INTERNAL DUALITY FOR RESOLUTIONS OF RINGS

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Section 0. Introduction

The following example provided the motivation for this paper. Let $\mathbf{F}_8$ be the field with 8 elements, and let $\mathbf{F}_8^*$ be its group of units which acts on $\mathbf{F}_8$ via left multiplication. The cohomology ring $H^*(\mathbf{F}_8 \rtimes \mathbf{F}_8^*; \mathbf{F}_2) \cong \mathbf{F}_2[x_1, x_2, x_3]^{\mathbf{Z}/7}$ is of much interest to topologists ([A], [CS], [M]). It is a straight forward exercise to construct minimal sets of homogeneous generators and relations for this ring: there are 13 generators and 54 relations. The surprising fact is that the minimal relations exhibit a certain *internal duality*:

$$\begin{array}{rccccccc} d: & 8 & 9 & 10 & 11 & 12 & 13 & 14 \\ \beta_{1d}: & 3 & 6 & 12 & 12 & 12 & 6 & 3. \end{array}$$

Here β_{1d} denotes the number of minimal relations of degree d. In fact this internal duality holds for each of the terms of the minimal Hilbert syzygy resolution of this ring (see Section 7).

In this paper, we exhibit an infinite family of rings of invariants whose minimal resolutions exhibit internal duality. Specifically, fix a positive integer n, let $N = 1 + n + n^2$, and let $\mathbf{k}$ be a field containing a primitive N-th root of unity, ω. Let G be the cyclic group generated by $\mathrm{diag}(\omega, \omega^n, \omega^{n^2}) \in \mathrm{SL}_3(\mathbf{k})$, the special linear group of 3×3 matrices over $\mathbf{k}$, and let $B_n = \mathbf{k}[x_1, x_2, x_3]^G$ be the associated ring of invariants.

Theorem 0.1. *The minimal resolution of B_n exhibits internal duality.*

To prove this, we construct an explicit minimal resolution for B_n. Along the way, we construct minimal resolutions for the following two families of monomial rings: $B_1^s = A/I_1$ and $B_2^s = A/I_2$, where $A = \mathbf{k}[y_1, \ldots, y_s]$, I_1 is the ideal generated by $\{y_i y_j \mid i \neq j\}$, and I_2 is the ideal generated by $\{y_i y_j \mid i \not\equiv j-1, j, j+1 \bmod s\}$. These resolutions are of independent interest. For example, there are many other rings of invariants whose resolutions can be determined from our resolution of B_2^s.

The organization of the paper follows. In Section 1, we recall the definitions and general properties of Hilbert syzygy resolutions and define *internal duality*. In Section 2, we discuss resolutions for rings of invariants and give a couple of examples. In Sections 3 and 4, we construct minimal resolutions for B_1^s and B_2^s. At the ends of these sections, our resolutions are compared to the resolutions constructed by Eagon–Northcott ([EN] or [ERS]) and Behnke ([B]). In Section 5, we show how to construct the minimal resolution of B_n from the resolution of B_2^{3n+6}, and we prove that B_n's resolution exhibits internal duality. The proof of Proposition (5.4) is delayed until Section 6. Finally, in Section 7 we investigate the motivating example $H^*(\mathbf{F}_8 \rtimes \mathbf{F}_8^*; \mathbf{F}_2)$ in detail.

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In [CHPS], a minimal set of generators for the ring of invariants $H^*(\mathbf{F}_{2^n} \rtimes \mathbf{F}_{2^n}^*; \mathbf{F}_2)$ was characterized by analyzing the associated ring of Laurant polynomials. In [CHW], similar results are obtained for $\mathbf{k}[x_1, \ldots, x_n]^G$, when $G \subseteq \mathrm{GL}_n(\mathbf{k})$ is any non-modular abelian group.

The computer programs Macaulay, Maple, and Mathematica were used for various computations during this work; however, none of our proof rely on these calculations.

Section 1. Minimal resolutions, Gorenstein rings, and internal duality

A reference for much of this section and the next is [S]. Let $\mathbf{k}$ be any field, and let $B = B_0 \oplus B_1 \oplus \cdots$ be an $\mathbf{N}$-graded $\mathbf{k}$-algebra. Given a set $\gamma_1, \gamma_2, \ldots, \gamma_s$ of homogeneous generators for B, with $\deg \gamma_i > 0$, form new indeterminants $y_1, \ldots, y_s$ and let A denote the polynomial ring $A = \mathbf{k}[y_1, \ldots, y_s]$ with an $\mathbf{N}$-grading $A_0 \oplus A_1 \oplus \cdots$ given by $\deg y_i = \deg \gamma_i$. Define an A-module structure on B by the conditions $y_i f = \gamma_i f$ for all $f \in B$. As an A-module B is generated by the single element 1. Hence B is isomorphic to a quotient ring A/J of A, where J is a homogeneous ideal of A. The elements of J are called the *syzygies of the first kind.*

The Hilbert syzygy theorem implies that there is an exact sequence of A-modules,

$$0 \longrightarrow M_h \xrightarrow{d_h} M_{h-1} \longrightarrow \cdots \longrightarrow M_1 \xrightarrow{d_1} M_0 \xrightarrow{d_0} B \longrightarrow 0, \tag{1.1}$$

where $h \leq s$ and each M_i is a finitely-generated *free* A-module. This exact sequence, sometimes denoted (M_*, d_*), is called a *finite free resolution* of B (as an A-module). By an appropriate choice of the degrees of the free generators of each M_i, the M_i's become $\mathbf{N}$-graded A-modules, and the homomorphisms d_i preserve degree. We will always suppose that (1.1) has been chosen so that each d_i preserves degree.

The homomorphisms d_i may be regarded as specifying the syzygies of the ith kind. We may think of constructing (1.1) by finding, $M_0, M_1, \ldots, M_h$ in turn. Once we have found M_i and d_i, pick any set of homogeneous generators for $\ker d_i$ and let a basis for M_{i+1} map onto these generators. If at each stage we chose a minimal set of generators for $\ker d_i$, then (1.1) is called a *minimal free resolution* of B (as an A-module). A minimal free resolution (1.1) of B is unique, in the sense that if

$$0 \longrightarrow N_j \xrightarrow{c_h} N_{j-1} \longrightarrow \cdots \longrightarrow N_1 \xrightarrow{c_1} N_0 \xrightarrow{c_0} B \longrightarrow 0$$

is another one, with $M_h \neq 0$ and $N_j \neq 0$, then $h = j$ and there are degree-preserving A-module isomorphisms $M_i \longrightarrow N_i$ making the following diagram commute:

$$\begin{array}{ccccccccccccc}
0 \longrightarrow & M_h & \xrightarrow{d_h} & M_{h-1} & \longrightarrow \cdots \longrightarrow & M_1 & \xrightarrow{d_1} & M_0 & \xrightarrow{d_0} & B & \longrightarrow 0 \\
& \downarrow\cong & & \downarrow\cong & & \downarrow\cong & & \downarrow\cong & & \downarrow\cong & \\
0 \longrightarrow & N_h & \xrightarrow{c_h} & N_{h-1} & \longrightarrow \cdots \longrightarrow & N_1 & \xrightarrow{c_1} & N_0 & \xrightarrow{c_0} & B & \longrightarrow 0.
\end{array}$$

In particular, the minimum number of generators of M_i and the degrees of these generators are uniquely determined in a minimal free resolution. The minimum number of generators (or *rank*) of M_i is called the ith *Betti number* of B (as an A-module) and is denoted $\beta_i^A(B)$. The number of generators of M_i having degree j is called the (i,j)th *internal Betti number* of B and is denoted $\beta_{i,j}^A(B)$. If we have chosen $\gamma_1, \gamma_2, \ldots, \gamma_s$ to be a minimal set of generators for B (as a $\mathbf{k}$-algebra), then it turns out that the $\beta_i^A(B)$ and the $\beta_{i,j}^A(B)$ depend only on B, not on the choice of the γ_i's. In this case we write $\beta_i(B)$ for $\beta_i^A(B)$ and $\beta_{i,j}(B)$ for $\beta_{i,j}^A(B)$. Note that since B is generated as an A-module by the single element 1, we always have $\beta_0(B) = \beta_{0,0}(B) = 1$.

The least integer h for which (1.1) exists (equivalently, the greatest integer h for which $\beta_h^A(B) \neq 0$) is called the *homological dimension* of B (as an A-module), denoted $\mathrm{hd}_A(B)$. As before we write $\mathrm{hd}(B)$ when $\{\gamma_1, \ldots, \gamma_s\}$ is minimal.

Once we have chosen bases for the M_i's in (1.1), we may regard elements of M_i as column vectors and represent the map d_i $(i \geq 1)$ as a $t \times r$ matrix, where $r = \mathrm{rank}(M_i)$ and $t = \mathrm{rank}(M_{i-1})$. The entries of the matrix d_i will be homogeneous elements of A. It is easy to see that the resolution (1.1) is minimal if and only if all of the entries of each d_i have positive degree (allowing the element 0 as an entry). Equivalently, no entry of any d_i can be a nonzero element of $\mathbf{k}$. We will use this fact a number of times in this paper, so we state it formally:

Minimality Criterion 1.2. *The resolution (1.1) is minimal if and only if the non-zero entries of each of the d_i have positive degree.*

Remark 1.3. When the resolution (1.1) is minimal, it follows that the rank of M_i, that is, the i-th Betti number β_i, equals the dimension of the torsion group $\mathrm{Tor}_i^A(B, \mathbf{k})$. Since the resolution is graded, this torsion group has an internal grading and the (i,j)-th internal Betti number, $\beta_{i,j}$, equals the dimension of the degree j subspace of $\mathrm{Tor}_i^A(B, \mathbf{k})$.

The *Krull dimension* of B, denoted $\dim(B)$, is the maximum number of elements of B which are algebraically independent over $\mathbf{k}$. A set $\{\theta_1, \ldots, \theta_m\}$ of $m = \dim(B)$ homogeneous elements of positive degree is said to be a *homogeneous system of parameters*, if B is a finitely generated module over the subalgebra $\mathbf{k}[\theta_1, \ldots, \theta_m]$. For this to happen, the $\theta_1, \ldots, \theta_m$ must be algebraically independent. The Noether Normalization Lemma implies that a homogeneous system of parameters for B always exists. The algebra B is called *Cohen-Macaulay* if B is a free module (necessarily finitely generated) over $\mathbf{k}[\theta_1, \ldots, \theta_m]$.

A Cohen-Macaulay $\mathbf{N}$-graded $\mathbf{k}$-algebra B is called *Gorenstein* if its highest non-zero Betti number, $\beta_h(B)$ equals 1. In this case, one can show that the minimal free resolution (1.1) for B is self dual; that is, with the correct choice of bases for the modules M_i and $M_j^* = \mathrm{Hom}(M_j, \mathbf{k})$, the matrices d_i and d_{h+1-i}^* are identical. In particular, $\mathrm{rank}(M_i) = \mathrm{rank}(M_{h-i})$, so we obtain the result that if B is Gorenstein, then

$$\beta_i(B) = \beta_{h-i}(B), \quad 0 \leq i \leq h. \tag{1.4}$$

Definition 1.5. *We will say that the Gorenstein algebra B exhibits internal duality if, for each i, there exists an integer h_i, such that $\beta_{i,j} = \beta_{i,h_i-j}$ for all j.*

The following is one of the simplest and most useful complexes of free modules.

Example 1.6. *Let J be a homogeneous ideal in the graded ring $A = \mathbf{k}[y_1, \ldots, y_s]$ and let $d_0\colon A \longrightarrow B = A/J$ be the canonical surjection. Let $\{z_1, \ldots, z_t\}$ be a minimal set of (homogeneous) positive degree generators for J. The Koszul complex of $\{z_1, \ldots, z_t\}$ has the form:*

$$0 \longrightarrow M_t \xrightarrow{d_t} M_{t-1} \longrightarrow \cdots \longrightarrow M_1 \xrightarrow{d_1} M_0 \xrightarrow{d_0} B \longrightarrow 0, \tag{1.7}$$

where M_i is a free A-module of rank $\binom{t}{i}$ with basis $\{\, (a_1, \ldots, a_i) \mid 1 \le a_1 < \cdots < a_i \le t \,\}$. The maps $d_i\colon M_i \longrightarrow M_{i-1}$ are given by

$$d_i\big((a_1, \ldots, a_i)\big) \;=\; \sum_{j=1}^{i} (-1)^{j+1} z_{a_j} (a_1, \ldots, \widehat{a_j}, \ldots, a_i),$$

where $\widehat{a_j}$ denotes that a_j is missing.

It is straightforward to check that $d_i d_{i+1} = 0$, so (M_*, d_*) is a complex. If (1.7) is exact, then Criterion (1.2) implies that (M_*, d_*) is a minimal resolution of B (as an A-module). This is the case, for example, when $B = \mathbf{k}$ and $\{z_1, \ldots, z_t\}$ is $\{y_1, \ldots, y_s\}$ or when $B = \mathbf{k}[y_l]$ and $\{z_1, \ldots, z_t\}$ is $\{y_1, \ldots, \widehat{y_l}, \ldots, y_s\}$. These two resolutions will be used later in the paper.

It is sometimes convenient to identify the Koszul complex with $\Lambda(z_1, \ldots, z_t)$, the exterior algebra over A. The basis element $(a_1, \ldots, a_i)$ in M_i corresponds to the wedge product $z_{a_1} \wedge \cdots \wedge z_{a_i}$ in Λ^i. Using this identification, we see that, if $1 \le a_1 < \cdots < a_i \le s$, then we can let the symbol $(a_{\sigma(1)}, \ldots, a_{\sigma(i)})$ stand for the element $(-1)^{\sigma}(a_1, \ldots, a_i)$ in M_i.

Proposition 1.8. *Let B be a graded $\mathbf{k}$-algebra minimally generated by s homogeneous elements of positive degree. Let $A = \mathbf{k}[y_1, \ldots, y_s]$ and let $d_0\colon A \to B$ be the usual surjection. Let $\{z_1, \ldots, z_t\}$ be a minimal set of (homogeneous) generators for the kernel of d_0 and suppose that the Koszul complex of $\{z_1, \ldots, z_t\}$ is a minimal resolution of B. Then (i) B is Gorenstein and (ii) B exhibits internal duality if and only if there exists an integer h_1 with $\beta_{1,j} = \beta_{1,h_1-j}$ for all j.*

Proof. The fact that B is Gorenstein follows directly from the definition.

Assuming $\beta_{1,j} = \beta_{1,h_1-j}$, we can define a pairing on the basis elements $\{(1), \ldots, (t)\}$ of M_1, $(a) \mapsto (a^*)$, having the properties (i) $\deg(a) + \deg(a^*) = h_1$ and (ii) $(a^{**}) = (a)$. Then on M_i we have the pairing $(a_1, \ldots, a_i) \leftrightarrow (a_1^*, \ldots, a_i^*)$. Since the degree of the generator $(a_1, \ldots, a_i)$ of M_i is $\sum_{j=1}^{i} \deg(z_{a_j})$ it follows that $\beta_{i,j} = \beta_{i,h_i-j}$ for all i and j, where $h_i = ih_1$. □

This result suggests the possibility that B exhibits internal duality whenever h_1 can be found. However, this is false as shown by Example (2.3) below.

Section 2. Rings of invariants

Let $R = \mathbf{k}[x_1, \ldots, x_m]$, and let V denote the vector space of linear forms in R. Then $\mathrm{GL}(V) = \mathrm{GL}_m(\mathbf{k})$ acts on V and the action of $M \in \mathrm{GL}(V)$ extends uniquely to an algebra automorphism of R. The set of all polynomials $f \in R$ satisfying $Mf = f$ for all M in some subgroup G of $\mathrm{GL}(V)$ forms a subalgebra R^G of R called the *algebra of invariants* of G. When the group G is finite and its order is relatively prime to the characteristic of $\mathbf{k}$, the algebra R^G has many nice properties.

Proposition 2.1. (see [S]) *Let $R = \mathbf{k}[x_1, \ldots, x_m]$, let G be a finite subgroup of $\mathrm{GL}_m(\mathbf{k})$, and let $\mathrm{char}(\mathbf{k})$ be relatively prime to $g = |G|$. Then*
(i) R^G is Cohen-Macaulay and has Krull dimension m,
(ii) if $G \subseteq \mathrm{SL}(V)$, then R^G is Gorenstein, and
(iii) if R^G is minimally generated by s homogeneous elements, then $\mathrm{hd}(R^G) = s - m$.

The following ring of invariants illustrates the above proposition and also exhibits internal duality.

Example 2.2. *Let $\mathbf{k}$ be a field containing a primitive sixth root of unity ω, let $G \subseteq \mathrm{SL}_3(\mathbf{k})$ be the cyclic group generated by diag$(\omega, \omega^2, \omega^3)$, and let $B = \mathbf{k}[x_1, x_2, x_3]^G$. Then B has a vector space basis consisting of the monomials*

$$\{x_1^{\epsilon_1} x_2^{\epsilon_2} x_3^{\epsilon_3} \mid \epsilon_1 + 2\epsilon_2 + 3\epsilon_3 \equiv 0 \bmod 6\}.$$

A minimal algebra generating set for B is given by $\gamma_1 = x_3^2$, $\gamma_2 = x_2^3$, $\gamma_3 = x_1x_2x_3$, $\gamma_4 = x_1^2x_2^2$, $\gamma_5 = x_1^3x_3$, $\gamma_6 = x_1^4x_2$, and $\gamma_7 = x_1^6$. Let $A = \mathbf{k}[y_1, \ldots, y_7]$ with $\deg(y_j) = \deg(\gamma_j)$. Then the minimal A resolution of B has the form:

$$0 \longrightarrow A \xrightarrow{d_4} A^9 \xrightarrow{d_3} A^{16} \xrightarrow{d_2} A^9 \xrightarrow{d_1} A \xrightarrow{d_0} B \longrightarrow 0,$$

with internal Betti numbers as follows.

$d:$	0	...	6	7	8	9	10	11	12	13	14	15	16	17	18	...	24
$\beta_{0d}:$	1																
$\beta_{1d}:$			1	2	3	2	1										
$\beta_{2d}:$							2	4	4	4	2						
$\beta_{3d}:$											1	2	3	2	1		
$\beta_{4d}:$																	1

The maps are given in the Appendix. Note that the maps d_1 and d_4 have polynomial degree 2 while d_2 and d_3 are linear. The resolutions of B_2^s and B_n which will be constructed in later sections also have this property – the first and last maps are quadratic while the other maps are linear. Resolutions where each map has a fixed polynomial degree are called *pure* (see [W], Remarks 1.8).

The following is an example of a ring of invariants which is Gorenstein, has M_1 internally dual, but doesn't have M_2 internally dual. This example was constructed by tensoring two rings of invariants together in such a way that their relations added up to a dual M_1. We thank Don Stanley for suggesting this approach. Explicitly, take $B_1 = \mathbf{k}[x_1, x_2, x_3]^{G_1}$ where G_1 is the cyclic group generated by diag$(\omega_1, \omega_1^4, \omega_1^5)$ with ω_1 a primitive 10-th root of unity, and take $B_2 = \mathbf{k}[x_4, x_5]^{G_2}$ where G_2 is the group generated by diag(ω_2, ω_2^8) with ω_2 a primitive 9-th root of unity. Then $B = B_1 \otimes B_2$ below.

Example 2.3. *Let $\mathbf{k}$ be a field containing a primitive* 90*-th root of unity ω, let $G \subseteq \mathrm{SL}_5(\mathbf{k})$ be the group generated by* diag$(\omega^9, \omega^{36}, \omega^{45}, \omega^{10}, \omega^{80})$*, and let $B = \mathbf{k}[x_1, x_2, x_3, x_4, x_5]^G$. Then B is minimally generated by* 10 *elements having degrees* $2, 2, 3, 4, 5, 6, 7, 9, 9, 10$ *and* *the minimal resolution of B as a module over $A = \mathbf{k}[y_1, \ldots, y_{10}]$ has length 5, Betti numbers $1, 10, 25, 25, 10, 1$ and the following set of internal Betti numbers.*

$d:$	0
$\beta_{0d}:$	1

$d:$	6	7	8	9	10	11	12	13	14	15	16	17	18
$\beta_{1d}:$	1			1	1	1	2	1	1	1			1

$d:$	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30	31	32	33
$\beta_{2d}:$	1	1	1	2	2	2	2	2	1	1	1		1			1	1	1	2	1	1	1

$d:$	19	20	21	22	23	24	25	26	27	28	29	30	31	32	33	34	35	36	37	38	39	40
$\beta_{3d}:$	1	1	1	2	1	1	1			1		1	1	1	2	2	2	2	2	1	1	1

$d:$	34	35	36	37	38	39	40	41	42	43	44	45	46
$\beta_{4d}:$	1			1	1	1	2	1	1	1			1

$d:$	52
$\beta_{5d}:$	1

Section 3. The minimal resolution of B_1^s

Let $A = \mathbf{k}[y_1, \ldots, y_s]$, and let I_1 be the ideal generated by $\{y_i y_j \mid i \neq j\}$. In this section, we determine a minimal resolution of $B_1^s = A/I_1$. This resolution will not be used elsewhere in the paper. The main reason for including it is to introduce certain constructions that will be used to construct the resolution of B_2^s in the next section. For the remainder of this section, we fix $s \geq 2$ and write $B = B_1^s$. Also, we take A and B graded via $\deg(y_i) = 1$.

To begin the resolution, let $N_0 = A$ and let $c_0 \colon A \longrightarrow B = A/I$ be the quotient map. Let N_1 be the free module with basis $\{(a_1, a_2) \mid 1 \leq a_1 < a_2 \leq s\}$ and let $c_1\big((a_1, a_2)\big) = y_{a_1} y_{a_2}$. Then $N_1 \xrightarrow{c_1} N_0 \xrightarrow{c_0} B \longrightarrow 0$ is exact. The kernel of c_1 is minimally spanned by the elements $\kappa_1 = y_{a_3}(a_1, a_2) - y_{a_2}(a_1, a_3)$ and $\kappa_2 = y_{a_3}(a_1, a_2) - y_{a_1}(a_2, a_3)$ where $a_1 < a_2 < a_3$. For N_2, we take ordered triples (a_1, a_2, a_3) as basis elements. However, we need two elements associated to the entries a_1, a_2, a_3 to hit the two kernel elements κ_1 and κ_2 listed above. We choose to decorate the triples with bars to allow for this. That is, N_2 is spanned by the symbols $(\overline{a_1}, a_2, a_3)$ and $(a_1, \overline{a_2}, a_3)$ mapping via c_2 to κ_1 and κ_2, respectively. (A bar over a_i indicates that y_{a_i} does not appear as a coefficient in the differential.) Continuing in this way, we can determine the complete resolution.

Example 3.1. *The minimal resolution of B has the form*

$$0 \longrightarrow N_{s-1} \xrightarrow{c_{s-1}} N_{s-2} \xrightarrow{c_{s-2}} \cdots \longrightarrow N_2 \xrightarrow{c_2} N_1 \xrightarrow{c_1} N_0 \xrightarrow{c_0} B \longrightarrow 0,$$

where rank $N_i = i\binom{s}{i+1}$, *the map* c_1 *is quadratic, and the maps* c_i, $i = 2, \ldots, s-1$, *are linear. For* $i \geq 1$, *the basis elements of* N_i *have degree* $i+1$.

Explicitly, for $1 \leq i \leq s-1$, N_i is the free module of rank $i\binom{s}{i+1}$ with basis

$$\{\, (a_1, \ldots, \overline{a_j}, \ldots, a_{i+1}) \mid 1 \leq a_1 < \cdots < a_{i+1} \leq s \text{ and } 1 \leq j \leq i \,\},$$

and with maps

$$\begin{aligned} c_i\big((a_1, \ldots, \overline{a_j}, \ldots, a_{i+1})\big) = &\sum_{k=1}^{j-1} (-1)^{k+1} y_{a_k} (a_1, \ldots, \widehat{a_k}, \ldots, \overline{a_j}, \ldots, a_{i+1}) \\ &+ \sum_{k=j+1}^{i+1} (-1)^{k+1} y_{a_k} (a_1, \ldots, \overline{a_j}, \ldots, \widehat{a_k}, \ldots, a_{i+1}), \end{aligned}$$

where the $\widehat{a_k}$ means omit a_k.

It is straightforward to check that $c_i c_{i+1} = 0$. Since none of the maps c_i involve constants from the field $\mathbf{k}$, Criterion (1.2) implies that if (N_*, c_*) is exact, then it is a minimal resolution of B.

One way to show it is exact is to construct vector space homomorphisms $t_i \colon N_i \longrightarrow N_{i+1}$ with $c_{i+1}t_i + t_{i-1}c_i = 1$. Such a map t_* is called a contracting homotopy. The details are left to the reader.

We choose to give an indirect proof that (N_*, c_*) is exact involving certain constructions that will be useful when we construct the resolution of B_2^s below.

To this end, let L_i, for $i = 0, \ldots, s-1$, be the free A-module of rank $(i+1)\binom{s}{i+1}$ with basis $\{\, (a_1, \ldots, \overline{a_j}, \ldots, a_{i+1}) \mid 1 \leq a_1 < \cdots < a_{i+1} \leq s \,\}$. This time the bar can occur over *any* entry. For $i = 1, \ldots, s-1$, define $e_i \colon L_i \longrightarrow L_{i-1}$ by the formula

$$\begin{aligned} e_i\big((a_1, \ldots, \overline{a_j}, \ldots, a_{i+1})\big) = &\sum_{k=1}^{j-1} (-1)^{k+1} y_{a_k} (a_1, \ldots, \widehat{a_k}, \ldots, \overline{a_j}, \ldots, a_{i+1}) \\ &+ \sum_{k=j+1}^{i+1} (-1)^{k+1} y_{a_k} (a_1, \ldots, \overline{a_j}, \ldots, \widehat{a_k}, \ldots, a_{i+1}), \end{aligned}$$

Clearly, (N_*, c_*) is a subcomplex of (L_*, e_*). It is also isomorphic to the quotient complex (M_*, d_*) of (L_*, e_*) given below. We will prove that this quotient complex is exact.

The complex (L_*, e_*) is easily seen to be a direct sum of s Koszul complexes as follows. Let L_i^l be the A-span of the basis elements of the form $(a_1, \ldots, a_k, \bar{l}, a_{k+2}, \ldots, a_{i+1})$. Since e_i preserves the value of the barred entry, (L_*^l, e_*) is a subcomplex of (L_*, e_*). It is easy to check that the restriction of e_* to L_*^l corresponds to the Koszul differential in the resolution of $\mathbf{k}[y_l]$ over A where the element $(-1)^k(a_1, \ldots, a_k, \bar{l}, a_{k+2}, \ldots, a_{i+1})$ in L_i^l corresponds to the basis element $(a_1, \ldots, a_k, a_{k+2}, \ldots, a_{i+1})$ in the Koszul resolution. (The $(-1)^k$ is needed here to ensure that the signs in the differentials alternate as we move along the entries.) Since these Koszul complexes are exact, the complex $(L_*, e_*) = \bigoplus_l (L^l_*, e_*)$ is exact.

For $i = 0, \ldots, s-1$, let K_{i+1} be the submodule of L_i spanned by the elements $(\overline{a_1, \ldots, a_{i+1}}) \equiv \sum_{j=1}^{i+1} (a_1, \ldots, \overline{a_j}, \ldots, a_{i+1})$. Also, let $K_0 = A$ and let $f_1 \colon K_1 \longrightarrow K_0$ be given by $f_1\big((\overline{a_1})\big) = y_{a_1}$. If we let f_{i+1} be the restriction of e_i to K_{i+1}, then it is easy to check that

$$f_{i+1}\big((\overline{a_1, \ldots, a_{i+1}})\big) = \sum_{j=1}^{i+1} (-1)^{j+1} y_{a_j}\, (\overline{a_1, \ldots}, \widehat{a_j}, \overline{\ldots, a_{i+1}})\,.$$

That is, that (K_*, f_*) is the Koszul resolution of $\mathbf{k}$ over A.

Finally, we define a resolution (M_*, d_*) of $B = A/I$. For $i = 1, \ldots, s-1$, let M_i be the quotient module L_i/K_{i+1}, let $q_i \colon L_i \longrightarrow M_i$ be the quotient map, and let $d_{i+1} \colon M_{i+1} \longrightarrow M_i$ be induced by e_{i+1}. Let $M_0 = A$ and let $d_1 \colon M_1 \longrightarrow M_0$ be given by $d_1\left(q_1\left((\overline{a_1}, a_2)\right)\right) = y_{a_1} y_{a_2}$. (Note that $d_1\left(q_1\left((a_1, \overline{a_2})\right)\right) = -y_{a_1} y_{a_2}$.)

The following commutative diagram illustrates what we have so far:

$$\begin{array}{cccccccccccc}
0\longrightarrow & K_s & \xrightarrow{f_s} & K_{s-1} & \xrightarrow{f_{s-1}} & \cdots\longrightarrow & K_3 & \xrightarrow{f_3} & K_2 & \xrightarrow{f_2} & K_1 & \\
& \big\downarrow{\scriptstyle i_s} & & \big\downarrow{\scriptstyle i_{s-1}} & & & \big\downarrow{\scriptstyle i_3} & & \big\downarrow{\scriptstyle i_2} & & \| & \\
0\longrightarrow & L_{s-1} & \xrightarrow{e_{s-1}} & L_{s-2} & \xrightarrow{e_{s-2}} & \cdots\longrightarrow & L_2 & \xrightarrow{e_2} & L_1 & \xrightarrow{e_1} & L_0 = & K_1 \\
& \big\downarrow{\scriptstyle q_{s-1}} & & \big\downarrow{\scriptstyle q_{s-2}} & & & \big\downarrow{\scriptstyle q_2} & & \big\downarrow{\scriptstyle q_1} & & & \big\downarrow{\scriptstyle f_1} \\
0\longrightarrow & M_{s-1} & \xrightarrow{d_{s-1}} & M_{s-2} & \xrightarrow{d_{s-2}} & \cdots\longrightarrow & M_2 & \xrightarrow{d_2} & M_1 & \xrightarrow{d_1} & M_0 = K_0 & = A
\end{array}$$

with (L_*, e_*) exact, (K_*, f_*) exact, and (L_*, e_*) satisfying Criterion (1.2). By construction, the sequence $M_1 \xrightarrow{d_1} M_0 \xrightarrow{d_0} B$ is the beginning of a minimal free resolution. The following lemma implies that (M_*, d_*) is a minimal resolution of B.

Lemma 3.2. *Assume that the following diagram commutes.*

$$\begin{array}{cccccccccccc}
0\longrightarrow & K_{h+1} & \xrightarrow{f_{h+1}} & K_h & \xrightarrow{f_h} & \cdots\longrightarrow & K_3 & \xrightarrow{f_3} & K_2 & \xrightarrow{f_2} & K_1 & \\
& \big\downarrow{\scriptstyle i_{h+1}} & & \big\downarrow{\scriptstyle i_{h+1}} & & & \big\downarrow{\scriptstyle i_3} & & \big\downarrow{\scriptstyle i_2} & & \| & \\
0\longrightarrow & L_h & \xrightarrow{e_h} & L_{h-1} & \xrightarrow{e_{h-1}} & \cdots\longrightarrow & L_2 & \xrightarrow{e_2} & L_1 & \xrightarrow{e_1} & L_0 = & K_1 \\
& & & & & & & & \big\downarrow{\scriptstyle q_1} & & & \big\downarrow{\scriptstyle f_1} \\
& & & & & & & & M_1 & \xrightarrow{d_1} & M_0 = K_0 & = A
\end{array}$$

Also assume that (L_*, e_*) and (K_*, f_*) are free resolutions with (L_*, e_*) satisfying Criterion (1.2), that $K_{i+1} \longrightarrow L_i$ is the inclusion of a direct summand, that M_1 is free, and that $K_2 \longrightarrow L_1 \longrightarrow M_1$ is short exact. For $2 \le i \le h$, let $M_i = L_i/K_{i+1}$ and let $d_i \colon M_i \longrightarrow M_{i-1}$ be induced by e_i. Then (M_*, d_*) is a minimal free resolution.

Proof. M_i is free since K_{i+1} is a direct summand of L_i. The sequence (M_*, d_*) is exact by a diagram chase and minimal by Criterion (1.2). □

It now follows that (N_*, c_*) is the minimal resolution of B since it maps isomorphically to (M_*, d_*) under the map q_*.

We remark that a minimal resolution for B_1^s can also be determined as follows. Consider the regular element $a = y_1 + y_2 + \cdots + y_s$ in B_1^s. It is easy to see that $B_1^s/(a)$ is isomorphic to $\mathbf{k}[y_1, \ldots, y_{s-1}]/(y_1, \ldots, y_{s-1})^2$. A minimal free resolution of this ring as a $\mathbf{k}[y_1, \ldots, y_{s-1}]$-module was discussed by Wahl ([W], p.240) and may be described as the Eagon–Northcott complex associated to the 2×2 minors of the matrix $\begin{pmatrix} y_1 & y_2 & \cdots & y_{s-1} & 0 \\ 0 & y_1 & \cdots & y_{s-2} & y_{s-1} \end{pmatrix}$ (see [EN] or [ERS]).

Section 4. The minimal resolution of B_2^s

Let $A = \mathbf{k}[y_1, \ldots, y_s]$, and let I_2 be the ideal generated by $\{y_i y_j \mid i \not\equiv j-1, j, j+1 \bmod s\}$. In this section, we determine the minimal resolution of $B_2^s = A/I_2$. The construction will be similar to the quotient construction given in Section 3. For the remainder of this section, we fix $s \geq 4$ and write $B = B_2^s$. We will say the indices i and j are *adjacent* if i is congruent to either $j-1$ or $j+1$ modulo s. We take A and B graded via $\deg(y_i) = 1$.

Before giving the quotient construction, we describe the beginning of the resolution directly. Let $N_0 = A$ and let $c_0: A \longrightarrow B = A/I$ be the quotient map. Let N_1 be the free module with basis $\{(a_1, a_2) \mid 1 \leq a_1 < a_2 \leq s,\ a_1 \text{ and } a_2 \text{ not adjacent}\}$, and let $c_1\big((a_1, a_2)\big) = y_{a_1} y_{a_2}$. Then $N_1 \xrightarrow{c_1} N_0 \xrightarrow{c_0} B \longrightarrow 0$ is exact.

The kernel of c_1 is spanned (not minimally) by elements of the form $\kappa_1 = y_{a_3}(a_1, a_2) - y_{a_2}(a_1, a_3)$, $\kappa_2 = y_{a_3}(a_1, a_2) - y_{a_1}(a_2, a_3)$, and $\kappa_3 = y_{a_2}(a_1, a_3) - y_{a_1}(a_2, a_3)$, where $a_1 < a_2 < a_3$. However, if some of these a's are adjacent, then some of the κ's don't make sense. There are three cases: if none of the a's are adjacent, then any two of the κ's will be linearly independent; if exactly two of the a's are adjacent, then only one of the κ's makes sense; and if the three a's are adjacent, then none of the κ's makes sense. So a basis for N_2 will consist of ordered triples (a_1, a_2, a_3), say with bars as in the previous section, where the number of basis elements associated to (a_1, a_2, a_3) depends on the adjacencies between a_1, a_2, and a_3.

This direct description becomes quite complicated as we try to construct the N_i for $i > 2$. For this reason, we use the quotient construction below. Essentially, the idea is to allow more basis elements than we need (the J_i's), and then quotient out the extra ones (the K_i's). The construction will imply the following.

Example 4.1. *The minimal resolution of B has the form*

$$0 \longrightarrow M_{s-2} \xrightarrow{d_{s-2}} M_{s-3} \xrightarrow{d_{s-3}} \cdots \longrightarrow M_2 \xrightarrow{d_2} M_1 \xrightarrow{d_1} M_0 \xrightarrow{d_0} B \longrightarrow 0,$$

where $\operatorname{rank}(M_0) = \operatorname{rank}(M_{s-2}) = 1$, *and, for* $1 \leq i \leq s-3$, $\operatorname{rank}(M_i) = s\binom{s-2}{i} - \binom{s}{i+1} = \frac{i(s-i-2)}{s-1}\binom{s}{i+1}$. *The basis element of M_0 is in degree 0; the basis elements of M_i, $1 \leq i \leq s-3$, have degree $i+1$; and the basis element of M_{s-2} has degree s. The maps d_1 and d_{s-2} are quadratic, and the maps d_i, $i = 2, \ldots, s-3$, are linear.*

To construct the resolution, we apply Lemma (3.2) to the following diagram, where $d_0: M_0 = A \longrightarrow B$ is the canonical projection and $M_1 \xrightarrow{d_1} M_0 \xrightarrow{d_0} B$ is the beginning of a minimal free resolution.

$$\begin{array}{ccccccccccc}
0 \longrightarrow & K_s & \xrightarrow{f_s} & K_{s-1} & \xrightarrow{f_{s-1}} \cdots \longrightarrow & K_3 & \xrightarrow{f_3} & K_2 & \xrightarrow{f_2} & K_1 & \\
& \downarrow{\scriptstyle i_s} & & \downarrow{\scriptstyle i_{s-1}} & & \downarrow{\scriptstyle i_3} & & \downarrow{\scriptstyle i_2} & & \| & \\
0 \longrightarrow & J_{s-1} & \xrightarrow{e_{s-1}} & J_{s-2} & \xrightarrow{e_{s-2}} \cdots \longrightarrow & J_2 & \xrightarrow{e_2} & J_1 & \xrightarrow{e_1} & J_0 = & K_1 \\
& & & & & & & \downarrow{\scriptstyle q_1} & & & \downarrow{\scriptstyle f_1} \\
& & & & & & & M_1 & \xrightarrow{d_1} & M_0 = K_0 & = A
\end{array}$$

We first define the modules J_i. Most of the basis elements of J_i are of the form $(a_1, \ldots, a_{i+1})$, $1 \leq a_1 < \cdots < a_{i+1} \leq s$, with bars over some of the entries. The bars must satisfy the conditions: (i) if a_k and a_l are adjacent, then either they both have bars or neither of them has, and (ii) there is exactly one adjacent group of indices having bars. In addition, J_{s-2} has one extra basis element which we call X. The elements of J_i have degree $i+1$, except for X which has degree s. It is fairly easy to show that

$$\operatorname{rank}(J_i) = \begin{cases} s\binom{s-2}{i}, & \text{for } i = 0, \ldots, s-3; \\ s+1, & \text{for } i = s-2; \text{ and} \\ 1, & \text{for } i = s-1. \end{cases}$$

To be more explicit, the basis elements of J_i (other than X) have one of the following forms:

(i) $(a_1, \ldots, a_{i-j}, \overline{s-j, \ldots, s})$, with $1 < a_1 < \cdots < a_{i-j} < s-j-1$,

(ii) $(a_1, \ldots, a_m, \overline{n, n+1, \ldots, n+j}, a_{m+j+2}, \ldots, a_{i+1})$, with $1 \leq a_1 < \cdots < a_m < n-1$ and $n+j+1 < a_{m+j+2} < \cdots < a_{i+1} \leq s$,

(iii) $(\overline{1, \ldots, j+1}, a_{j+2}, \ldots, a_{i+1})$, with $j+2 < a_{j+2} < \cdots < a_{i+1} \leq s$, or

(iv) $(\overline{1, \ldots, n}, a_{n+1}, \ldots, a_{i-j+n}, \overline{s-j+n, \ldots, s})$, with $n+1 < a_{n+1} < \cdots < a_{i-j+n} < s-j+n-1$.

In each of these, $0 \leq j \leq i$ and there are $j+1$ barred adjacent indices. In (ii), $0 \leq m \leq i-j$, $n > 1$, and $n+j < s$; and in (iv), $1 \leq n \leq j$.

Given a basis element $(a_1, \ldots, a_{i+1})$ with bars on a set of adjacent indices and given an element σ in the symmetric group on $i+1$ letters, we let $(a_{\sigma(1)}, \ldots, a_{\sigma(i+1)})$ with bars over the same indices, represent the element $(-1)^\sigma(a_1, \ldots, a_{i+1})$. For example, $(3, \overline{1}, 5, \overline{s})$ represents $(-1)(\overline{1}, 3, 5, \overline{s})$.

Given a basis element $(a_1, \ldots, a_{i+1})$ with bars on a set of adjacent indices, we let $\tau\big((a_1, \ldots, a_{i+1})\big) = (a_1+1, \ldots, a_{i+1}+1)$ with bars over the corresponding indices (where $s+1$ is replaced by 1 when necessary). For example, $\tau(\overline{1}, 3, 5, \overline{s}) = (\overline{2}, 4, 6, \overline{1})$, which equals $(-1)(\overline{1, 2}, 4, 6)$ by the above. We can extend τ to a $\mathbf{k}$-linear mapping on J_i via the formulas $\tau(y_j) = y_{j+1}$ (with $y_{s+1} \equiv y_1$), and $\tau(pZ) = \tau(p)\tau(Z)$, where $p \in A$ and Z is a basis element of J_i. (When $i = s-2$, we let $\tau(X) = X$.)

With these definitions, it is easy to see that any basis element of J_i (other than X) can be written in the form

$$\pm\tau^k(a_1, \ldots, a_{i-j}, \overline{s-j, \ldots, s})$$

for some k with $0 \le k \le s-1$. This observation will be useful below when we describe the differential e_* and the contracting homotopy t_*.

It is sometimes convenient to regard the basis elements of J_i (other than X) as certain sums of the basis elements of L_i (see Section 3). For example,

$$(a_1, \ldots, a_{i-j}, \overline{s-j, \ldots, s}) = \sum_{r=s-j}^{s} (a_1, \ldots, a_{i-j}, s-j, \ldots, r-1, \overline{r}, r+1, \ldots, s).$$

With this interpretation, the value of e_* on the basis elements of J_* (other than X) is just the restriction of the e_* defined on L_*. To be explicit,

$$\begin{aligned} e_i\big((a_1, \ldots, a_{i-j}, \overline{s-j, \ldots, s})\big) &= \sum_{k=1}^{i-j} (-1)^{k+1} y_{a_k} (a_1, \ldots, \widehat{a_k}, \ldots, a_{i-1}, \overline{s-j, \ldots, s}) \\ &+ \sum_{k=s-j}^{s-1} (-1)^{i-s+k} y_k (a_1, \ldots, a_{i-j}, s-j, \ldots, \widehat{k}, \overline{k+1, \ldots, s}) \\ &+ \sum_{k=s-j+1}^{s} (-1)^{i-s+k} y_k (a_1, \ldots, a_{i-j}, \overline{s-j, \ldots, k-1}, \widehat{k}, \ldots, s), \end{aligned}$$

$$e_i\big(\tau^k (a_1, \ldots, a_{i-j}, \overline{s-j, \ldots, s})\big) = \tau^k e_i\big((a_1, \ldots, a_{i-j}, \overline{s-j, \ldots, s})\big),$$

and

$$\begin{aligned} e_{s-2}(X) = y_1 y_2 (\overline{3, \ldots, s}) &+ \sum_{k=3}^{s-1} (-1)^k y_1 y_k (2, \ldots, \widehat{k}, \overline{\ldots, s}) \\ &+ \sum_{l=2}^{s-2} \sum_{k=l+2}^{s} (-1)^{k+l} y_l y_k (1, \ldots, \widehat{l}, \overline{l+1, \ldots, k-1}, \widehat{k}, \ldots, s). \end{aligned}$$

The appearence of X is something of a mystery (hence its name). Note that $e_{s-2}(X)$ is a quadratic combination of basis elements, while all other basis elements map to linear combinations.

To complete the picture, let (K_*, f_*) be the same as the (K_*, f_*) defined in Section 3. That is, K_{i+1} is spanned by the elements

$$(\overline{a_1, \ldots, a_{i+1}}) \equiv \sum_{k=1}^{i+1} (a_1, \ldots, \overline{a_k}, \ldots, a_{i+1}).$$

To apply Lemma (3.2), we need to show that (J_*, e_*) is exact. We do this by finding a contracting homotopy; that is, a collection of vector space maps $t_i \colon J_i \longrightarrow J_{i+1}$ with $t_{i+1} e_i +$

$e_{i-1}t_i = 1$. If $E = (\epsilon_1, \ldots, \epsilon_s)$ is a sequence of nonnegative integers, let $y^E = y_1^{\epsilon_1} \cdots y_s^{\epsilon_s}$. Then the set $\{y^E Z\}$ where Z is a basis element in J_i forms a **k**-basis for J_i. We must define $t_i(y^E Z)$. The idea is similar to what one does to construct a contracting homotopy for the Koszul resolution of **k**. For example, when Z has the form $(a_1, \ldots, a_{i+1})$ with appropriate bars, the homotopy (usually) sends $y^E(a_1, \ldots, a_{i+1})$ to some linear combination of terms of the form $(y^E/y_j)(j, a_1, \ldots, a_{i+1})$ where y_j is a variable occuring in y^E.

Below we give explicit formulas for $t_i(y^E Z)$, for $-1 \leq i \leq s-2$. In many cases, t_i is given only on elements of the form $y^E(a_1, \ldots, a_{i-j}, \overline{s-j, \ldots, s})$. In these cases, the general formula is gotten using the action of τ via the formula

$$t_i\Big(y^E \cdot \tau^k\big((a_1, \ldots, a_{i-j}, \overline{s-j, \ldots, s})\big)\Big) = \tau^k\Big(t_i\big(\tau^{s-k}(y^E) \cdot (a_1, \ldots, a_{i-j}, \overline{s-j, \ldots, s})\big)\Big). \tag{4.2}$$

In all of the following formulas, the exponents ϵ_j on the variables y_j stand for nonnegative integers. Therefore an exponent of $\epsilon_j + 1$ on y_j means that y_j actually occurs.

Let $e_0 \colon J_0 \longrightarrow J_0/\mathrm{Im}(e_1)$ be the projection. To describe $t_{-1} \colon J_0/\mathrm{Im}(e_1) \longrightarrow J_0$, we write its composition with e_0.

For $2 \leq k \leq s-2$:

$$t_{-1} \circ e_0 \left(y_{s-1}^{\epsilon_{s-1}} y_s^{\epsilon_s}(\overline{s})\right) = y_{s-1}^{\epsilon_{s-1}} y_s^{\epsilon_s}(\overline{s}).$$
$$t_{-1} \circ e_0 \left(y_1^{\epsilon_1+1} y_s^{\epsilon_s}(\overline{s})\right) = y_1^{\epsilon_1} y_s^{\epsilon_s+1}(\overline{1}).$$
$$t_{-1} \circ e_0 \left(y_1^{\epsilon_1+1} y_{s-1}^{\epsilon_{s-1}+1} y_s^{\epsilon_s}(\overline{s})\right) = 0.$$
$$t_{-1} \circ e_0 \left(y_1^{\epsilon_1} y_k^{\epsilon_k+1} \cdots y_s^{\epsilon_s}(\overline{s})\right) = 0.$$

The general definition of t_{-1} follows from Equation (4.2).

For $0 \leq i \leq s-4$ there are a number of formulas.

$$t_i\left(y_{s-i}^{\epsilon_{s-i}} \cdots y_s^{\epsilon_s}(\overline{s-i, \ldots, s})\right) = 0.$$
$$t_i\left(y_{s-i-1}^{\epsilon_{s-i-1}+1} \cdots y_s^{\epsilon_s}(\overline{s-i, \ldots, s})\right) = 0.$$
$$t_i\left(y_1^{\epsilon_1+1} y_{s-i}^{\epsilon_{s-i}} \cdots y_s^{\epsilon_s}(\overline{s-i, \ldots, s})\right) = y_1^{\epsilon_1} y_{s-i}^{\epsilon_{s-i}} \cdots y_s^{\epsilon_s}(\overline{1}, \overline{s-i, \ldots, s}).$$
$$t_i\left(y_1^{\epsilon_1+1} y_{s-i-1}^{\epsilon_{s-i-1}+1} \cdots y_s^{\epsilon_s}(\overline{s-i, \ldots, s})\right) = y_1^{\epsilon_1} y_{s-i-1}^{\epsilon_{s-i-1}+1} \cdots y_s^{\epsilon_s}(\overline{1}, \overline{s-i, \ldots, s})$$
$$+ \sum_{r=s-i}^{s} (-1)^{s-i+r+1} y_1^{\epsilon_1} y_{s-i-1}^{\epsilon_{s-i-1}} \cdots y_r^{\epsilon_r+1} \cdots y_s^{\epsilon_s}(\overline{1}, s-i-1, \ldots, r-1, \widehat{r}, \overline{r+1, \ldots, s}).$$

Continuing with $0 \leq i \leq s-4$ and taking $2 \leq k \leq s-i-2$,

$$t_i\left(y_k^{\epsilon_k+1} \cdots y_s^{\epsilon_s}(\overline{s-i, \ldots, s})\right) = y_k^{\epsilon_k} \ldots y_s^{\epsilon_s}(k, \overline{s-i, \ldots, s}).$$
$$t_i\left(y_1^{\epsilon_1+1} y_k^{\epsilon_k+1} \cdots y_s^{\epsilon_s}(\overline{s-i, \ldots, s})\right) = y_1^{\epsilon_1+1} y_k^{\epsilon_k} \cdots y_s^{\epsilon_s}(k, \overline{s-i, \ldots, s})$$
$$+ \sum_{r=s-i+1}^{s} (-1)^{s-i+r} y_1^{\epsilon_1} y_k^{\epsilon_k} \cdots y_r^{\epsilon_r+1} \cdots y_s^{\epsilon_s}(1, k, \overline{s-i, \ldots, r-1}, \widehat{r}, r+1, \ldots, s).$$

Still continuing with $0 \leq i \leq s-4$, taking $0 \leq j < i$ and $2 \leq k < a_1$,

$$t_i\left(y_1^{\epsilon_1} y_{s-j-1}^{\epsilon_{s-j-1}} \cdots y_s^{\epsilon_s}(a_1, \ldots, a_{i-j}, \overline{s-j, \ldots, s})\right) = 0.$$
$$t_i\left(y_k^{\epsilon_k+1} \cdots y_s^{\epsilon_s}(a_1, \ldots, a_{i-j}, \overline{s-j, \ldots, s})\right) = y_k^{\epsilon_k} \cdots y_s^{\epsilon_s}(k, a_1, \ldots, a_{i-j}, \overline{s-j, \ldots, s}).$$
$$t_i\left(y_1^{\epsilon_1+1} y_k^{\epsilon_k+1} \cdots y_s^{\epsilon_s}(a_1, \ldots, a_{i-j}, \overline{s-j, \ldots, s})\right) = y_1^{\epsilon_1+1} y_k^{\epsilon_k} \cdots y_s^{\epsilon_s}(k, a_1, \ldots, a_{i-j}, \overline{s-j, \ldots, s}) + \sum_{r=s-j+1}^{s} (-1)^{s-i+r} y_1^{\epsilon_1} y_k^{\epsilon_k} \cdots y_r^{\epsilon_r+1} \cdots y_s^{\epsilon_s} (1, k, a_1, \ldots, a_{i-j}, \overline{s-j, \ldots, r-1}, \widehat{r}, r+1 \ldots, s).$$

And continuing with $0 \leq i \leq s-4$ and $0 \leq j < i$, and taking $a_1 \leq k \leq s-j-2$,

$$t_i\left(y_k^{\epsilon_k+1} \cdots y_s^{\epsilon_s}(a_1, \ldots, a_{i-j}, \overline{s-j, \ldots, s})\right) = 0.$$
$$t_i\left(y_1^{\epsilon_1+1} y_k^{\epsilon_k+1} \cdots y_s^{\epsilon_s}(a_1, \ldots, a_{i-j}, \overline{s-j, \ldots, s})\right) = 0.$$

The general definition of t_i for $0 \leq i \leq s-4$ follows from Equation (4.2).

Now we give the formulas for t_{s-3}. At this point, some of the formulas do not commute with the action of τ.

$$t_{s-3}\left(y_2^{\epsilon_2} \cdots y_s^{\epsilon_s}(\overline{3, \ldots, s})\right) = 0.$$
$$t_{s-3}\left(y_1^{\epsilon_1+1} y_3^{\epsilon_3} \cdots y_s^{\epsilon_s}(\overline{3, \ldots, s})\right) = y_1^{\epsilon_1} y_3^{\epsilon_3} \cdots y_s^{\epsilon_s}(\overline{1}, \overline{3, \ldots, s}).$$

The above two formulas extend by Equation (4.2).
For $1 \leq k \leq s-1$,

$$t_{s-3}\left(y_1^{\epsilon_1} \cdots y_k^{\epsilon_k+1} y_{k+1}^{\epsilon_{k+1}+1} \cdots y_s^{\epsilon_s}(\overline{1, \ldots, k-1}, \overline{k+2, \ldots, s})\right) = X + \sum_{r=2}^{k} (-1)^r y_1^{\epsilon_1} \cdots y_r^{\epsilon_r+1} \cdots y_s^{\epsilon_s}(\overline{1, \ldots}, \widehat{r}, \overline{\ldots, s}).$$
$$t_{s-3}\left(y_1^{\epsilon_1+1} \cdots y_s^{\epsilon_s+1}(\overline{2, \ldots, s-1})\right) = (-1)^{s-1}\left(X + \sum_{r=2}^{s} (-1)^r y_1^{\epsilon_1} \cdots y_r^{\epsilon_r+1} \cdots y_s^{\epsilon_s}(\overline{1, \ldots}, \widehat{r}, \overline{\ldots, s})\right).$$

For $1 \leq j \leq s-3$,

$$t_{s-3}\left(y_1^{\epsilon_1} \cdots y_s^{\epsilon_s}(2, \ldots, j+1, \overline{j+3, \ldots, s})\right) = 0.$$

The above formula extends using Equation (4.2).

Here are the formulas for t_{s-2}.

$$t_{s-2}\left(y_2^{\epsilon_2}\cdots y_s^{\epsilon_s}(\overline{2,\ldots,s})\right)=0.$$

This formula extends using Equation (4.2).
For $2\le k\le s-1$,

$$t_{s-2}\left(y_1^{\epsilon_1+1}\cdots y_s^{\epsilon_s}(\overline{2,\ldots,s})\right)=y_1^{\epsilon_1}\cdots y_s^{\epsilon_s}(\overline{1,\ldots,s}).$$
$$t_{s-2}\left(y_1^{\epsilon_1}\cdots y_k^{\epsilon_k+1}\cdots y_s^{\epsilon_s}(\overline{1,\ldots},\widehat{k},\overline{\ldots,s})\right)=0.$$

Finally,

$$t_{s-2}\big(y_1^{\epsilon_1}\cdots y_s^{\epsilon_s}X\big)=0.$$

The verification that t_* is a contracting homotopy is tedious, but straight forward and left to the (masochistic) reader.

We remark that a minimal resolution for B_2^s can also be determined as follows. Consider the regular sequence $a=y_1+y_3+y_5+\cdots$, $b=y_2+y_4+\cdots$ in B_2^s. One can show that $B_2^s/(a,b)$ is isomorphic to $\mathbf{k}[z_1,\ldots,z_{s-2}]/(z_{ij},\, z_k^2-z_1^2 \mid i<j,\, k>1)$, where the z_i's are certain linear combinations of the y_i's. A minimal resolution of this ring as a $\mathbf{k}[z_1,\ldots,z_{s-2}]$-module was discussed by Wahl ([W], p.241) and determined by Behnke ([B]). In our application to rings of invariants in the next section, we assign certain *internal* degrees to the variables $y_1,\ldots,y_s$. It will be evident that the regular sequence (a,b) is not homogeneous in this internal degree. Since our main result concerns the internal Betti numbers, our resolution of B_2^s is more useful than Behnke's.

Section 5. The minimal resolution of B_n.

In this section we describe the minimal resolutions for a particular family of rings of invariants and show that each of these resolutions exhibits internal duality. Fix an integer $n\ge 1$, let $N=1+n+n^2$, and let $\mathbf{k}$ be a field containing a primitive N-th root of unity ω. Let $G\subseteq \mathrm{SL}_3(\mathbf{k})$ be the cyclic group generated by the matrix $\mathrm{diag}(\omega,\omega^n,\omega^{n^2})$. Let $B=B_n$ denote the ring of invariants $\mathbf{k}[x_1,x_2,x_3]^G$.

The section is organized as follows. First we construct a minimal set of homogeneous algebra generators and a minimal set of homogeneous algebra relations for B. Then we quotient out a (homogeneous) regular element and show that the resulting ring B' is of the form $\mathbf{k}[y_1,\ldots,y_{3n+6}]/J'$ where J' is minimally generated by polynomials whose leading terms are $\{y_iy_j \mid i\not\equiv j-1,j,j+1 \bmod 3n+6\}$. We then show that a minimal resolution of B' has the same structure as the minimal resolution of B_2^{3n+6} constructed in Section 4. Finally, we prove the main theorem (5.5) using this explicit resolution.

Since G acts diagonally on $\mathbf{k}[x_1,x_2,x_3]$, the ring of invariants is spanned by monomials. The invariant monomials are

$$\{\, x_1^{\epsilon_1}x_2^{\epsilon_2}x_3^{\epsilon_3} \mid \epsilon_1+n\epsilon_2+n^2\epsilon_3\equiv 0 \bmod N\,\}.$$

To such a monomial, we associate the exponent sequence $(\epsilon_1,\epsilon_2,\epsilon_3)$, and we call the integer $(\epsilon_1+n\epsilon_2+n^2\epsilon_3)/N$ its *multiplicity*. Note that the sequence $(\epsilon_1,\epsilon_2,\epsilon_3)$ is invariant if and only if its *cycled* sequences $(\epsilon_2, \epsilon_3, \epsilon_1)$ and $(\epsilon_3, \epsilon_1, \epsilon_2)$ are (this follows from the fact that $n^3 \equiv 1 \bmod N$). Also note that the sequence $(\epsilon_1, \epsilon_2, \epsilon_3)$ corresponds to an algebra generator if and only if its cycled sequences correspond to algebra generators.

It is clear that any monomial with multiplicity 1 is a generator of B. For example, the exponent sequence $(1, 1, 1)$ corresponds to a degree three generator $\alpha = x_1x_2x_3$. Since any invariant monomial with ϵ_1, ϵ_2, and ϵ_3 all non-zero is divisible by α, any other algebra generators must have some zero ϵ's.

Suppose that $(\epsilon_1, \epsilon_2, 0)$ corresponds to a generator. If it has multiplicity 1, then $\epsilon_1 + n\epsilon_2 = N$, so $\epsilon_1 \equiv 1 \bmod n$. It follows that $(\epsilon_1, \epsilon_2, 0) = (1 + nk, n + 1 - k, 0)$ for some $k = 0, \ldots, n + 1$; we call the corresponding generators μ_k. The cycled sequences $(n + 1 - k, 0, 1 + nk)$ and $(0, 1 + nk, n + 1 - k)$ correspond to generators which we call ν_k and ω_k respectively. Note that the degree of μ_k, ν_k, and ω_k is $3 + (n - 1)(k + 1)$.

Proposition 5.1. *The elements α, μ_k, ν_k, and ω_k, for $k = 0, \ldots, n + 1$ form a minimal generating set for B.*

Proof. From their construction it is clear that none of these elements divide each other. So, if they generate, then they are minimal. We must show that every invariant is a product of them.

Suppose $E = (\epsilon_1, \epsilon_2, \epsilon_3) \neq (0, 0, 0)$ corresponds to the invariant monomial, x^E. If all of its entries are non-zero, then it is divisible by α and we are done. If at least one of its entries is zero, then it can be cycled until the third entry is zero. So assume $(\epsilon_1, \epsilon_2, 0)$ is an invariant. If it has multiplicity 1 then it is one of the μ_k, so assume it has multiplicity greater than 1. There are three cases.

(1) If $\epsilon_1 = 0$, then $\epsilon_2 \neq 0$ and $n\epsilon_2 \equiv 0 \bmod N$. Consequently, $N \leq \epsilon_2$, so ω_{n+1}, which corresponds to $(0, N, 0)$, divides x^E.
(2) If $1 \leq \epsilon_1 < N$, then since $\epsilon_1 + n\epsilon_2 \geq 2N$, we have $\epsilon_2 > n+1$. It follows that μ_0, which corresponds to $(1, n + 1, 0)$, divides x^E.
(3) If $\epsilon_1 \geq N$, then μ_{n+1}, which corresponds to $(N, 0, 0)$, divides x^E. □

Note that the degrees of the generators are all less that or equal to N, the order of G.

We now determine a minimal set of relations for B. Let $A = \mathbf{k}[a, u_k, v_k, w_k]$ be the graded polynomial ring on indeterminants a, u_k, v_k, and w_k, for $k = 0, \ldots, n + 1$, where a has degree 3, and u_k, v_k, and w_k have degree $3 + (n - 1)(k + 1)$. Let $e_0 \colon A \longrightarrow B$ be the map sending a to α, u_k to μ_k, v_k to ν_k, and w_k to ω_k. Then $B = A/J$ for an appropriate homogeneous ideal J. We describe a minimal generating set for J.

Given a sequence $E = (\epsilon_0, \ldots, \epsilon_{3n+6})$ of nonnegative integers, let m^E stand for the monomial $a^{\epsilon_0} u_0^{\epsilon_1} v_0^{\epsilon_2} w_0^{\epsilon_3} \cdots w_{n+1}^{\epsilon_{3n+6}}$. If $F = (\phi_0, \ldots, \phi_{3n+6})$ is another such sequence, we will say that $m^E < m^F$ if $\deg(m^E) < \deg(m^F)$ or if $\deg(m^E) = \deg(m^F)$ and $\epsilon_j < \phi_j$ for the smallest j where E and F differ. (This is sometimes called the *graded lexicographical* order on A.) The degree of m^E is clearly the degree of $e_0(m^E)$, a monomial in the x's. The *polynomial degree* of m^E is defined as $\sum_{i=0}^{3n+6} \epsilon_j$. The *leading term* of a polynomial p in A will be the largest monomial occurring in A.

Since the generators α, μ_k, ν_k, and ω_k of B are monomials in the x's, there is a basis for J consisting of differences $m^E - m^F$ of monomials in A. The following elements of J

are of the form $m^E - m^F$, where $m^E > m^F$ and m^E has polynomial degree 2.

$u_i u_j - u_{i-1}u_{j+1}$, for $1 \le i \le j \le n$;
$u_0 v_0 - a u_1$;
$u_0 v_i - a^{n+2-i} w_0^{i-1}$, for $1 \le i \le n+1$;
$u_i v_i - a^{n+1-i} v_0^{i-1} v_{i-1}$, for $1 \le i \le n+1$;
$u_i v_j - a^{n+1-i} v_0^{i-1} v_{j-1}$, for $1 \le i < j \le n+1$;
$u_0 w_i - a w_{i+1}$, for $1 \le i \le n$;
$u_1 w_j - a^{n+1-j} u_0^j$, for $2 \le j \le n+1$; and
$u_i w_j - a^{n+1-i} u_0^{j-1} u_{i-1}$, for $2 \le i < j \le n+1$.

Each of these elements can be *cycled* ($a \mapsto a$, $u_k \mapsto v_k$, $v_k \mapsto w_k$, and $w_k \mapsto u_k$) twice to give two more elements in J of the same form. The total number of elements (including cycles) is $(3n+6)(3n+3)/2$. We say that a monomial is *A-admissible* if it occurs as a leading term in the above list (cycles included).

Proposition 5.2. *The above polynomials (cycles included) minimally generate the ideal J.*

Proof. The above list of polynomials minimally generates some ideal, call it K. Assume $K \neq J$ and let $p \in J - K$ be a (monic) polynomial having the smallest leading term, l. If any A-admissible m^E divides l, then the polynomial $p - (l/m^E)(m^E - m^F)$, with $m^E - m^F$ in the above list, is in $J - K$, has smaller leading term than p, and therefore contradicts the choice of p.

It follows that the leading term l of p must be one of (or a cycle of one of) the following: (1) $a^{\epsilon_1} u_0^{\epsilon_2} u_k^{\epsilon_3}$, (2) $a^{\epsilon_1} u_k^{\epsilon_3} u_{n+1}^{\epsilon_4}$, or (3) $a^{\epsilon_1} u_0^{\epsilon_2} w_{n+1}^{\epsilon_5}$, where $1 \le k \le n+1$, $\epsilon_1, \epsilon_2, \epsilon_4, \epsilon_5$ are nonnegative integers, and ϵ_3 equals 0 or 1.

Consider the map $e_0 \colon A \longrightarrow B$. Since p is in J, $e_0(p) = 0$. Now $e_0(l)$ is some monomial in the x's, so for $e_0(p) = 0$, there must be other monomials in p (necessarily smaller than l in the ordering) which map to $e_0(l)$. However, it is easy to check that the possible l's above are the minimal monomials in A mapping to their respective x_1, x_2, x_3 monomials. □

We should remark that the above basis for J is a Gröbner basis; that is, the A-admissible monomials minimally generate the ideal of leading terms of J.

Now let $A' = A/(a) = \mathbf{k}[u_k, v_k, w_k]$ and let $B' = B/(a)$. Then $B' = A'/J'$, where J' is minimally generated by the above list of generators for J with all of the a's replaced by zero. The following well known lemma implies that the minimal resolution of B as an A-module has the same structure as the minimal resolution of B' as an A'-module.

Lemma 5.3. *Let $A = \mathbf{k}[y_1, \ldots, y_s]$, let $B = A/J$ with J homogeneous, and let a be an element of A which is not a zero divisor for A or for B. Let (M_*, e_*) be a minimal resolution of B as an A-module. Then the complex $(M_*/(a), e_*/(a))$ is a minimal resolution of $B/(a)$ as an $A/(a)$-module. In particular, the internal Betti numbers of these resolutions are the same.*

Proof. The sequence $0 \longrightarrow A \xrightarrow{\cdot a} A \longrightarrow A/(a) \longrightarrow 0$ is exact since a is A-regular. Tensoring this sequence with B yields an exact sequence $B \xrightarrow{\cdot a} B \longrightarrow B/(a) \longrightarrow 0$. However, since

a is B-regular, this sequence is in fact short exact ($\cdot a$ is injective). It follows that $\mathrm{Tor}_m^A(B, A/(a)) = 0$ for all $m > 0$. By a change of rings theorem ([CE], p.117) $\mathrm{Tor}_m^A(B, \mathbf{k})$ is isomorphic to $\mathrm{Tor}_m^{A/(a)}(B/(a), \mathbf{k})$ for all m. □

To describe the minimal A'-resolution of B', it is convenient to use the following ordering on the monomials of A'. Given $E = (\epsilon_1, \ldots, \epsilon_{3n+6})$ and $F = (\phi_1, \ldots, \phi_{3n+6})$, we say $m^E < m^F$ if $\deg(m^E) < \deg(m^F)$ or if $\deg(m^E) = \deg(m^F)$ and $\epsilon_j < \phi_j$ for the *largest* j where E and F differ. (This is sometimes called the *graded reverse lexicographical* order on A'.) With this ordering, the generators of J' are the following (and their cycles):

$u_{i-1}u_{j+1} - u_i u_j$, for $1 \le i \le j \le n$;
$u_0 v_0$;
$u_0 v_i$, for $1 \le i \le n+1$;
$u_i v_i$, for $1 \le i \le n$; $u_{n+1}v_{n+1} - v_0^n v_n$;
$u_i v_j$, for $1 \le i < j \le n+1$;
$u_0 w_i$, for $1 \le i \le n$;
$u_1 w_j$, for $2 \le j \le n$; $u_1 w_{n+1} - u_0^{n+1}$; and
$u_i w_j$, for $2 \le i < j \le n+1$.

Let us say that a monomial is A'-*admissible* if it appears as a leading term in the above generating set for J'. Let I' be the ideal generated by the A'-admissible monomials and let $B'' = A'/I'$.

If we rename the variables u_0, u_1, $\ldots$, u_{n+1}, v_0, $\ldots$, v_{n+1}, w_0, $\ldots$, w_{n+1} by y_1, $\ldots$, y_{3n+6}, then it is easy to see that the product $y_i y_j$ is A'-admissible if and only if $i \not\equiv j-1, j, j+1 \bmod 3n+6$. That is, B'' is isomorphic to the ring B_2^{3n+6} of Section 4, where the y_i's are given appropriate internal degrees. The following Proposition will be proved in the next section.

Proposition 5.4. *The minimal resolutions of B' and B'' as A'-modules have the same internal Betti numbers.*

The proof of this proposition relies on the fact that the resolution of B'' is pure (each differential has a homogeneous polynomial degree), so can be lifted to a minimal resolution of B'.

To end this section, we show that the resolution of B'' (hence the resolution for B) exhibits internal duality. Recall that this means that for each free module M_i in the minimal resolution, there exists an h_i such that $\beta_{i,j} = \beta_{i,h_i-j}$ for all j.

Theorem 5.5. *The minimal resolution of B'',*

$$0 \longrightarrow M_{3n+4} \xrightarrow{d_{3n+4}} M_{3n+3} \longrightarrow \cdots \longrightarrow M_1 \xrightarrow{d_1} M_0 \longrightarrow B'' \longrightarrow 0$$

exhibits internal dualty with $h_0 = 0$; $h_i = (i+1)(n^2+2n+3)$, for $1 \le i \le 3n+3$; and $h_{3n+4} = (3n+6)(n^2+2n+3)$.

Proof. The result clearly holds for $M_0 = A'$. For $1 \le i \le 3n+3$, the basis elements for M_i are quotients of elements of the form $(a_1, \ldots, a_{i+1})$, $1 \le a_1 < \cdots < a_{i+1} \le 3n+6$, with bars over some adjacent entries (see Section 4). The pairing $y_j \leftrightarrow y_{3n+7-j}$ on the variables of A' associates variables of complementary degrees: $\deg(y_j) + \deg(y_{3n+7-j}) = n^2+2n+3$. This pairing extends to the M_i basis elements via $(a_1,\ldots,a_{i+1}) \leftrightarrow (3n+7-a_{i+1},\ldots,3n+7-a_1)$ where the bars are over corresponding indices. For example, when $n=2$, then $3n+7=13$ and $(\overline{1},3,\overline{11},\overline{12})$ pairs with $(\overline{1},\overline{2},9,\overline{12})$. Since $\deg(a_1,\ldots,a_{i+1})+\deg(3n+7-a_{i+1},\ldots,3n+7-a_1)=(i+1)(n^2+2n+3)$, these M_i exhibit internal duality. The basis element X of M_{3n+4} has degree $\sum_{j=1}^{3n+6}\deg(y_j)$ which equals $(3n+6)(n^2+2n+3)/2$. So M_{3n+4} exhibits internal duality with $h_{3n+4}=(3n+6)(n^2+2n+3)$. □

Remark 5.6. The example which motivated this work corresponds to the $n=2$ case of the above theorem (see Section 7).

Remark 5.7. Computer calculations support the possibility that all rings of the form $\mathbf{k}[x_1,x_2,x_3]^G$, with $G \subseteq \mathrm{SL}_3(\mathbf{k})$ cyclic, have resolutions with the same structure as B_2^s for some s. Only three families of these seem to exhibit internal duality: the ring in Example (2.2), the family of Theorem (5.5), and the following family. For $n \geq 1$, let $\mathbf{k}$ be a field containing a primitive $(4n)$-th root of unity ω, and let $G \subseteq \mathrm{SL}_3(\mathbf{k})$ be the cyclic group generated by $\mathrm{diag}(\omega,\omega^{2n-1},\omega^{2n})$. Then, for each n, $B_n=\mathbf{k}[x_1,x_2,x_3]^G$ has 9 generators and 20 relations. The minimal resolutions of the B_n's each exhibit internal duality – the proof is similar to the above.

Section 6. Relation between resolutions of A/J and resolutions of A/I

In this section, $A=\mathbf{k}[y_1,\ldots,y_s]$ with $\deg(y_j)=1$. We fix a well ordering on $\mathrm{Mono}(A)$, the set of (monic) monomials in $A=\mathbf{k}[y_1,\ldots,y_s]$, which is multiplicative (for $\alpha_1,\alpha_2,\alpha_3 \in \mathrm{Mono}(A)$, $\alpha_1>\alpha_2$ implies $\alpha_1\alpha_3>\alpha_2\alpha_3$). The largest monomial of a given polynomial will be called the *leading term*. Let J be an ideal minimally generated by $\{z_{01},\ldots,z_{0\beta_1}\}$, and let I be the ideal generated by the leading terms $y_{0,j}$ of the $z_{0,j}$. We assume that the set $\{z_{0j}\}$ is a *Gröbner basis* for J which means that the ideal I is minimally generated by the y_{0j} and contains the leading terms of all the elements of J. Our goal is to give conditions under which the minimal resolutions of A/J and A/I have the same structure.

Let M be a free A-module with basis $\{x_j \mid j=1,\ldots,\beta\}$. We call the $\mathbf{k}$-basis $\{\alpha x_j \mid \alpha \in \mathrm{Mono}(A)\}$ of M the set of *(monic) monomials* of M. For later induction proofs, we will need a $\mathrm{Mono}(A)$-grading (called *type*) on the non-zero elements of M and an ordering on the monomials of M which is compatible with the above fixed order on $\mathrm{Mono}(A)$.

For a monic polynomial p in A, $\mathrm{type}(p)$ is the leading term of p, and for general $p \neq 0$, $\mathrm{type}(p)$ is $\mathrm{type}(kp)$ where $k \in \mathbf{k}^*$ and kp is monic. For each $j=1,\ldots,\beta$, choose an element $\mathrm{type}(x_j) \in \mathrm{Mono}(A)$, and for $x=\sum p_j x_j$, let $\mathrm{type}(x)=\max_j\{\mathrm{type}(p_j)\,\mathrm{type}(x_j)\}$. For an ordering on the monomials of M we say $\alpha x_j > \alpha' x_{j'}$, for $\alpha,\alpha' \in \mathrm{Mono}(A)$, if one of the following holds:

(i) $\mathrm{type}(\alpha x_j) > \mathrm{type}(\alpha' x_{j'})$, or
(ii) $\mathrm{type}(\alpha x_j) = \mathrm{type}(\alpha' x_{j'})$ and $\mathrm{type}(x_j) > \mathrm{type}(x_{j'})$, or
(iii) $\mathrm{type}(\alpha x_j) = \mathrm{type}(\alpha' x_{j'})$, $\mathrm{type}(x_j) = \mathrm{type}(x_{j'})$, and $j > j'$.

The maximal monomial of an element $x=\sum p_j x_j$ in M will be called the *leading term* of x, denoted $l.t.(x)$. The sum of all the monomials of x having type α will be called the α *type-term*, denoted x_α. The type-term of x having the same type as the leading term will be called the *leading type-term*, denoted $l.t.t.(x)$. We write M^α for the $\mathbf{k}$-span of the monomials in M of type less than or equal to α and $M^{(\alpha)}$ for the $\mathbf{k}$-span of monomials of type exactly α.

Let $0 \longrightarrow M_h \xrightarrow{e_h} \cdots \xrightarrow{e_2} M_1 \xrightarrow{e_1} M_0 = A$ be a sequence of free A-modules and A-linear maps, and for each i, let $\{x_{ij} \mid j = 1, \ldots, \beta_i\}$ be an A-basis for M_i. We say (M_*, e_*) has a *type grading* if each M_i has a Mono(A)-grading as above. By convention, we always take $x_{01} = 1$ and type$(x_{01}) = 1$. A type graded sequence (M_*, e_*) is said to be *filtered by type* if $e_i(M_i^\alpha) \subseteq M_{i-1}^\alpha$ and to *preserve type* if $e_i(M_i^{(\alpha)}) \subseteq M_{i-1}^{(\alpha)}$. We define the *associated* sequence of a type graded sequence (M_*, e_*) to be $0 \longrightarrow M_h \xrightarrow{d_h} \cdots \xrightarrow{d_2} M_1 \xrightarrow{d_1} M_0 = A$, where d_i is the A-linear map which sends x_{ij} to the type(x_{ij})–term of $e_i(x_{ij})$. Clearly, the associated sequence (M_*, d_*) preserves type.

Lemma 6.1. *Let (M_*, e_*) be filtered by type, and let (M_*, d_*) be its associated sequence. If $x \in M_i$ has type α, then $d_i(x_\alpha) = (e_i(x_\alpha))_\alpha = (e_i(x))_\alpha$. In particular, if $e_i(x) = 0$ then $d_i(x_\alpha) = 0$.*

Proof. Let $x = \sum_{j \in S} p_j x_{ij}$, let $T = \{j \in S \mid \text{type}(p_j)\,\text{type}(x_{ij}) = \alpha\}$, and let $U = \{j \in S \mid \text{type}(p_j)\,\text{type}(e_i(x_{ij})) = \alpha\}$. Then $x_\alpha = \sum_{j \in T} l.t.(p_j) x_{ij}$, and

(i) $d_i(x_\alpha) = \sum_{j \in T} l.t.(p_j) d_i(x_{ij}) = \sum_{j \in U} l.t.(p_j) l.t.t.(e_i(x_{ij}))$,

(ii) $(e_i(x_\alpha))_\alpha = \left(\sum_{j \in T} l.t.(p_j) e_i(x_{ij})\right)_\alpha = \sum_{j \in U} l.t.(p_j) l.t.t.(e_i(x_{ij}))$, and

(iii) $(e_i(x))_\alpha = \left(\sum_{j \in S} p_j e_i(x_{ij})\right)_\alpha = \sum_{j \in U} l.t.(p_j) l.t.t.(e_i(x_{ij}))$. □

Suppose that (M_*, e_*) is a sequence of A-modules and A-linear maps as above satisfying the additional property that $e_i(x_{ij}) \neq 0$ for each i, j. Then we can define an *induced type grading* on M_* so that (M_*, e_*) is filtered by type. For this, first recall by convention that $x_{01} = 1$ has type 1, so non-zero elements of $M_0 = A$ have the usual types (a scalar multiple of the leading term). Once the type has been defined on M_{i-1}, we let type$(x_{ij}) = $ type$(e_i(x_{ij}))$ and then extend to the rest of M_i in the obvious way.

We now return to the ideals J and I with generators $\{z_{0j}\}$ and $\{y_{0j}\}$ respectively. Our first result constructs a resolution of A/I from one of A/J.

Proposition 6.2. *Let (M_*, e_*) be a free resolution of A/J with $e_0 \colon M_0 = A \longrightarrow B$ the projection, and give (M_*, e_*) the induced type grading defined above. If each subcomplex (M_*^α, e_*) is exact, then the associated complex (M_*, d_*) is a resolution of A/I. In addition, if (M_*, e_*) is a minimal resolution of A/J, then (M_*, d_*) is a minimal resolution of A/I.*

Proof. Since the image of e_1 is J and $\{z_{0j}\}$ is a Gröbner basis, the image of d_1 is I. Hence $M_1 \xrightarrow{d_1} M_0 \xrightarrow{d_0} A/I$ is exact.

For $i \geq 1$, $d_i d_{i+1} = 0$, since $d_i d_{i+1}(x_{i+1,j})$ is the type$(x_{i+1,j})$–term of $e_i e_{i+1}(x_{i+1,j})$, which is zero. Assume $d_i(x) = 0$ for some $x \neq 0$ in M_i. We may assume x is type-homogeneous, say with type α. $d_i(x) = 0$ implies that $\beta = $ type$(e_i(x))$ is less than α. Since $e_i(x)$ is in the kernel of e_{i-1}, and (M_*^β, e_*) is exact, there is a $y \in M_i$ with type$(y) \leq \beta$ and $e_i(y) = e_i(x)$. Now, $e_i(x - y) = 0$, so there is a $z \in M_{i+1}^\alpha$ with $e_{i+1}(z) = x - y$. From Lemma (6.1), we have $d_{i+1}(z_\alpha) = (e_{i+1}(z))_\alpha = x$. We have shown that (M_*, d_*) is exact.

By the Minimality Criterion (1.2), (M_*, e_*) minimal implies that the matrices of the e_i relative to the bases $\{x_{ij}\}$ and $\{x_{i-1,j}\}$ involve no non-zero constants. But the matrices of the d_i come from the matrices of the e_i (by choosing only some of the non-zero entries), so they have no non-zero constants either. □

Our next result will construct a resolution of A/J from one of A/I. Since I is a monomial ideal, we can choose a minimal resolution (M_*, d_*) of A/I which preserves types (construct the d_i inductively writing the elements of $\ker(d_{i-1})$ as sums of homogeneous type-terms).

Proposition 6.3. *Let (M_*, d_*) be a free resolution of A/I preserving types with $M_1 = \oplus_{j=1}^{\beta_1} Ax_{1j}$, $M_0 = A$, and $d_1(x_{1j}) = y_{0j}$. Then there is a resolution (M_*, e_*) of A/J, beginning with $e_1(x_{1j}) = z_{0j}$, which satisfies the following properties.*
(i) (M_*, e_*) is filtered by type,
(ii) (M_*, d_*) is the associated resolution to (M_*, e_*), and
(iii) For each $\alpha \in \mathrm{Mono}(A)$, the subcomplex (M_*^α, e_*) is exact.

Proof. We will construct the e_i's inductively. We first check that $M_1 \xrightarrow{e_1} M_0 \xrightarrow{e_0} A/J$ satisfies properties (i) to (iii).

$$\begin{aligned}\operatorname{type}\Big(e_1(\sum p_j x_{1j})\Big) &= \operatorname{type}\Big(\sum p_j z_{0j}\Big)\\ &\le \max_j\{\operatorname{type}(p_j z_{0j})\}\\ &= \max_j\{\operatorname{type}(p_j y_{0j})\}\\ &= \max_j\{\operatorname{type}(p_j x_{1j})\}\\ &= \operatorname{type}\Big(\sum p_j x_{1j}\Big)\end{aligned}$$

So (i) is satisfied. Property (ii) is trivial.

For (iii), let $\alpha \in \mathrm{Mono}(A)$ and assume that we have shown that $M_1^\beta \xrightarrow{e_1} M_0^\beta \xrightarrow{e_0} A/J$ is exact at M_0^β for each $\beta < \alpha$. Suppose that $p \in M_0^\alpha$ has type α and $e_0(p) = 0$. Then $p \in J$ and the leading term of p is $k\alpha$ for some k in $\mathbf{k}$. Since $\{z_{0j}\}$ is a Gröbner basis, there exist $q_j \in A$ with $k\alpha = \sum q_j y_{0j}$. But $k\alpha$ and the y_{0j} are monomials, so $k\alpha = qy_{0j_0}$ for some $q \in \mathrm{Mono}(A)$. Let $x = qx_{1j_0}$ in M_1. Then $e_1(x) = qz_{0j_0}$ has leading term $k\alpha$. Now $e_0(p - e_1(x)) = 0$ with $\operatorname{type}(p - e_1(x)) < \alpha$, so there exists $y \in M_1^\beta$, $\beta < \alpha$, with $e_1(y) = p - e_1(x)$. So $e_1(x + y) = p$ with $\operatorname{type}(x + y) = \alpha = \operatorname{type}(p)$.

Now, assume that e_{i-1} has been defined having properties (i) to (iii). Then

$$e_{i-1}(d_i(x_{ij})) = d_{i-1}(d_i(x_{ij})) + y = y$$

with $\operatorname{type}(y) < \operatorname{type}(x_{ij})$ and $e_{i-2}(y) = 0$. Since e_{i-1} satisfies (iii), there exists $z \in M_{i-1}$ with $e_{i-1}(z) = y$ and $\operatorname{type}(z) \le \operatorname{type}(y)$. Let $e_i(x_{ij}) = d_i(x_{ij}) - z$. Extend e_i linearly to M_i and notice that $e_{i-1}e_i = 0$. It is clear that e_i satisfies (i) and (ii); we have to verify (iii).

Suppose we've shown $M_i^\beta \xrightarrow{e_i} M_{i-1}^\beta \xrightarrow{e_{i-1}} M_{i-2}^\beta$ is exact at M_{i-1}^β for all $\beta < \alpha$, and suppose $e_{i-1}(x) = 0$ for some $x \in M_{i-1}^\alpha$ having type α. Then $d_{i-1}(x_\alpha) = 0$, so there exists $y \in M_i^{(\alpha)}$ with $d_i(y) = x_\alpha$. Let $z = e_i(y) - x$. Then $e_{i-1}(z) = 0$ and $\text{type}(z) < \alpha$. By induction, there exists $w \in M_i$ with $e_i(w) = z$ and $\text{type}(w) < \alpha$. It follows that $e_i(y - w) = x$ with $y - w \in M_i^\alpha$. □

As the following example shows, it is not generally true that (M_*, e_*) is minimal when (M_*, d_*) is.

Example 6.4. *Let $A = \mathbf{k}[y_1, \ldots, y_6]$, $J = \langle y_3^2 - y_1y_5,\ y_3y_4 - y_1y_6,\ y_2y_4 - y_3y_5,\ y_5^2 - y_2y_6,\ y_4y_5 - y_3y_6\rangle$, and $I = \langle y_3^2,\ y_3y_4,\ y_2y_4,\ y_5^2,\ y_4y_5\rangle$. The internal Betti numbers for the minimal resolutions of A/I and A/J are as follows.*

A/I

$d:$	0	1	2	3	4	5
$\beta_{0d}:$	1					
$\beta_{1d}:$			5			
$\beta_{2d}:$				5	1	
$\beta_{3d}:$					1	1

A/J

$d:$	0	1	2	3	4	5
$\beta_{0d}:$	1					
$\beta_{1d}:$			5			
$\beta_{2d}:$				5		
$\beta_{3d}:$						1

However, with some further conditions on the d_i's in Proposition (6.3), we can conclude that (M_*, e_*) is minimal. Let A_+ be the ideal of positive degree elements in the polynomial ring A.

Proposition 6.5. *Let (M_*, d_*) satisfy the hypotheses of Proposition (6.3), and assume that, for each $i \geq 1$, there is a positive integer k_i such that the entries of the matrices representing d_i are all homogeneous polynomials of degree k_i (i.e. (M_*, d_*) is pure). Also, assume that each of the monomials in z_{0j} has polynomial degree greater than or equal to k_1. Then the maps e_i can be chosen so that $e_i(M_i) \subseteq (A_+)^{k_i} M_{i-1}$.*

Proof. We use induction on i. By hypothesis, $e_1(M_1) \subseteq (A_+)^{k_1} M_0$. Suppose that for each $j < i$ we have chosen e_j so that $e_j(M_j) \subseteq (A_+)^{k_j} M_{j-1}$ for each j less than i. In the proof of Proposition (6.3), we set $e_i(x_{ij}) = d_i(x_{ij}) - z$, where z was chosen with $e_{i-1}(z) = y = e_{i-1}(d_i(x_{ij}))$ and $\text{type}(z) < \text{type}(x_{ij})$. By hypothesis, $d_i(x_{ij}) \in (A_+)^{k_i} M_{i-1}$, but what about z?

Since $y = e_{i-1}(d_i(x_{ij}))$, we know by induction that $y \in (A_+)^{k_i + k_{i-1}} M_{i-2}$. Since the matrix for e_{i-1} may contain polynomials of degree greater than k_{i-1}, it is conceivable that z could contain monomials which are not in $(A_+)^{k_i} M_i$ and still have $y \in (A_+)^{k_i + k_{i-1}} M_{i-2}$. To prove the proposition, we must choose z more carefully than before.

Since $e_{i-2}(y) = 0$, we know by Lemma (6.1) that $d_{i-2}(l.t.t.(y)) = 0$. By the exactness of (M_*, d_*) and the fact that this resolution preserves type, there is a $z_1 \in M_{i-1}$ of homogeneous type such that $d_{i-1}(z_1) = l.t.t.(y)$ (which is in $(A_+)^{k_i + k_{i-1}} M_{i-2}$). Since d_{i-1} is homogeneous of degree k_{i-1}, it follows that $z_1 \in (A_+)^{k_i} M_{i-1}$. And from this it follows that $e_{i-1}(z_1) \in (A_+)^{k_i + k_{i-1}} M_{i-2}$.

Now let $y_1 = e_{i-1}(d_i(x_{ij}) - z_1) = y - e_{i-1}(z_1)$. Then $\text{type}(y_1) < \text{type}(y)$, $e_{i-2}(y_1) = 0$, and $y_1 \in (A_+)^{k_i + k_{i-1}} M_{i-2}$. Applying the above argument again with y_1 in the role of y, we construct z_2 and y_2, then z_3 and y_3, etc, until $y_r = 0$. We then set $z = z_1 + \cdots z_r$. By construction, $e_{i-1}(z) = y$ and z is in $(A_+)^{k_i} M_{i-1}$. □

Remark 6.6. This proposition should be compared to Wahl ([W], Remarks 1.8).

Section 7. Motivating example.

In this section we describe the ring $B = H^*(\mathbf{F}_8 \rtimes \mathbf{F}_8^*; \mathbf{F}_2) = \mathbf{F}_2[x, y, z]^{\mathbf{Z}/7}$. We fix a primitive 7-th root of unity, ω, in $\mathbf{F}_8$ by insisting ω is a root of the polynomial $t^3 + t^2 + 1 \in \mathbf{F}_2[t]$. Then the action of left multiplication by ω on $\mathbf{F}_8 = \mathbf{F}_2(1, \omega, \omega^2)$ is given by the 3×3 matrix

$$T = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix}.$$

Here are the 13 generators for B.

$a = x^3 + y^3 + z^3 + x^2y + y^2z + xz^2 + xyz,$
$b = x^4 + y^4 + z^4 + x^2y^2 + y^2z^2 + x^2z^2 + x^2yz + xy^2z + xyz^2,$
$c = x^3z + x^2y^2 + x^2yz + xy^3 + xy^2z + xz^3 + y^3z + y^2z^2,$
$d = x^4 + x^3y + x^2y^2 + x^2yz + xy^3 + xz^3 + y^4 + yz^3 + z^4,$
$e = x^5 + y^5 + z^5 + x^4y + y^4z + xz^4 + x^2y^2z + x^2yz^2 + xy^2z^2,$
$f = x^4z + x^3z^2 + x^2y^3 + x^2yz^2 + x^2z^3 + xy^4 + xy^2z^2 + xz^4 + y^4z + y^3z^2,$
$g = x^4y + x^3y^2 + x^2y^3 + x^2y^2z + x^2yz^2 + x^2z^3 + xy^4 + xz^4 + y^2z^3 + yz^4,$
$h = x^4y^2 + x^2y^4 + y^4z^2 + y^2z^4 + x^4z^2 + x^2z^4 + x^4yz + xy^4z + xyz^4 + x^2y^2z^2,$
$i = x^5z + x^4z^2 + x^4y^2 + x^4yz + x^2z^4 + xy^5 + xy^4z + xz^5 + y^5z + y^2z^4,$
$j = x^5y + x^4yz + x^2y^2z^2 + x^2z^4 + xy^5 + xz^5 + y^2z^4 + yz^5,$
$k = x^4y^2z + y^4xz^2 + z^4x^2y + x^4yz^2 + y^4x^2z + z^4xy^2,$
$l = x^6z + x^5z^2 + x^4y^2z + x^4yz^2 + y^5x^2 + z^5x^2 + y^4x^2z + y^6x + y^4xz^2 + xz^6 + y^6z + y^5z^2,$
$m = x^5z^2 + y^3x^4 + x^4yz^2 + z^3x^4 + x^3z^4 + z^5x^2 + z^4x^2y + y^4xz^2 + z^4xy^2 + y^5z^2 + y^3z^4 + y^5x^2.$

And here are the 54 relations (listed by degree).

$R1 = ag + b^2 + bd + c^2,$
$R2 = ae + af + b^2 + cd,$
$R3 = af + ag + bc + bd + d^2,$

$R4 = ah + aj + bf + bg + ce,$
$R5 = a^3 + ah + ai + aj + be + bf + bg + cf,$
$R6 = a^3 + aj + be + bg + cg,$
$R7 = ah + ai + aj + be + bf + de,$
$R8 = aj + bf + bg + df,$
$R9 = a^3 + ai + be + bf + bg + dg,$

$R10 = a^2c + am + bi + ch,$
$R11 = a^2b + a^2c + a^2d + ak + al + bh + bj + ci,$
$R12 = a^2c + al + bh + cj,$
$R13 = a^2b + a^2c + a^2d + al + am + bj + dh,$
$R14 = ak + al + am + bh + di,$

$R15 = ak + al + bi + bj + dj,$
$R16 = a^2b + ak + bh + e^2,$
$R17 = a^2b + a^2c + a^2d + am + bh + bj + ef,$
$R18 = a^2b + a^2d + al + am + bh + bi + bj + f^2,$
$R19 = al + am + bh + bi + bj + eg,$
$R20 = a^2b + a^2d + ak + am + bj + fg,$
$R21 = al + bi + g^2,$

$R22 = a^2f + a^2g + ab^2 + abd + bl + bm + cm + dk,$
$R23 = a^2e + a^2f + ab^2 + abc + bk + bl + bm + ck + cl + dl,$
$R24 = a^2g + ab^2 + abd + bk + bm + cl + cm + dm,$
$R25 = a^2e + ab^2 + bm + ck + cl + cm + eh,$
$R26 = a^2f + a^2g + ab^2 + abc + abd + bk + bl + ck + cm + ei,$
$R27 = a^2e + abc + abd + bl + cl + cm + ej,$
$R28 = a^2f + a^2g + ab^2 + abd + bl + bm + ck + fh,$
$R29 = a^2f + a^2g + ab^2 + abd + bk + bl + bm + cl + fi,$
$R30 = a^2e + ab^2 + abc + bk + bl + ck + cm + fj,$
$R31 = a^2g + ab^2 + abd + bk + bm + cm + gh,$
$R32 = a^2e + a^2f + a^2g + abc + abd + bk + ck + cl + gi,$
$R33 = a^2g + ab^2 + abc + abd + bk + bl + bm + cl + gj,$

$R34 = a^2i + b^2c + em + fk,$
$R35 = a^2i + abf + b^2c + el + h^2 + fl,$
$R36 = a^4 + a^2j + b^2c + b^2d + el + em + fm,$
$R37 = abf + abg + b^3 + b^2d + el + em + gk,$
$R38 = a^4 + abg + b^2c + b^2d + el + em + h^2 + gl,$
$R39 = a^4 + abe + abg + b^3 + b^2d + em + h^2 + gm,$
$R40 = a^4 + a^2h + b^3 + ek + h^2,$
$R41 = a^2h + a^2i + a^2j + b^3 + b^2d + el + hi,$
$R42 = a^2h + a^2j + abf + b^3 + b^2c + b^2d + em + h^2 + hj,$
$R43 = a^2i + abg + b^3 + b^2c + b^2d + em + i^2,$
$R44 = a^4 + a^2j + abe + abg + h^2 + ij,$
$R45 = b^2c + el + em + h^2 + j^2,$

$R46 = a^3b + a^3d + a^2m + b^2g + hl + ik,$
$R47 = a^3d + a^2k + a^2m + abh + abi + abj + b^2e + b^2f + hm + il,$
$R48 = a^3c + a^2k + abh + abj + b^2f + b^2g + hk + hl + im,$
$R49 = a^3c + abh + abj + b^2f + b^2g + hk + hm + jk,$
$R50 = a^3b + a^3c + a^2k + a^2l + a^2m + abi + abj + b^2e + b^2f + b^2g + hk + jl,$
$R51 = a^3b + a^2l + a^2m + abh + abi + abj + b^2e + b^2g + hl + jm,$

$R52 = a^3g + a^2b^2 + a^2bc + a^2bd + abk + abm + acm + b^2i + km + l^2,$
$R53 = a^3e + a^3g + a^2b^2 + abl + ack + acl + b^2i + b^2j + k^2 + kl + lm,$
$R54 = a^3g + a^2bc + abk + abl + acm + b^2h + b^2j + kl + km + m^2.$

Now consider the ring $B \otimes \mathbf{F}_8$. If we tensor a minimal $\mathbf{F}_2$ resolution of B with $\mathbf{F}_8$, we will have a minimal resolution of $B \otimes \mathbf{F}_8$ (since $\mathbf{F}_8$ is flat over $\mathbf{F}_2$). Over $\mathbf{F}_8$ the matrix

T diagonalizes to $\mathrm{diag}(\omega, \omega^2, \omega^4)$, where ω is a primitive 7-th root of unity. It follows that $B \otimes \mathbf{F}_8 = \mathbf{F}_8[x_1, x_2, x_3]^{\mathbf{Z}/7}$ is isomorphic to the ring B_2 discussed in Section 5. Its minimal resolution is determined there. The Betti numbers for this ring are

$$(1,\ 54,\ 320,\ 945,\ 1728,\ 2100,\ 1728,\ 945,\ 320,\ 54,\ 1).$$

The internal Betti numbers are given on the next page.

Remark 7.1. We remark that the rings $H^*\big((\mathbf{F}_8 \rtimes (\mathbf{F}_8^* \rtimes Gal)); \mathbf{F}_2\big) = \mathbf{F}_2[x, y, z]^{H_{21}}$ and $H^*(BG_2; \mathbf{F}_2) = \mathbf{F}_2[x, y, z]^{\mathrm{GL}_3(\mathbf{F}_2)}$, where Gal is the Galois group of F_8 over F_2 and G_2 is the exceptional Lie group, are also of interest to topologists. The first of these has generators $\{a, b, e, h, k\}$ and relations $\{a^2b + ak + bh + e^2,\ a^4 + a^2h + b^3 + ek + h^2\}$. The second is the polynomial ring $\mathbf{F}_2[b, h, k]$, which is also known as the rank 3 Dickson algebra.

The internal Betti numbers for $H^*(\mathbf{F}_8 \rtimes \mathbf{F}_8^*; \mathbf{F}_2)$.

d	β_{0d}	β_{1d}	β_{2d}	β_{3d}	β_{4d}	β_{5d}	β_{6d}	β_{7d}	β_{8d}	β_{9d}	$\beta_{10,d}$
0	1										
…											
8		3									
9		6									
10		12									
11		12									
12		12	2								
13		6	12								
14		3	30								
15			50								
16			66								
17			66	6							
18			50	24							
19			30	66							
20			12	117							
21			2	168							
22				183	6						
23				168	30						
24				117	84						
25				66	168						
26				24	258						
27				6	318	2					
28					318	21					
29					258	66					
30					168	152					
31					84	258					
32					30	357					
33					6	388	6				
34						357	30				
35						258	84				
36						152	168				
37						66	258				
38						21	318				
39						2	318	6			
40							258	24			
41							168	66			
42							84	117			
43							30	168			
44							6	183			
45								168	2		
46								117	12		
47								66	30		
48								24	50		
49								6	66		
50									66		
51									50		
52									30	3	
53									12	6	
54									2	12	
55										12	
56										12	
57										6	
58										3	
…											
66											1

Appendix. Homomorphisms for Example (2.2).

The minimal resolution of the ring B described in Example (2.2) has the form

$$0 \longrightarrow A \xrightarrow{d_4} A^9 \xrightarrow{d_3} A^{16} \xrightarrow{d_2} A^9 \xrightarrow{d_1} A \xrightarrow{d_0} B \longrightarrow 0,$$

where the homomorphisms d_1, d_2, d_3, d_4 are given respectively by the following matrices:

$$\begin{pmatrix} y_3^2-y_1y_4 & y_3y_5-y_1y_6 & y_3y_4-y_2y_5 & y_4^2-y_2y_6 & y_4y_5-y_3y_6 & y_5^2-y_1y_7 & y_4y_6-y_2y_7 & y_5y_6-y_3y_7 & y_6^2-y_4y_7 \end{pmatrix},$$

$$\begin{pmatrix}
y_4 & y_5 & 0 & y_6 & 0 & 0 & y_7 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-y_2 & -y_3 & y_5 & -y_4 & 0 & y_4 & -y_6 & 0 & y_6 & 0 & 0 & y_7 & 0 & 0 & 0 & 0 \\
-y_3 & 0 & 0 & 0 & y_4 & -y_5 & 0 & y_6 & 0 & 0 & y_7 & 0 & 0 & 0 & 0 & 0 \\
y_1 & 0 & 0 & 0 & -y_3 & 0 & 0 & 0 & 0 & -y_5 & 0 & 0 & y_6 & 0 & y_7 & 0 \\
0 & y_1 & 0 & y_3 & y_2 & 0 & 0 & 0 & y_5 & y_4 & 0 & 0 & 0 & y_6 & 0 & y_7 \\
0 & 0 & -y_3 & 0 & 0 & -y_2 & 0 & 0 & -y_4 & 0 & 0 & -y_6 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & y_1 & 0 & -y_3 & 0 & y_3 & -y_5 & 0 & -y_4 & 0 & -y_6 & 0 \\
0 & 0 & y_1 & 0 & 0 & 0 & y_3 & y_2 & 0 & -y_2 & y_4 & y_5 & 0 & -y_4 & 0 & -y_6 \\
0 & 0 & 0 & 0 & 0 & 0 & -y_1 & 0 & y_1 & 0 & 0 & 0 & y_2 & y_3 & y_4 & y_5
\end{pmatrix},$$

$$\begin{pmatrix}
-y_5 & y_6 & 0 & 0 & y_7 & 0 & 0 & 0 & 0 \\
y_4 & 0 & y_6 & 0 & 0 & y_7 & 0 & 0 & 0 \\
-y_2 & 0 & -y_4 & 0 & 0 & -y_6 & 0 & 0 & 0 \\
0 & -y_4 & -y_5 & 0 & 0 & 0 & 0 & y_7 & 0 \\
0 & -y_5 & 0 & y_6 & 0 & 0 & y_7 & 0 & 0 \\
y_3 & -y_4 & 0 & 0 & -y_6 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -y_4 & -y_5 & 0 & -y_6 & 0 \\
0 & y_3 & 0 & -y_4 & -y_5 & 0 & 0 & 0 & y_7 \\
0 & y_2 & y_3 & 0 & 0 & 0 & 0 & -y_6 & 0 \\
-y_1 & y_3 & 0 & 0 & 0 & 0 & y_6 & 0 & y_7 \\
0 & 0 & 0 & 0 & y_3 & 0 & -y_4 & 0 & -y_6 \\
0 & 0 & 0 & 0 & y_2 & y_3 & 0 & y_4 & 0 \\
0 & -y_1 & 0 & y_3 & 0 & 0 & y_5 & 0 & 0 \\
0 & 0 & -y_1 & -y_2 & 0 & 0 & -y_4 & y_5 & 0 \\
0 & 0 & 0 & 0 & -y_1 & 0 & y_3 & 0 & y_5 \\
0 & 0 & 0 & 0 & 0 & -y_1 & -y_2 & -y_3 & -y_4
\end{pmatrix},$$

$$\begin{pmatrix}
y_6^2-y_4y_7 \\
y_5y_6-y_3y_7 \\
-y_4y_6+y_2y_7 \\
y_5^2-y_1y_7 \\
-y_4y_5+y_3y_6 \\
y_4^2-y_2y_6 \\
-y_3y_5+y_1y_6 \\
-y_3y_4+y_2y_5 \\
y_3^2-y_1y_4
\end{pmatrix}.$$

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