When is a Ring of Torus Invariants a Polynomial Ring?

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Let $\rho : T \to GL(V)$ be a finite dimensional rational representation of a torus over an algebraically closed field $\mathbf{k}$. We give necessary and sufficient conditions on the arrangement of the weights of V within the character lattice of T for the ring of invariants, $\mathbf{k}[V]^T$, to have a homogeneous system of parameters consisting of monomials (Theorem 4.1). Using this we give two simple constructive criteria each of which gives necessary and sufficient conditions for $\mathbf{k}[V]^T$ to be a polynomial ring (Theorem 5.8 and Theorem 5.10).

1. Introduction

Let $\rho : G \to GL(V)$ be a finite dimensional rational representation of a reductive algebraic group defined over an algebraically closed field $\mathbf{k}$. Denote by $\mathbf{k}[V]$ the algebra of polynomial functions on V. The action of G on V given by ρ induces an action of G on $\mathbf{k}[V]$. An element of $\mathbf{k}[V]$ fixed by this action is called an invariant and the set of invariants forms a finitely generated subalgebra called the *ring of invariants*, denoted $\mathbf{k}[V]^G$.

The fundamental problem in invariant theory is to describe the ring of invariants. A first step toward understanding $\mathbf{k}[V]^G$ is to determine for which representations ρ is $\mathbf{k}[V]^G$ a polynomial ring (see [11]).

G.C. Shephard and J.A. Todd showed in 1954 [13] that the ring of invariants of a finite group over a field of characteristic zero is a polynomial ring if and only if the action of G on V is generated by pseudo-reflections. Their proof of the "if" half of this theorem was by classification. In 1955 C. Chevalley [3] gave an elegant algebraic proof that, if the action of a finite group G on V is generated by reflections (still assuming $\mathrm{char}(\mathbf{k}) = 0$), then $\mathbf{k}[V]^G$ is a polynomial ring. Soon thereafter, J.P. Serre observed that Chevalley's proof was also valid for actions generated by pseudo-reflections and that it therefore provided an algebraic proof of the "if" half of the Shephard-Todd theorem.

A method for classifying those reductive groups having a polynomial ring of invariants was suggested in 1976 by V. Kac, V. Popov and E. Vinberg [5]. Us-

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ing this method G. Schwarz [12] in 1978, and independently O. Adamovich and E. Golovina [1] in 1979, enumerated the representations of complex connected simple groups having a polynomial ring of invariants. In 1989, P. Littelmann [9] listed (up to the equivalence relation called *castling*) the irreducible representations of connected semi-simple complex groups whose rings of invariants are polynomial rings. While the result of Shephard and Todd gives simple conditions for $\mathbf{k}[V]^G$ to be polynomial if G is finite, there is no known similar simple characterization if G is semi-simple.

Here we will consider the situation when $G = T$ is a torus. We first give a number of conditions on a representation of a torus each of which is equivalent to the condition that the ring of invariants, $\mathbf{k}[V]^T$, have a system of parameters consisting of monomials (Theorem 4.1). This last condition is clearly a necessary condition for $\mathbf{k}[V]^T$ to be polynomial. One of these equivalent conditions (4) is entirely in terms of the geometric arrangement of the weights of V within the character lattice. We also give two simple constructive numerical criteria each of which determines those representations satisfying one (all) of these first conditions actually has $\mathbf{k}[V]^T$ a polynomial ring (Theorem 5.8 and Theorem 5.10).

2. Preliminaries

Proofs of the results in this section may be found for example in [7]. Let $\mathbf{k}$ be an algebraically closed field of any characteristic and let $T \cong (\mathbf{k}^*)^r$ be an algebraic torus of dimension r. The character group of T, $X^*(T) := \{\, \omega : T \to \mathbf{k}^* \mid \omega$ is a group homomorphism $\}$, is isomorphic to $\mathbb{Z}^r$. Suppose $\rho : T \to GL(V)$ is a finite dimensional rational representation of T. There is a unique direct sum decomposition of V into its weight spaces: $V = \oplus_{\omega \in X^*(T)} V_\omega$ where $V_\omega = \{\, v \in V \mid \rho(t)v = \omega(t)v \ \forall t \in T \,\}$. The elements of the (finite) set $\Omega(V) := \{\, \omega \in X^*(T) \mid V_\omega \neq \{0\} \,\}$ are called the *weights* of T. If we choose a basis of V_ω for each weight $\omega \in \Omega(V)$ and then take the union of these bases we get a basis $\{v_1, \ldots, v_n\}$ of V with respect to which $\rho(t)$ is a diagonal matrix for all $t \in T$. Conversely any basis with respect to which all the elements of $\rho(T)$ are diagonal must be a union of bases for the V_ω. We call any such basis a *basis of eigenvectors*. Fix such a basis of eigenvectors, $\{v_1, \ldots, v_n\}$ and let $\omega_i \in \Omega(V)$ denote the weight of v_i for $1 \leq i \leq n$, i.e., $\rho(t)v_i = \omega_i(t)v_i$ for all $t \in T$. Let V^* denote the vector space dual of V and let $\{x_1, \ldots, x_n\}$ be the basis of V^* which is dual to $\{v_1, \ldots, v_n\}$.

In the following we will always use $\{v_1, \ldots, v_n\}$ to denote a basis of eigenvectors for V, $\{x_1, \ldots, x_n\}$ to denote the dual basis of semi-invariants of V^* and ω_i to denote the weight of v_i.

Denote by $\mathbf{k}[V]$ the ring of polynomial functions on V. The T-action on V given by ρ induces an action of T on V^*. This action is given in terms of the x_i by $t \cdot x_i = -\omega_i(t)x_i$ for all $t \in T$. Extending this action on V^* multiplicatively and additively we see that ρ induces an action of T on all of $\mathbf{k}[V] \cong \mathbf{k}[x_1, \ldots, x_n]$. The elements of $\mathbf{k}[V]$ which are fixed by this T-action form a finitely generated $\mathbf{k}$-subalgebra of $\mathbf{k}[V]$, called the *ring of invariants* and denoted $\mathbf{k}[V]^T$.

Each invariant polynomial $f(x_1, \ldots, x_n) \in \mathbf{k}[V]^T$ is a linear combination of invariant monomials. In particular, there are minimal generating sets for the ring $\mathbf{k}[V]^T$ containing only monomials.

Clearly T acts on a monomial, $X^A = x_1^{a_1} x_2^{a_2} \ldots x_n^{a_n}$ where $A = (a_1, \ldots, a_n) \in \mathbb{N}^n$, by the character $\chi := -(a_1\omega_1 + \ldots + a_n\omega_n) \in X^*(T)$, i.e., $t \cdot X^A = \chi(t)X^A$ for all $t \in T$. Thus the invariant monomials are in one-to-one correspondence with the semi-group, $S_\rho = S := \{ A \in \mathbb{N}^n \mid X^A \in \mathbf{k}[V]^T \} = \{ A \in \mathbb{N}^n \mid a_1\omega_1 + \ldots + a_n\omega_n = \mathbf{o} \}$ where $\mathbf{o}$ is the trivial character in $X^*(T)$. This semi-group was studied in the nineteenth century by Gordan. He used it to show that $\mathbf{k}[V]^T$ is a finitely generated algebra by showing that S is finitely generated as a semi-group (see [4]).

Recall that a representation $\rho : G \to GL(V)$ is called *stable* if the union of the closed G-orbits in V contains an open dense subset of V.

Let $\rho : T \to GL(V)$ be a representation of a torus. It is not hard to see that V contains a T-invariant subspace V' such that ρ restricted to V' is stable and $\mathbf{k}[V]^T = \mathbf{k}[V']^T$ [15, Lemma 2]. Furthermore ρ is a faithful representation if and only if $\operatorname{span}_{\mathbb{Z}}\{\omega_1, \ldots, \omega_n\} = X^*(T)$. Thus it is clear that in order to study the properties of $\mathbf{k}[V]^T$, it suffices to suppose that ρ is both faithful and stable and we shall do this from now on. In general, the Krull dimension of $\mathbf{k}[V]^T$ is equal to the rank of the semi-group S. The two hypotheses that ρ is faithful and stable together imply that this dimension is equal to $\dim V - \dim T$.

3. Completely Fundamental Generators

Now we are ready to investigate how the properties of $\mathbf{k}[V]^T$ are related to the geometric arrangement of the weights of V. We begin by introducing some notation. We will use $\mathbf{o}$ to denote the origin in $X^*(T) \otimes \mathbb{Q} \cong \mathbb{Q}^r$. If $A = (a_1, \ldots, a_n) \in S$ we define $\operatorname{supp}(A) := \{ i \mid 1 \leq i \leq n, a_i \neq 0 \}$. If $\{Z_1, \ldots, Z_q\} \subset \mathbb{Q}^q$ then $\mathbf{C}(Z_1, \ldots, Z_q)$ denotes the closed cone generated by the points $Z_1, \ldots, Z_q$: $\mathbf{C}(Z_1, \ldots, Z_q) = \sum_{i=1}^m \mathbb{Q}^+ Z_i$ where $\mathbb{Q}^+ = \{ y \in \mathbb{Q} \mid y \geq 0 \}$.

Define a partial order on $S \setminus \{(0, \ldots, 0)\}$ by inclusion of supports, i.e., if $A, B \in S$ with $\operatorname{supp}(A) \subset \operatorname{supp}(B)$ then $A \prec B$. Furthermore if $A = (a_1, \ldots, a_n), B = (b_1, \ldots, b_n) \in S$ with $\operatorname{supp}(A) = \operatorname{supp}(B)$ then we set $A \preceq B$ if $a_i \leq b_i$ for all $1 \leq i \leq n$. If $R \in S$ is a minimal element in this partial ordering then R is called a *completely fundamental generator* of S (and by abuse of terminology) we also say that the monomial X^R is a completely fundamental generator of $\mathbf{k}[V]^T$. The following proposition shows the importance of the completely fundamental generators.

Proposition 3.1 (Stanley [14, Theorem 3.7]. *Suppose $\{X^{A_1}, \ldots, X^{A_s}\}$ is any minimal set of monomials such that $\mathbf{k}[V]^T$ is integral over $\mathbf{k}[X^{A_1}, \ldots, X^{A_s}]$. Then there are exactly s completely fundamental generators of $\mathbf{k}[V]^T$, $R_1, \ldots, R_s$ and there exists a permutation π of $\{1, \ldots, s\}$ such that $\operatorname{supp}(R_j) = \operatorname{supp}(A_{\pi(j)})$. In fact, there exist positive integers $q_1, \ldots, q_s$ such that $A_{\pi(j)} = q_j R_j$ for all $1 \leq j \leq s$.*

Remark 3.2. Recall that ρ is stable implies that $\text{span}_{\mathbb{Z}}\{\omega_1, \ldots, \omega_n\} = X^*(T)$. Hence the above proposition implies that $\cup_{j=1}^{s} \text{supp}(R_j) = \{1, 2, \ldots, n\}$.

Corollary 3.3. $s \geq n - r$

Proof. This result follows immediately from the fact that the Krull dimension of $\mathbf{k}[V]^T$ is given by $\dim V - \dim T = n - r$. Therefore $s \geq \dim \mathbf{k}[X^{R_1}, \ldots, X^{R_s}] = \dim \mathbf{k}[V]^T$. □

4. Representations having a MSOP

Definition. We say an n dimensional representation, V, of an r dimensional torus, T, has a *monomial system of parameters* (MSOP) if there exists a homogeneous system of parameters for $\mathbf{k}[V]^T$ consisting entirely of monomials, i.e., if there are $n - r$ monomials: $f_1, \ldots, f_{n-r} \in \mathbf{k}[V]^T$ such that $\mathbf{k}[V]^T$ is integral over the subring $\mathbf{k}[f_1, \ldots, f_{n-r}]$.

We shall also need to recall the following definition.

Definition. The *nullcone of* V is the set $V^\circ := \{ v \in V \mid 0 \in \overline{\rho(T)v} \} = \{ v \in V \mid f(v) = 0 \text{ for all homogeneous non-constant } f \in \mathbf{k}[V]^T \}$.

To see that these two characterizations of the nullcone coincide we observe that the inclusion $\supseteq$ follows since the invariants separate closed T-orbits and that the other inclusion is the content of the Hilbert-Mumford criteron (see for example [7, §III.2.4]).

We remark that using the second description of the nullcone it is easy to see that for a torus representation the nullcone is a union of subspaces of V various dimensions. One characterization of these subspaces is given in Proposition 4.4 below.

Theorem 4.1. *Let $\rho : T \to V$ be an n dimensional representation of a rank r torus. Suppose that $\{R_1, \ldots, R_s\}$ are the completely fundamental generators of $\mathbf{k}[V]^T$. Then the following conditions are all equivalent:*

(1) V has a MSOP
(2) $s = n - r$
(3) $\text{supp}(R_j) \not\subseteq \cup_{\substack{i=1 \\ i \neq j}}^{s} \text{supp}(R_i)$ for all $1 \leq j \leq s$
(4) There exists a partition $\{1, 2, \ldots, n\} = \Gamma_1 \cup \Gamma_2$ with $\#\Gamma_1 = n - r$ and $\#\Gamma_2 = r$ such that $\{ \omega_i \mid i \in \Gamma_2 \}$ is a basis for $X^*(T) \otimes \mathbb{Q}$ and $\omega_j \in \mathbf{C}(-\omega_i \mid i \in \Gamma_2)$ for all $j \in \Gamma_1$
(5) V° contains an irreducible component of dimension r.

Proof. The proof that (1) $\iff$ (2) follows immediately from Proposition 3.1.

Now we show (2) $\Rightarrow$ (3) Suppose then that $s = n-r$ and assume $\text{supp}(R_1) \subseteq \cup_{j=2}^{n-r} R_j$. Write $R_1 = (r_1, \ldots, r_n)$. Choosing $c_2, \ldots, c_{n-r} \in \mathbb{N}$ sufficiently large and defining $A = (a_1, \ldots, a_n) := \sum_{j=2}^{n-r} c_j R_j$ we may ensure that $a_i \geq r_i$ for all $1 \leq i \leq n$. Hence $A' = A - R_1 \in \mathbb{N}^n$ and thus $A' \in S$. By Proposition 3.1 we may write $q A' = \sum_{j=1}^{n-r} d_j R_j$ with $d_j \in \mathbb{N}$ for q sufficiently large. But then $q A = q A' + q R_1 = \sum_{j=1}^{n-r} d_j R_j + q R_1$. Therefore $(q+d_1)R_1 = \sum_{j=2}^{n-r}(qc_j - d_j)R_j$. Thus $\{R_1, \ldots, R_{n-r}\}$ is linearly dependent. But this implies that the Krull dimension of $k[X^{R_1}, \ldots, X^{R_{n-r}}]$ is strictly less than $n-r$, the Krull dimension of $\mathbf{k}[V]^T$, which contradicts Proposition 3.1.

To show that (3) $\Rightarrow$ (4) we assume (3). Reordering the ω_i we may assume that $j \in \text{supp}(R_j) \subseteq \{j\} \cup \{s+1, \ldots, n\}$ for all $1 \leq j \leq s$. Then setting $\Gamma_1 := \{1, \ldots, s\}$ and $\Gamma_2 := \{s+1, \ldots, n\}$ we have $\text{supp}(R_j) \subseteq \Gamma_2 \cup \{j\}$. Now take j with $1 \leq j \leq s$ and write $R_j = (r_1, \ldots, r_n)$. Then $R_j \in S$ implies that $r_j \omega_j = -r_{s+1}\omega_{s+1} - \ldots - r_n \omega_n$. Thus $\omega_j \in \mathbf{C}(-\omega_{s+1}, \ldots, -\omega_n)$. Moreover since this is true for all $1 \leq j \leq s$ we get $\text{span}_{\mathbb{Q}}\{\omega_{s+1}, \ldots, \omega_n\} = \text{span}_{\mathbb{Q}}\{\omega_1, \ldots, \omega_n\} = X^*(T) \otimes \mathbb{Q} \cong \mathbb{Q}^r$. In particular, $n - s \geq r$. But then $\#\Gamma_1 = s = n - r$ by Corollary 3.3.

Now we show that (4) $\Rightarrow$ (2). Since $\omega_j \in \mathbf{C}(-\omega_i \mid i \in \Gamma_2)$ we see that there is a corresponding completely fundamental generator R_j with $j \in \text{supp}(R_j) \subseteq \Gamma_2 \cup \{j\}$. Conversely from the hypothesis that $\{\omega_i \mid i \in \Gamma_2\}$ is a basis for $X^*(T) \otimes \mathbb{Q}$ it follows easily that these are the only completely fundamental generator. Hence $s = \#\Gamma_1 = n - r$.

Finally the equivalence (5) $\iff$ (4) is a special case of Proposition 4.4 below. □

Example 4.2. Here we consider the special case when $T \cong \mathbf{k}^*$, i.e., $r = 1$. By (4) of the above theorem we see that V has a MSOP if and only if exactly one weight of V is negative or exactly one weight of V is positive. Suppose then that $\omega_n < 0$ and $\omega_i \geq 0$ for $i = 1, \ldots, n-1$. Then the $n-1$ completely fundamental generators which form a MSOP for V are $R_i = (x_i)^{\omega_n/g_i} \cdot (x_n)^{\omega_i/g_i}$ for $i = 1, \ldots, n-1$ where $g_i = \gcd(\omega_i, \omega_n)$. The subspace of V spanned by v_n is the one dimensional component of the nullcone.

Remark 4.3. If V has a MSOP then there is, in general, more than one way to choose the partition $\{1, 2, \ldots, n\} = \Gamma_1 \cup \Gamma_2$ satisfying the conditions of (4) of Theorem 4.1.

If $I \subseteq \{1, 2, \ldots, n\}$ denote $\mathbf{C}(\omega_i \mid i \in I)$ by $\mathbf{C}_I$ and set $V_I := \text{span}_{\mathbf{k}}\{v_i \mid i \in I\}$. Now there is an invariant monomial X^A with $I = \text{supp}(A)$ if and only if $\mathbf{C}_I$ contains a non-zero subspace of $\mathbb{Q}^r$, i.e., $V_I \subseteq V^\circ$ if and only if $\mathbf{C}_I$ contains no non-zero subspace of $\mathbb{Q}^r$. Moreover, $V_I \subseteq V^\circ$ is maximal if and only if $-\omega_j \in \mathbf{C}_I \ \forall j \notin I$. Thus we have the following proposition.

Proposition 4.4. *The components of V° are the sets $\text{span}_{\mathbf{k}}\{v_i \mid i \in I\}$ where $\mathbf{C}_I$ contains no non-zero subspace of $\mathbb{Q}^r$ and $-\mathbf{C}_{I^c} \subseteq \mathbf{C}_I$ where I^c denotes the*

complement of $I \subseteq \{1, 2, \ldots, n\}$. *Furthermore, the dimension of the component is given by* $\#I$.

5. From Tori to Finite Abelian Groups

In this section we describe the connection between representations of tori having a MSOP and representations of finite abelian groups.

Suppose V has a MSOP and let Γ_1 and Γ_2 be sets as in part (4) of Theorem 4.1. Without loss of generality we may assume that $\Gamma_1 = \{1, 2, \ldots, n-r\}$ and $\Gamma_2 = \{n-r+1, \ldots, n\}$. Let us relabel our eigenbasis by setting $w_j = v_j$ for $1 \leq j \leq n-r$ and $u_i = v_{n-r+i}$ for $1 \leq i \leq r$. Then define $W := \operatorname{span}_{\mathbf{k}}\{w_1 \ldots w_{n-r}\}$ and $U := \operatorname{span}_{\mathbf{k}}\{u_1, \ldots, u_r\}$. Also let μ_i denote the weight of u_i and ν_j the weight of w_j. Take $p = (p_1, \ldots, p_r) \in U \subset V$ such that $p_i \neq 0$ for $1 \leq i \leq r$ and let $H := T_p$, be the subgroup of T which fixes p. Set $L := \mathbb{Z}\mu_1 \oplus \ldots \oplus \mathbb{Z}\mu_r$, the (free) rank r sublattice of $X^*(T)$ spanned by the μ_i.

Lemma 5.1.
(1) $H = \cap_{i=1}^r \ker \mu_i = \cap_{\chi \in L} \ker \chi$.
(2) $\#H = |\det(\mu_1 \ \ldots \ \mu_r)|$. *In particular,* H *is finite.*
(3) If $H \subseteq \ker \chi$ *for some* $\chi \in X^*(T)$, *then* $\chi \in L$.

Proof. By the theory of finitely generated abelian groups, there exists a $\mathbb{Z}$-basis $\{\alpha_1, \ldots, \alpha_r\}$ of $X^*(T)$, a $\mathbb{Z}$-basis $\{\lambda_1, \ldots, \lambda_r\}$ of L and positive integers $m_1, \ldots, m_r$ such that $\lambda_i = m_i\alpha_i$ for $1 \leq i \leq r$ and $|\det(\mu_1 \ \ldots \ \mu_r)| = m_1 m_2 \ldots m_r$ and m_i divides m_{i+1} for $1 \leq i \leq r-1$. Now for all $t \in T$ we have $\rho(t)p = (\mu_1(t)p_1, \ldots, \mu_r(t)p_r)$. Therefore $t \in H \iff t \in \cap_{i=1}^r \ker \mu_i \iff t \in \cap_{\chi \in L} \ker \chi \iff t \in \cap_{i=1}^r \ker \lambda_i$. The basis $\{\alpha_1, \ldots, \alpha_r\}$ of $X^*(T)$ determines an explicit isomorphism $T \cong (\mathbf{k}^*)^r$ such that $\alpha_i(t_1, \ldots, t_r) = t_i$. Relative to this isomorphism we then have $\lambda_i(t_1, \ldots, t_r) = t_i^{m_i}$ and $t \in \ker \lambda_i \iff t_i$ is an $m_i{}^{\text{th}}$ root of unity. Therefore $H = \{(t_1, \ldots, t_r) \in T \mid t_i \text{ is an } m_i{}^{\text{th}} \text{ root of unity for } i = 1, \ldots, r\}$ and $H \cong \mathbb{Z}/m_1\mathbb{Z} \oplus \ldots \oplus \mathbb{Z}/m_r\mathbb{Z}$ is a finite group of order $m_1 m_2 \ldots m_r = |\det(\mu_1 \ \ldots \ \mu_r)|$.

To prove (3) suppose $H \subseteq \ker \chi$ and write $\chi = a_1\alpha_1 + \ldots + a_r\alpha_r$ where $a_i \in \mathbb{Z}$. Take h_i a primitive $m_i{}^{\text{th}}$ root of unity and consider $h := (1, \ldots, 1, h_i, 1, \ldots, 1) \in H$. Then $1 = \chi(h) = h_i^{a_i}$. Hence a_i is divisible by m_i and writing $a_i = b_i m_i$ for each i we have $\chi = b_1\lambda_1 + \ldots + b_r\lambda_r \in L$. □

Remark 5.2. The above proof shows that we have a short exact sequence of abelian groups: $1 \to L \hookrightarrow X^*(T) \xrightarrow{\text{Res}} X^*(H) \to 1$ where Res is just the restriction mapping. Thus $H \cong X^*(H) \cong X^*(T)/L$.

Remark 5.3. From the above proof it is clear that the subgroup $H \subset T$ is independent of the choice of the $p_i \in \mathbf{k}^*$. Furthermore, the fact that H acts

both diagonally and faithfully on W implies that if $\operatorname{char}\mathbf{k}$ is not zero then $\operatorname{char}\mathbf{k}$ does not divide $\#H$.

Proposition 5.4. *The map* $\operatorname{Res}_p : \mathbf{k}[V]^T \to \mathbf{k}[W]^H$ *given by* $\operatorname{Res}_p(f)(w) = f(p+w)$ *is a* $\mathbf{k}$*-algebra isomorphism.*

Proof. Since Res_p is just the restriction to the subset $\{p\} \times W \cong W$, Res_p is clearly a morphism of $\mathbf{k}$-algebras.

To see that Res_p is surjective, let $y_j := x_j$ for $1 \leq j \leq n-r$ and $z_i = x_{n-r+i}$ for $1 \leq i \leq r$. Then $\{y_1, \ldots, y_{n-r}\}$ and $\{z_1, \ldots, z_r\}$ are the bases of semi-invariants dual to $\{w_1, \ldots, w_{n-r}\}$ and $\{u_1, \ldots, u_r\}$ respectively. If the monomial $f = y_1^{a_1} \ldots y_{n-r}^{a_{n-r}}$ lies in $\mathbf{k}[W]^H$, then the T-weight of f, $\chi := -\sum_{j=1}^{n-r} a_j \nu_j$ must lie in the sublattice L. Therefore we may write $\chi = b_1\mu_1 + \ldots b_r\mu_r$ with the $b_i \in \mathbf{Z}$. Since $\nu_j \in \mathbf{C}(-\mu_1, \ldots, -\mu_r)$ for all $1 \leq j \leq n-r$, each b_i must in fact be a non-negative integer. Hence if we define $g := z_1^{b_1} \ldots z_r^{b_r}$, then the weight of g is $-\chi$ and $gf \in \mathbf{k}[V]^T$ with $\operatorname{Res}_p(gf) = g(p)\, f = p_1^{b_1} \ldots p_r^{b_r} f$ is a non-zero scalar multiple of f. Thus Res_p is surjective.

Now suppose that $\operatorname{Res}_p(f) \equiv 1$, i.e., $f(p+w) = 1$ for all $w \in W$. Then $f \in \mathbf{k}[U] \subset \mathbf{k}[V]$. But $\mathbf{k}[U]^T = \mathbf{k}$ and therefore $f \equiv 1$. Hence Res_p is injective. □

Remark 5.5. A proof of the above theorem, for the case $T \cong \mathbf{k}^*$, may be found in [8, §4.1].

Putting together the above results we have a characterization of those representations of tori whose ring of invariants is a polynomial ring.

Theorem 5.6. *Suppose that* $\rho : T \to GL(V)$ *is a faithful stable representation of a torus. Then* $\mathbf{k}[V]^T$ *is a polynomial ring if and only if* V *has a MSOP and the action of the finite group* H *on* W *is generated by pseudo-reflections where* H *and* W *are as defined above.*

In the above we have used the theorem of Shephard and Todd, and Chevalley (which is also true in positive characteristic if the characteristic does not divide the order of the group — see for example [2, Théorème 4,pg. 115]) to characterize those representations for which $\mathbf{k}[W]^H$ is a polynomial ring. However there is also a criterion for determining whether certain sets of invariants of a finite group form a set of generators for the ring of invariants (see [2, Exercise 5(b), pg. 137]). This criterion is given in the following proposition.

Proposition 5.7. *Let* H *be any finite group and let* $\tau : H \to GL(W)$ *be a faithful* q *dimensional representation of* H *defined over some field* $\mathbf{k}$*. Suppose that* $f_1, \ldots, f_q \in \mathbf{k}[W]^H$ *are* q *homogeneous invariants such that* $\mathbf{k}[W]$ *is integral over the subring* $\mathbf{k}[f_1, \ldots, f_q]$*. Then* $\mathbf{k}[W]^H = \mathbf{k}[f_1, \ldots, f_q]$ *if and only if* $\#H = \prod_{j=1}^{q} \deg f_j$.

For our $n-r$ dimensional representation $\tau : H \to GL(W)$ we have obvious candidates for the generators $f_1, \ldots, f_{n-r}$. Just take f_j to be (a scalar multiple of) $\mathrm{Res}_p(X^{R_j})$. We may do this explicitly by first choosing a basis of semi-invariants for W^*, $\{y_1, \ldots, y_{n-r}\}$. Then set $S_\tau := \{ A = (a_1, \ldots, a_{n-r}) \in \mathbb{N}^{n-r} \mid \sum_{j=1}^{n-r} a_j \nu_j = 0 \in X^*(H) \} = \{ A = (a_1, \ldots, a_{n-r}) \in \mathbb{N}^{n-r} \mid Y^A \in \mathbf{k}[W]^H \}$. The fact that $\mathrm{Res}_p : \mathbf{k}[V]^T \to \mathbf{k}[W]^H$ is an isomorphism implies that the induced map $\Pi : S_\rho \to S_\tau$ given by $\Pi(a_1, \ldots, a_n) = (a_1, \ldots, a_{n-r})$ is an isomorphism of semi-groups. Let $\hat{R}_j := \Pi(R_j)$ for $1 \leq j \leq n-r$. Then $\hat{R}_j$ has all coordinates zero except for the j^{th} which is a positive integer we will denote by m_j. Hence we may take $f_j = Y^{\hat{R}_j} = y_j^{m_j}$. From the fact that Π is an isomorphism we may deduce that m_j is nothing other than the order of ν_j as an element of $X^*(H)$. Our assumption that ρ is faithful implies that τ is also faithful. Therefore Proposition 5.7 applies and we have the following theorem.

Theorem 5.8. *Let $\rho : T \to GL(V)$ be a faithful stable representation of a torus and let the notation be as above. Then $\mathbf{k}[V]^T$ is a polynomial ring if and only if V has a MSOP and $m_1 m_2 \ldots m_{n-r} = |\det(\mu_1 \ \ldots \ \mu_r)|$.*

Remark 5.9. Although m_j is the j^{th} coordinate of R_j, we may use the characterization of m_j as the order of $\nu_j \in X^*(H) \cong X * (T)/L$ to easily compute m_j without calculating the whole of R_j. Expressing ν_j as $\nu_j = (a_{j1}/b_{j1})\mu_1 + \ldots + (a_{jr}/b_{jr})\mu_r$ with $a_{ji}, b_{ji} \in \mathbb{Z}$ and $\gcd(a_{ji}, b_{ji}) = 1$ we have $m_j = \mathrm{lcm}(b_{j1}, \ldots, b_{jr})$.

We finish this section with another constructive criterion for a torus representation having a MSOP to have a polynomial ring of invariants. This last criterion relies on an important result from the theory of torus embeddings. Let D be the subgroup of diagonal matrices in $GL(V)$ with respect to our basis $\{v_1, \ldots, v_n\}$. Then the torus $\overline{D} := D/T$ acts on our ring $\mathbf{k}[V]^T$. Geometrically this gives a corresponding action of $\overline{D}$ on the variety $X := \mathrm{MaxSpec}(\mathbf{k}[V]^T)$ having a dense $\overline{D}$-orbit which is isomorphic to $\overline{D}$. The variety X together with its $\overline{D}$-action is called is an embedding of the torus $\overline{D}$.

The result we need from the theory of torus embeddings is that X is a smooth variety (i.e., $\mathbf{k}[V]^T$ is a regular ring) if and only if $\{R_1, \ldots, R_S\} \subseteq \mathbb{N}^n$ can be extended to a $\mathbb{Z}$-basis of $\mathbb{Z}^n$ (see for example [10, Theorem 1.10]).

Now consider the $s \times n$ integer matrix M whose j^{th} row is the vector R_j. Using integer elementary row and column operations on M we may reduce M to the $\mathbb{Z}$-equivalent "diagonal" matrix

$$\widetilde{M} = \begin{pmatrix} d_{11} & 0 & \ldots & 0 & \ldots & 0 \\ 0 & d_{22} & \ldots & 0 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots & \ldots & \vdots \\ 0 & 0 & \ldots & d_{ss} & \ldots & 0 \end{pmatrix}$$

where d_{ii} divides d_{jj} for all $i < j$. Now the subset $\{R_1, \ldots, R_s\}$ may be extended to a basis of $\mathbb{Z}^n$ if and only if all the d_{ii} are equal to 1. This is also equivalent to the condition that the gcd of all the $s \times s$ minors of M be 1.

Since $\mathbf{k}[V]^T$ is generated by *homogeneous* polynomials we see that $\mathbf{k}[V]^T$ is regular if and only if $\mathbf{k}[V]^T$ is polynomial. Thus we have the folowing criterion.

Theorem 5.10. *Let $\rho : T \to GL(V)$ be a faithful stable representation of a torus having a MSOP. Form the matrices M and $\widetilde{M}$ from $R_1, \ldots, R_{n-r}$ as above. Then $\mathbf{k}[V]^T$ is a polynomial ring if and only if the "diagonal" entries of $\widetilde{M}$, d_{ii}, are all equal to 1 if and only if the gcd of all the $(n-r) \times (n-r)$ minors of M is 1.*

Remark 5.11. There is an algorithm for finding the completely fundamental generators $\{R_1, \ldots, R_s\}$ [16, Proposition 1] and thus both Theorem 5.8 and Theorem 5.10 give a constructive method for determining whether a ring of torus invariants is a polynomial ring.

6. From Finite Abelian Groups Back to Tori

In the previous section we began with a faithful stable torus representation, $\rho : T \to GL(V)$ having a MSOP and we found a finite abelian group H and a faithful diagonalizable representation $\tau : H \to GL(W)$ such that $\mathbf{k}[V]^T \cong \mathbf{k}[W]^H$. It is also possible to go in the opposite direction as the next proposition shows.

Proposition 6.1. *Suppose H is a finite abelian group and $\tau : H \to GL(W)$ is a finite dimensional faithful representation of H. Further suppose that either* $\operatorname{char}\mathbf{k}$ *is zero or* $\operatorname{char}\mathbf{k}$ *does not divide $\#H$. Then there is a torus T and a finite dimensional representation $\rho : T \to V$ of T having a MSOP and such that $\mathbf{k}[V]^T \cong \mathbf{k}[W]^H$.*

Proof. Write $H \cong \mathbb{Z}/n_1\mathbb{Z} \oplus \ldots \oplus \mathbb{Z}/n_r\mathbb{Z}$ where each n_i is a positive integer and n_i divides n_{i+1} for $1 \leq i \leq r-1$. Let $\{w_1, \ldots, w_q\}$ be an eigenbasis of W and let $\nu_j \in X^*(H)$ denote the weight of w_j. Since $X^*(H) \cong H$ we may write $\nu_j = (\nu_{j1}, \ldots, \nu_{jr})$ where $\nu_{ji} \in \mathbb{N}$ and $0 \leq \nu_{ji} \leq n_j - 1$. We define $T := (\mathbf{k}^*)^r$ and take V to be the $q + r$ dimensional representation of T having weights $(\nu_{11}, \ldots, \nu_{1r})$, $(\nu_{21}, \ldots, \nu_{2r})$,..., $(\nu_{q1}, \ldots, \nu_{qr})$ together with $\mu_1 := (-n_1, 0, \ldots, 0)$, $\mu_2 := (0, -n_2, 0, \ldots, 0)$, ..., $\mu_r := (0, \ldots, 0, -n_r)$. Now it is clear that given $(a_1, \ldots, a_q) \in \mathbb{N}^q$ that there exists $(b_1, \ldots, b_r) \in \mathbb{N}^r$ with $\sum_{j=1}^q a_j\nu_j + \sum_{i=1}^r b_i\mu_i = \mathbf{o}$ if and only if $\sum_{j=1}^q a_j\nu_{ji} \equiv 0 \pmod{n_i}$ for all $1 \leq i \leq r$. Thus $\mathbf{k}[V]^T \cong \mathbf{k}[W]^H$. □

Remark 6.2. If we begin with a representation $\rho : T \to GL(V)$ of a torus having a MSOP then using the method of section 5 we get a representation

$\tau : H \to GL(W)$ of a finite abelian group H. Then using Proposition 6.1 we get a representation of another torus, $\rho_1 : T_1 \to GL(V_1)$. Usually ρ and ρ_1 will be inequivalent representations. Also notice that there may be more more than one such τ corresponding to different choices for the partition $\{1, 2, \ldots, n\} = \Gamma_1 \cup \Gamma_2$.

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