Rings of Invariants and Varieties of Representations

Dr James Shank, University of Kent, June 2010
Introduction and Notation
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$$F[V]^G := \{ f \in F[V] \mid (f)g = f \ \forall g \in G \}.$$ 

If $G$ is finite then $F[V]^G$ is a finitely generated algebra.
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However, if $p$ divides $|G|$ (the modular case) there may be infinitely many isomorphism classes of $n$-dimensional $\mathbb{F}G$-modules.
Example: \( \mathbb{F} = \overline{\mathbb{F}}_p \), \( G = \mathbb{Z}/p \times \mathbb{Z}/p = \langle g_1, g_2 \rangle \), \( n = 2 \). Define

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\rho(g_1) = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad \rho(g_2) = \begin{pmatrix} 1 & \lambda \\ 0 & 1 \end{pmatrix}.
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Identify \( x \leftrightarrow [0 \ 1] \) and \( y \leftrightarrow [1 \ 0] \) so \( x \in \mathbb{F}[x, y]^{\rho(G)} \). Define

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N y := \prod_{g \in G} y \rho(g) = \prod_{a, b \in \mathbb{F}_p} (y + (a + b\lambda)x) = y^{p^2} + xf(x, y, \lambda).
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Calculate over the function field $\mathbb{F}(t)$ and specialise $t$ to $\lambda$?

Since $\lambda \in \mathbb{F}_p$ iff $\lambda^p = \lambda$, specialisation gives a generating set as long as we avoid roots of $t^p - t$. 
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A representation \( \rho : G \to GL_n(\mathbb{F}) \) is determined by matrices \( \rho(g_i) \) subject to relations \( \rho(r_j) \).

Thus each \( n \)-dimensional \( \mathbb{F}G \)-module gives a point in \( GL_n(\mathbb{F})^r \).

Let \( X \) denote the subset of \( GL_n(\mathbb{F})^r \) consisting of representations.

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Then \( GL_n(\mathbb{F}) \) and \( GL_n(\mathbb{F})^r \) are algebraic varieties.

\( X \) is determined by the relations \( \rho(r_j) \), which are polynomials in the entries of the matrices, and is thus a subvariety of \( GL_n(\mathbb{F})^r \).
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The automorphism group of $G$, $Aut(G)$, acts on $X$ by pre-composition. For each element of $Aut(G) \backslash X/GL_n(\mathbb{F})$, there is a corresponding isomorphism class of subrings of $\mathbb{F}[x_1, \ldots, x_n]$, i.e., for $\varphi \in Aut(G)$ and $\sigma \in GL_n(\mathbb{F})$,

$$\mathbb{F}[x_1, \ldots, x_n]^\rho(G) \cong \mathbb{F}[x_1, \ldots, x_n]^\sigma \rho(\varphi(G))\sigma.$$
Example: $\mathbb{F} = \overline{\mathbb{F}}_p$, $G = \mathbb{Z}/p \times \mathbb{Z}/p = \langle g_1, g_2 \rangle$, $n = 3$. Define

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Furthermore, $\mathbb{F}[x, y, z]^{\rho(g_1)}$ is the hypersurface generated by $x$,

$$d = y^2 - x(2z + y),$$

$$N_1 = \prod_{a \in \mathbb{F}_p} (y + ax) = y^p - yx^{p-1},$$

$$N_2 = \prod_{a \in \mathbb{F}_p} \left( z + ay + \binom{a}{2} x \right) = z^p + \ldots,$$

subject to a relation $d^p - N_1^2 + 2x^p N_2 + f(x, d)$. 
Writing $\Delta = \rho(g_2) - 1$, we have

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(d)\Delta = (\lambda^2 - \lambda - 2\mu)x^2.
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We have four cases:

1. The generic case: Calculate over $\mathbb{F}(t, s)$ and specialise $t$ to $\lambda$ and $s$ to $\mu$. The ring of invariants is a hypersurface with generators in degrees $1, p, p + 2, p^2$ and a relation in degree $p(p + 2)$. The specialisation gives the correct ring of invariants for $\lambda^p - \lambda \neq 0$ and $\lambda^2 - \lambda - 2\mu \neq 0$. 
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2. \( \mu = \binom{\lambda}{2} \): Then \( d \) is invariant. Calculate over \( \mathbb{F}(t) \) and specialise \( t \) to \( \lambda \). The ring of invariants is a hypersurface with generators in degrees \( 1, 2, p^2, p^2 \) and a relation in degree \( 2p^2 \). The specialisation gives the correct ring of invariants for \( \lambda^p - \lambda \neq 0 \).
3. $\lambda^p = \lambda$: Then $N_1$ is invariant. By replacing $g_2$ with $g_1^{-\lambda}g_2$, we can assume $\lambda = 0$. Calculate over $\mathbb{F}(s)$ and specialise $s$ to $\mu$. The ring of invariants is a hypersurface with generators in degrees $1, p, p + 1, p^2$ and a relation in degree $p(p+1)$. The specialisation gives the correct ring of invariants for $\mu \neq 0$. 


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4. $\lambda^p = \lambda$ and $\mu = \binom{\lambda}{2}$: The action is not faithful. In fact

$$\mathbb{F}[x, y, z]^{\rho(G)} = \mathbb{F}[x, y, z]^{\rho(g_1)}.$$
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- These are a posteriori observations.
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Remarks:

- These are *a posteriori* observations.

- Can we develop a framework to identify the equations that describe the generic case without computing the ring of invariants?
In the $n = 2$ case, it is possible to embed the family of representations in a larger representation: $V(\lambda) \subset W$, giving $\mathbb{F}[W] \rightarrow \mathbb{F}[V(\lambda)]$. 
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The long exact sequence coming from group cohomology gives

$$\cdots \rightarrow \mathbb{F}[W]^G \rightarrow \mathbb{F}[V(\lambda)]^G \rightarrow H^1(G, \ker(\pi)) \rightarrow H^1(G, \mathbb{F}[W]) \rightarrow \cdots$$
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Choose the embedding so that $\pi(x_3) = y$, $\pi(x_2) = x$ and $\pi(x_1) = cx$ where $c = (\lambda - \alpha)/(\alpha^p - \alpha)$. 
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To verify $\pi$ is a $G$-map:

$$
\pi(x_3 \Delta_2) = \pi(\alpha x_2 + (\alpha^p - \alpha)x_1) = (\alpha + (\alpha^p - \alpha)c)x = \lambda x.
$$
It is easy to see that $\mathbb{F}[W]^G = \mathbb{F}[x_1, x_2, N(x_3)]$ with

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Hence $\mathbb{F}[W]^G/I \cong \mathbb{F}[x, \pi(N(x_3))]$. 
It is easy to see that $\mathbb{F}[W]^G = \mathbb{F}[x_1, x_2, N(x_3)]$ with

$$N(x_3) = \prod_{g \in G} x_3g = x_3^{b^2} + \cdots$$

and that $\ker(\pi) = (x_1 - cx_2)\mathbb{F}[W]$.

Thus $I := \ker(\pi)^G = (x_1 - cx_2)\mathbb{F}[W]^G$.

Hence $\mathbb{F}[W]^G/I \cong \mathbb{F}[x, \pi(N(x_3))]$.

Therefore, if $\lambda \in \mathbb{F} \setminus \mathbb{F}_p$ then $\pi^G$ is surjective.
It is easy to see that $\mathbb{F}[W]^G = \mathbb{F}[x_1, x_2, N(x_3)]$ with 

$$N(x_3) = \prod_{g \in G} x_3 g = x_3^{p^2} + \cdots$$

and that $\ker(\pi) = (x_1 - cx_2)\mathbb{F}[W]$.

Thus $I := \ker(\pi)^G = (x_1 - cx_2)\mathbb{F}[W]^G$.

Hence $\mathbb{F}[W]^G/I \cong \mathbb{F}[x, \pi(N(x_3))]$.

Therefore, if $\lambda \in \mathbb{F} \setminus \mathbb{F}_p$ then $\pi^G$ is surjective.

However, if $\lambda \in \mathbb{F}_p$ then $y^p - x^{p-1}y \in \mathbb{F}[V(\lambda)]^G \setminus \text{im}(\pi^G)$,
It is easy to see that $\mathbb{F}[W]^G = \mathbb{F}[x_1, x_2, N(x_3)]$ with
\[ N(x_3) = \prod_{g \in G} x_3 g = x_3^{p^2} + \cdots \]
and that $\ker(\pi) = (x_1 - cx_2)\mathbb{F}[W]$.
Thus $I := \ker(\pi)^G = (x_1 - cx_2)\mathbb{F}[W]^G$.
Hence $\mathbb{F}[W]^G/I \cong \mathbb{F}[x, \pi(N(x_3))]$.
Therefore, if $\lambda \in \mathbb{F} \setminus \mathbb{F}_p$ then $\pi^G$ is surjective.
However, if $\lambda \in \mathbb{F}_p$ then $y^p - x^{p-1}y \in \mathbb{F}[V(\lambda)]^G \setminus \text{im}(\pi^G)$,
i.e., there is a class $\chi \in H^1(G, \ker(\pi))$ which is zero in $H^1(G, \mathbb{F}[W])$. 

It is easy to see that $\mathbb{F}[W]^G = \mathbb{F}[x_1, x_2, N(x_3)]$ with

$$N(x_3) = \prod_{g \in G} x_3 g = x_3^p + \cdots$$

and that ker($\pi$) = $(x_1 - cx_2)\mathbb{F}[W]$.

Thus $I := \ker(\pi)^G = (x_1 - cx_2)\mathbb{F}[W]^G$.

Hence $\mathbb{F}[W]^G/I \cong \mathbb{F}[x, \pi(N(x_3))]$.

Therefore, if $\lambda \in \mathbb{F} \setminus \mathbb{F}_p$ then $\pi^G$ is surjective.

However, if $\lambda \in \mathbb{F}_p$ then $y^p - x^{p-1}y \in \mathbb{F}[V(\lambda)]^G \setminus \text{im}(\pi^G)$, i.e., there is a class $\chi \in H^1(G, \ker(\pi))$ which is zero in $H^1(G, \mathbb{F}[W])$.

I am still looking for the analog of $W$ for the three variable case.