrow \quad \beta_{\text{sep}}(G) = |G|.$

Corollary: $|G| = p^k m, (p, m) = 1 \Rightarrow \beta_{\text{sep}}(G) \geq p^k.$
**Notation**
- Degree bounds for generating sets
- Separating invariants

**New Results**
- Relative degree bounds
- Bounds for finite groups
- The additive group
- Unipotent groups
- Semisimple groups

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**Theorem (finite cyclic groups)**

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**$p$-groups and cyclic groups**

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**Theorem (finite cyclic groups)**

$G$ cyclic $\Rightarrow \beta_{\text{sep}}(G) = |G|.$

Corollary: $\beta_{\text{sep}}(G) \geq \max_{\sigma \in G} \text{ord}(\sigma)$.
Dihedral groups, the Hilbert ideal

**Theorem (dihedral groups)**

\[ p \geq 3, \ r \geq 1 \Rightarrow \beta_{\text{sep}}(D_{2p^r}) = 2p^r. \]
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**Corollary of Proofs:**

Let \( G \) be a \( p \)-group, or a cyclic group, or \( G = D_{2p^r} \). There exists a \( V \) such that the Hilbert ideal \( K[V]^G + K[V] \) is not generated by invariants of degree \(< |G|\).
Non-modular subquotients

Combine good bounds for non-modular generating invariants and the relative degree bounds for separating invariants:

Assume $N \trianglelefteq H \leq G$ s.t. $H/N$ is non-cyclic and non-modular of order $s \Rightarrow \beta_{\text{sep}}(G) \leq \begin{cases} 342 & \text{for } s \text{ even} \\ 58 & \text{for } s \text{ odd.} \end{cases}$

Ex.: $p \neq 2, n = p^m$, $(p, m) = 1, m > 1 \Rightarrow D_2^m < D_2^n \Rightarrow \beta_{\text{sep}}(D_2^n) \leq 32n = 32n$.

Recall: $\beta_{\text{sep}}(D_2^p^r) = 2p^r$.

Conjecture: $\text{char}(K) = 2, p \geq 3$ prime, $\Rightarrow \beta_{\text{sep}}(D_2^p^r) = p^r + 1$. 

Martin Kohls Technische Universität München
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**Theorem**

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$$
\beta_{\text{sep}}(G) \leq \begin{cases} 
\frac{3}{4} |G| & \text{for } s \text{ even} \\
\frac{5}{8} |G| & \text{for } s \text{ odd.}
\end{cases}
$$
Non-modal subquotients

Combine good bounds for non-modal generating invariants and the relative degree bounds for separating invariants:

**Theorem**

Assume \( N \trianglelefteq H \leq G \) s.t. \( H/N \) is non-cyclic and non-modal of order \( s \) \( \Rightarrow \)

\[
\beta_{\text{sep}}(G) \leq \begin{cases} 
\frac{3}{4} |G| & \text{for } s \text{ even} \\
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\]

- Ex.: \( p \neq 2, n = p^r m, (p, m) = 1, m > 1 \Rightarrow D_{2m} < D_{2n} \)
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- Ex.: $p \neq 2$, $n = p^r m$, $(p, m) = 1$, $m > 1 \Rightarrow D_{2m} < D_{2n}$

  $$\Rightarrow \quad \beta_{\text{sep}}(D_{2n}) \leq \frac{3}{4} 2n = \frac{3}{2} n$$
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**Recall:** $\beta_{\text{sep}}(D_{2p^r}) = 2p^r$. 

Conjecture: $\text{char}(K) = 2, p \geq 3$ prime, $\Rightarrow \beta_{\text{sep}}(D_{2p}) = p + 1$
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Combine good bounds for non-modular generating invariants and the relative degree bounds for separating invariants:

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- Conjecture: $\text{char}(K) = 2, p \geq 3$ prime, $\Rightarrow \beta_{\text{sep}}(D_{2p}) = p + 1$.
Example/Open questions

Ex.: $p = 3$. $A_4$ has the Klein four group as non-modular non-cyclic subgroup of even order.
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$\Rightarrow \beta_{\text{sep}}(A_4 \times A_4) \leq \beta_{\text{sep}}(A_4)^2 = 81 < 144 = |A_4 \times A_4|$. 
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1. Which groups do not contain a non-cyclic non-modular subquotient?
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Note: If $p = 2$ or $p = 3$, the group $S_3$ does not contain a non-cyclic non-modular subquotient. But:
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- \( p = 2 \): \( \beta_{\text{sep}}(S_3) = 4 < 6 = |S_3| \)
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$\Rightarrow \beta_{\text{sep}}(A_4) \leq \frac{3}{4} |A_4| = 9$

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- $p = 3$: $\beta_{\text{sep}}(S_3) = 6 = |S_3|$. 
Definitions

Basic $\mathbb{G}_a$-actions

$\mathbb{G}_a$ acts on the $n + 1$ dimensional vector space $V_n = \langle X_0, \ldots, X_n \rangle_K$

t $\ast$ $X_k = \sum_{i=0}^{k} \binom{k}{i} t^i X_{k-i}$ for $t \in \mathbb{G}_a$, $k = 0, \ldots, n$,
Definitions

Basic \( \mathbb{G}_a \)-actions

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t \ast X_k = \sum_{i=0}^{k} \binom{k}{i} t^i X_{k-i} \quad \text{for } t \in \mathbb{G}_a, \ k = 0, \ldots, n,
\]

Frobenius powers

For any \( G \)-module \( U \), define its \( p^m \)th Frobenius power

\[
F^{p^m}(U) := \{ f \in S^{p^m}(U) : \exists u \in U \text{ with } f = u^{p^m} \} \leq S^{p^m}(U).
\]
Definitions

**Basic $\mathbb{G}_a$-actions**

$\mathbb{G}_a$ acts on the $n + 1$ dimensional vector space $V_n = \langle X_0, \ldots, X_n \rangle_K$

\[ t \ast X_k = \sum_{i=0}^{k} \binom{k}{i} t^i X_{k-i} \quad \text{for } t \in \mathbb{G}_a, \ k = 0, \ldots, n, \]

**Frobenius powers**

For any $G$-module $U$, define its $p^m$th Frobenius power

\[ F^{p^m}(U) := \{ f \in S^{p^m}(U) : \exists u \in U \text{ with } f = u^{p^m} \} \leq S^{p^m}(U). \]

Ex.: \[ F^{p^m}(V_n) = \langle X_0^{p^m}, \ldots, X_n^{p^m} \rangle_K. \]
Lower bounds for the additive group

**Theorem**

Assume $\text{char}(K) = p > 0$. Then

$$\beta_{\text{sep}}(\mathbb{G}_a, V_1 \oplus F_p^n(V_1)) = p^n + 1$$

$$\Rightarrow \beta_{\text{sep}}(\mathbb{G}_a) = \infty.$$
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**Theorem**

Assume \( \text{char}(K) = 0 \). Then

\[
\beta_{\text{sep}}(\mathbb{G}_a, V_1 \oplus V_n) \geq n + 1
\]

\[\Rightarrow \beta_{\text{sep}}(\mathbb{G}_a) = \infty.\]
Roberts’ isomorphism

Let \( \langle X, Y \rangle \) be the natural representation of \( SL_2 \). For any \( SL_2 \)-module \( V \), there is the degree decreasing Roberts’-isomorphism

\[
\Phi : K [ V \oplus \langle X, Y \rangle^* ]^{SL_2} \to K[V]^{G_a}
\]
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\]

**Theorem**

Robert’s isomorphism maps \( SL_2 \)-separating sets to \( G_a \)-separating sets. In particular,

\[
\beta_{sep}(SL_2) = \infty.
\]
Warning

In general, isomorphisms of algebras do not take separating sets to separating sets!
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- Take $V^* := \langle X_1, Y_1 \rangle \oplus \langle X_2, Y_2 \rangle$ and $W := V^* \oplus \langle X, Y \rangle^*$. 
Warning

In general, isomorphisms of algebras do not take separating sets to separating sets!

- Take $V^* := \langle X_1, Y_1 \rangle \oplus \langle X_2, Y_2 \rangle$ and $W := V^* \oplus \langle X, Y \rangle^*.$
- Set $u_{12} := X_1 Y_2 - X_2 Y_1,$ $u_1 = X_1 Y - XY_1,$ $u_2 := X_2 Y - XY_2.$
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In general, isomorphisms of algebras do not take separating sets to separating sets!

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Set $u_{12} := X_1 Y_2 - X_2 Y_1$, $u_1 = X_1 Y - XY_1$, $u_2 := X_2 Y - XY_2$.
Set $S := \{X_1, X_2, X_1 u_{12}, X_2 u_{12}\} \subseteq K[V]^G$ is a separating set.
In general, isomorphisms of algebras do not take separating sets to separating sets!

- Take $V^* := \langle X_1, Y_1 \rangle \oplus \langle X_2, Y_2 \rangle$ and $W := V^* \oplus \langle X, Y \rangle^*$.  
- Set $u_{12} := X_1 Y_2 - X_2 Y_1$, $u_1 = X_1 Y - XY_1$, $u_2 := X_2 Y - XY_2$. 
- $S := \{X_1, X_2, X_1 u_{12}, X_2 u_{12}\} \subseteq K[V]^G$ is a separating set. 
- Though $\Phi : K[W]^{SL_2} \rightarrow K[V]^G$ is an isomorphism,  
  
  $\Phi^{-1}(S) = \{u_1, u_2, u_1 u_{12}, u_2 u_{12}\} \subseteq K[W]^{SL_2}$ 

is not a separating set!
Unipotent groups

- The connected component \( G^0 \trianglelefteq G \) is of finite index

\[ \Rightarrow \quad \beta_{\text{sep}}(G^0) \leq \beta_{\text{sep}}(G) \leq \beta_{\text{sep}}(G^0)(G : G^0). \]
Unipotent groups

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- For any connected nontrivial unipotent group, there is a closed normal subgroup $N \trianglelefteq G$ with $G/N \cong \mathbb{G}_a$. 

  Martin Kohls Technische Universität München
  Degree bounds for generating sets
  Relative degree bounds
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Unipotent groups

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**Theorem**

$G$ any infinite unipotent group (of arbitrary characteristic) $\Rightarrow$

$$\beta_{sep}(G) = \infty$$
Semisimple groups

Lemma
For any non-trivial torus $T$, we have $\beta_{\text{sep}}(T) = \infty$. 
Semisimple groups

Lemma
For any non-trivial torus $T$, we have $\beta_{\text{sep}}(T) = \infty$.

Theorem
For any non-trivial connected semi-simple group $G$, $\beta_{\text{sep}}(G) = \infty$. 
Lemma
For any non-trivial torus $T$, we have $\beta_{\text{sep}}(T) = \infty$.

Theorem
For any non-trivial connected semisimple group $G$, $\beta_{\text{sep}}(G) = \infty$.

Corollary (of unipotent & semisimple case)
Let $G$ be a linear algebraic group. Then

$$\beta_{\text{sep}}(G) < \infty \iff |G| < \infty.$$
Thank You!