5-Chromatic Steiner Triple Systems

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Abstract

We show that, up to an automorphism, there is a unique independent set in PG(5,2) that meet every hyperplane in 4 points or more. Using this result, we show that PG(5,2) is a 5-chromatic STS. Moreover, we construct a 5-chromatic STS(v) for every admissible $v \geq 127$.

Introduction

A *Steiner triple system* of order v (an STS(v)) is an ordered pair $\mathcal{S} = (V, T)$ where V is a set of order v and T is a collection of triples of V such that every pair from V is contained in exactly one triple of T. It is well known that a necessary and sufficient condition for an STS(v) to exist is that $v \equiv 1$ or $3 \pmod 6$. Such a v is said to be *admissible.* A (weak) n-colouring of an STS(v) is a map $\mathbf{c} : V \to \{1, \ldots, n\}$ such that $|\{c(x), c(y), c(z)\}| \geq 2$ for every triple $\{x, y, z\}$ of T (here we say that no triple of T is monochromatic). If $\mathbf{c}$ is a colouring of an STS(v), then it is customary to identify the different colourings obtained from $\mathbf{c}$ by permuting the colours. Thus an n-colouring of an STS(v) gives rise to a partition of its vertex set into n subsets so that none of

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them contains a triple of T. A Steiner triple system, $\mathcal{S}$, is *k-chromatic* if it admits a k-colouring but not a $(k-1)$-colouring. In this case we say that $\mathcal{S}$ has chromatic number k and we write $\chi(\mathcal{S}) = k$.

The problem of colouring Steiner triple systems is a generalization of that for graphs. Indeed we can consider a colouring of a graph as an assignment of colours to its vertices such that no edge is monochromatic. The theory of colouring Steiner triple systems differs from that of colouring graphs in certain fundamental ways. For example, any bipartite graph is 2-chromatic while it is well-known that every STS(v) with $v \geq 7$ is at least 3-chromatic [14]. Colouring problems for Steiner triple systems and for hypergraphs are hard in general. For example, no specific example of a k-chromatic Steiner triple system is known for any $k \geq 5$ and the existence of such objects is shown only by non-constructive methods [2]. On the other hand several examples of 3 and 4-chromatic Steiner triple systems are known [2,11,12,14].

An interesting class of Steiner triple systems is the family $\mathcal{S}_n^*$ for $n \geq 2$. We can construct $\mathcal{S}_n^*$ by considering $\mathbb{F}^{n+1}$, the vector space of dimension $n+1$ over the field of order 2. The elements of $\mathcal{S}_n^*$ are the one dimensional subspaces of $\mathbb{F}^{n+1}$ and the triples are the two dimensional subspaces. Expressed another way, the elements of $\mathcal{S}_n^*$ are the points of the projective n-space, PG(n,2), and the triples are the lines in PG(n,2). It is easy to see that $\chi(\mathcal{S}_2^*) = \chi(\mathcal{S}_3^*) = 3$. A. Rosa in [14] showed that $\chi(\mathcal{S}_4^*) = 4$, and $\chi(\mathcal{S}_n^*) \leq \chi(\mathcal{S}_n^* + 1) \leq \chi(\mathcal{S}_n^*) + 1$. Independently, J. Pelikán also showed that $\chi(\mathcal{S}_4^*) = 4$ [12]. More than twenty years ago Rosa [14] asked whether $\chi(\mathcal{S}_5^*) = 4$ or 5; the question was repeated recently in [16]. In this paper we prove that $\chi(\mathcal{S}_5^*) = 5$ and hence provide the first specific example of a 5-chromatic Steiner triple system. This is done by studying independent sets that meet every hyperplane in t points and we prove some results concerning these sets that are of interest in their own right (Theorem 2, Corollaries 3 and 4 and Proposition 5). In addition, we prove that if a k-chromatic STS(v) exists with $v \equiv 3 \pmod 6$ and $k \geq 5$ then a k-chromatic STS(u) exists for every admissible $u \geq 2v+1$. A combination of these two results constructs a 5-chromatic STS(v) for every admissible $v \geq 127$.

Preliminaries

Let $\mathbb{F}^{n+1}$ denote the $n+1$ dimensional vector space over $\mathbb{F} := GF(2)$, the field of order 2. The elements of $\mathcal{S}_n^*$ are the one dimensional subspaces of $\mathbb{F}^{n+1}$. Each such subspace is represented by the non-zero vector contained in it. For ease of notation, if $\{e_0, \ldots, e_n\}$ is a basis of $\mathbb{F}^{n+1}$ and x is an element of $\mathcal{S}_n^*$, then we denote x by $a_1 \cdots a_s$ where $x = e_{a_1} + \ldots + e_{a_s}$ is the unique expansion of x in the given basis. For example the element $x = e_0 + e_2 + e_3$ is denoted 023. We will call elements of $\mathcal{S}_n^*$ *words* and we say that $a_1 \cdots a_s$ is a word of *length* s (with respect to the basis $\{0, \ldots, n\}$). Note that these words are *commutative*. To complete the definition of $\mathcal{S}_n^*$ we still need to specify the set of triples T: $\{x, y, z\}$ is a triple if and only if $x + y + z = \vec{0}$, the zero vector. We remark in passing that $\mathcal{S}_n^*$ is the unique totally associative Steiner triple system of order $2^{n+1} - 1$ [14].

A subset X of $V(\mathcal{S}_n^*)$ is called *independent* if it is line free, i.e., if it does not contain any triple of T. Moreover it is called *linearly independent* if it is a linearly independent subset of $\mathbb{F}^{n+1}$. For $2 \leq k \leq n$, a subset $K \subseteq V(\mathcal{S}_n^*)$ is a *k-flat* if $K \cup \{\vec{0}\}$ is a $(k+1)$ dimensional subspace of $\mathbb{F}^{n+1}$. Note that the set of triples, T, restricted to a k-flat, K, induces on K the structure of a Steiner triple system isomorphic to $\mathcal{S}_k^*$.

If $X \subseteq \mathcal{S}_n^*$, then $\langle X \rangle$ represents the smallest k-flat containing X. A *plane* is a 2-flat while a *hyperplane* is an $(n-1)$-flat. Now it is well known that K is a hyperplane if and only if there is a $u_0 \in V(\mathcal{S}_n^*)$ such that

$$K = K_{u_0} := \{\, v \in V(\mathcal{S}_n^*) \ :\ v \cdot u_0 = 0 \,\},$$

where $\cdot$ represents the usual $\mathbb{F}$ inner (or dot) product on $\mathbb{F}^{n+1}$.

In the sequel we consider $n = 5$. We will often consider the hyperplanes $V_0 := K_{12345}, V_1 := K_{02345}, \ldots, V_5 := K_{01234}$ and the "even" hyperplane $V_e := K_{012345}$ (notice that this is the set of all words of even length). Moreover for $j = 1, \ldots, 6$ we will denote by X_j the subset of $V(\mathcal{S}_5^*)$ that consists of all words of length j. Note that a word of odd length belongs to V_i iff it contains i, and a word of even length belongs to V_i iff it does not contain i, $i = 0, \ldots, 5$.

Technical Results

Let C be an independent subset of $V(\mathcal{S}_5^*)$.

Lemma 1 *If C does not contain a basis, then there is a hyperplane K such that $|C \cap K| \leq 2$.*

Proof: We may assume w.l.o.g. that C contains no word with the letter 5.

Case 1. Suppose $\dim(\mathrm{Span}(C)) \leq 3$. Then C contains at most 3 linearly independent vectors, say $\{0{,}1{,}2\} \subseteq C$, and therefore $C \cap V_e = \emptyset$.

Case 2. $\dim(\mathrm{Span}(C)) = 4$. Let $\{0, 1, 2, 3\} \subseteq C$. As 0123 is the only element of V_e that might belong to C, we get $|C \cap V_e| \leq 1$.

Case 3. Now let $\dim(\mathrm{Span}(C)) = 5$. As above assume $\{0, 1, 2, 3, 4\} \subseteq C$. If $|C \cap V_e| \leq 2$, then we are done. So let $|C \cap V_e| \geq 3$, and w.l.o.g. we may assume that $\{0123, 0124, 0234\} \subset C$. This implies that C contains at most one more word, namely 134 and so $|C \cap V_0| \leq 2$. ∎

In the following Theorem, we suppose that the independent set C contains a basis denoted by $\{0,\ldots,5\}$.

Put $C_4 := \{0,1,2,3,4,5,0123,0124{,}0125,012345,034,035,045,134,135,145,234,$ $235,245,345\}$. Note that C_4 meets every hyperplane of $\mathcal{S}_5^*$ in at least 4 points.
An *automorphism* of a STS, $\mathcal{S}=(V,T)$, is a bijection $f:V\to V$ such that $\{f(x),f(y),f(z)\}\in T$ for every triple $\{x,y,z\}\in T$. Let $Aut(\mathcal{S})$ denote the *automorphism group* of $\mathcal{S}$. For $\mathcal{S}=\mathcal{S}_n^*$ (since we are working over the field of order 2) this automorphism group is just the group of vector space automorphisms of $\mathbb{F}^{n+1}$.

Theorem 2 *If the independent set C satisfies $|C\cap K|\geq 4$ for every hyperplane K, then $C=f(C_4)$ for some $f\in Aut(\mathcal{S}_5^*)$.*

Proof: The proof will be given as follows. In our first step we assume $|C\cap X_4|\leq 4$. First we start with the case where $|C\cap X_4|=3$. We show that in this case C is an automorphic image of C_4 (Claim 1). Next we consider the case where $|C\cap X_4|=4$. We show that the four words in $C\cap X_4$ must share a common letter (Claim 2) and again in this case C is an automorphic image of C_4 (Claim 3). In a second step we assume $|C\cap X_4|\geq 5$ and moreover that the following holds: For every basis $\{v_0,\ldots,v_5\}$ contained in C, we have $|C\cap f(V_e)|\geq 5$, where f is the automorphism defined by $f(i)=v_i$ for $i=0,1,\ldots,5$.

Step 1

Claim 1 *If the set C contains exactly 3 words of length four and moreover satisfies $|C\cap K|\geq 4$ for every hyperplane K, then $C=f(C_4)$ for some $f\in Aut(\mathcal{S}_5^*)$.*

Proof: Since $C\cap V_e\subset(C\cap X_4)\cup\{012345\}$ we must have $012345\in C$. Now by a simple counting argument we see that w.l.o.g. either $\{0123,0145,2345\}$ or $\{0123,0124\}$

is contained in $C \cap X_4$. Since the former set is a line we have $\{0123, 0124\} \subseteq C$. Hence $C \cap \{012, 013, 023, 014, 024, 123, 124\} = C \cap X_5 = \emptyset$. Note that the three letters 0, 1 and 2 on one hand and 3 and 4 on the other hand play symmetric roles. Let α_3 be the third word of length four that belongs to C.

Case 1. None of the pairs $\{01, 02, 12\}$ appears in α_3. Then $\alpha_3 = i345$, for some $i \in \{0, 1, 2\}$. This implies that $C \cap \{i34, i35, i45\} = \emptyset$ and so $|C \cap V_i| \leq 3$, a contradiction.

Case 2. One of the pairs $\{01, 02, 12\}$, say 01, appears in α_3 but 012 does not appear in α_3. First suppose that $0134 \in C$. Then neither 034 nor 134 belong to C. Hence the inequality $|C \cap V_0| \geq 4$ implies that $|C \cap \{015, 025, 035, 045\}| \geq 3$. A similar argument for V_1 gives $|C \cap \{015, 125, 135, 145\}| \geq 3$. If $025 \in C$, then $025 + 0123 = 135 \notin C$ and $025 + 0124 = 145 \notin C$, a contradiction. Thus $025 \notin C$. Similarly $035 \in C$ implies that neither 125 ($= 035 + 0123$) nor 145 ($= 035 + 0134$) belongs to C and again this is a contradiction. This shows that $0134 \notin C$. Suppose now that $0135 \in C$. Then $C \cap \{015, 035, 135\} = \emptyset$ and so the inequalities $|C \cap V_i| \geq 4$ for $i = 0, 1$ imply that $\{025, 034, 045, 125, 134, 145\} \subseteq C$, which is a contradiction since $025 + 145 = 0124 \in C$. Since the letters 3 and 4 play symmetric roles, $0145 \in C$ is also impossible.

Case 3. 012 appears in α_3, i.e., $\alpha_3 = 0125 \in C$. Hence $C \cap \{015, 025, 125\} = \emptyset$. In this case the inequalities $|C \cap V_i| \geq 4$ imply $\{i34, i35, i45\} \subseteq C$ for $i = 0, 1, 2$, and so $\{0, 1, 2, 3, 4, 5, 0123, 0124, 0125, 012345, 034, 035, 045, 134, 135, 145, 234, 235, 245\} = C_4 \subseteq C$. Moreover $|C \cap K_{012}| \geq 4$ implies $345 \in C$. This shows that $C = f(C_4)$ for some $f \in \mathrm{Aut}(\mathcal{S}_5^*)$ and completes the proof of Claim 1. ∎

Claim 2 *If the set C satisfies $|C \cap X_4| = 4$ and no single letter appears in all four* *of these words of length* 4*, then* $|C \cap K| \leq 3$ *for some hyperplane* K *of* $\mathcal{S}_5^*$.

Proof: Let $C \cap X_4 = \{\alpha_1, \ldots, \alpha_4\}$. We may assume w.l.o.g. that $\alpha_1 = 0123$ and $\alpha_2 = 0124$. Moreover, as none of 0,1,2 appears in both α_3 and α_4, we have that $\{\alpha_3, \alpha_4\} \cap \{0345, 1345, 2345\} \neq \emptyset$. Since 0, 1 and 2 play symmetric roles we may assume w.l.o.g. that $\alpha_3 = 0345 \in C$. Thus $C \cap \{012, 013, 023, 014, 024, 034, 035, 045, 123, 124, 345, 01234, 01235, 01245, 01345, 02345\} = \emptyset$. Note that 0 does not appear in α_4, and consequently $12345 \notin C$, i.e., $C \cap X_5 = \emptyset$. Now $|C \cap V_0| \geq 4$ implies that $C \cap V_0 = \{0, 015, 025, \alpha_4\}$. We have two cases.

Case 1. The pair 12 does not appear in α_4, and so we may assume $\alpha_4 = 1345$ (note that the case $\alpha_4 = 2345$ is equivalent). Then $C \cap \{134, 145, 12345\} = \emptyset$, and so $|C \cap V_4| \geq 4$ implies that $C \cap V_4 = \{4, 234, 245, 0123\}$, a contradiction as $015 + 245 = 0124 \in C$.

Case 2. $\alpha_4 = 12ij$ where $i \neq j \in \{3, 4, 5\}$. If $\alpha_4 = 1234$, then $C \cap \{134, 234, 12345\} = \emptyset$ and so $|C \cap V_4| \geq 4$ implies that $245 \in C$ and, as in case 1, $015 + 245 = 0124 \in C$, a contradiction. Now suppose $\alpha_4 = 1245$ (note that the case $\alpha_4 = 1235$ is equivalent). Then as before $|C \cap V_4| \geq 4$ implies $134 \in C$ which gives the contradiction $015 + 134 = 0345 \in C$. This completes the proof of Claim 2. ▪

Claim 3 *If the set* C *satisfies* $|C \cap X_4| = 4$ *and these four words all contain a common letter, and moreover if* $|C \cap K| \geq 4$ *for every hyperplane* K*, then* $C = f(C_4)$ *for some* $f \in Aut(\mathcal{S}_5^*)$.

Proof: Set $C \cap X_4 = \{\alpha_1, \ldots, \alpha_4\}$ and suppose that there is a letter, say 0, that appears in all of α_j, $j = 1, \ldots, 4$. Then w.l.o.g. we may assume that $\alpha_1 = 0123$ and $\alpha_2 = 0124$. Hence $C \cap \{012, 013, 023, 014, 024, 123, 124, 01234, 01235, 01245\} = \emptyset$. We distinguish four cases:

Case 1. $0345 \in C$. Then $C \cap \{034, 035, 045, 02345, 01345\} = \emptyset$, and so $C \cap V_0 \subseteq \{0, 015, 025\}$ contradicting $|C \cap K| \geq 4$ for every hyperplane K.

Case 2. The letter 5 appears in neither α_3 nor in α_4. Hence $C \cap X_4 = \{0123, 0124, 0134, 0234\}$. Then $C \cap (V_0 \cap X_5) = \emptyset$, and $C \cap V_0 \subseteq \{0, 015, 025, 035, 045\}$. Moreover $C \cap V_1 \subseteq \{1, 015, 125, 135, 145, 12345\}$. Note that the letters 1 and 2 on one hand and 3 and 4 on the other hand play symmetric roles. Now as $|C \cap V_0| \geq 4$, the set C must contain at least one element of each of $\{015, 025\}$ and $\{035, 045\}$. By symmetry we can assume that $\{015, 035\} \subseteq C$. Thus $015 + 0234 = 12345 \notin C$ and moreover $035 + 0123 = 125 \notin C$, $035 + 0134 = 145 \notin C$. These give $C \cap V_1 \subseteq \{1, 015, 135\}$ and thus $|C \cap V_1| \leq 3$, again this is a contradiction.

Case 3. $\alpha_3 = 0125$. Here $0, 1, 2$ on the one hand and $3, 4, 5$ on the other hand play symmetric roles and thus w.l.o.g. we may assume that $\alpha_4 = 0134$. Hence $C \cap \{015, 025, 034, 125, 134, 01345\} = \emptyset$, and so for $i = 0, 1$ the inequality $|C \cap V_i| \geq 4$ implies that $\{i35, i45, i2345\} \subseteq C$. This is impossible since $035 + 0124 = 12345 \notin C$.

Case 4. In this final case we assume that (i) neither 0125 nor 0345 belong to C and (ii) 5 appears in either α_3 or α_4. Let 5 appear in α_3. Now since α_3 has the form $0ij5$ where $i \in \{1, 2\}$ and $j \in \{3, 4\}$, we may assume w.l.o.g that $\alpha_3 = 0135$. This implies that $C \cap \{015, 035, 135, 01345\} = \emptyset$, and so $C \cap V_0 \subseteq \{0, 025, 034, 045, 02345\}$. Note that 1,125,134,145,12345 and α_4 (if it is not written with the letter 1) are the only elements of V_1 that may belong to C. If $045 \in C$ then $045 + 0124 = 125 \notin C$ and $045 + 0135 = 134 \notin C$. In this case $|C \cap V_1| \geq 4$ would imply that $12345 \in C$ and so $045 + 12345 = 0123 \in C$, a contradiction. So suppose $045 \notin C$, and thus $|C \cap V_0| \geq 4$ implies that $\{0, 025, 034, 02345\} \subseteq C$. As $02345 \in C$, we deduce that $\alpha_4 \notin \{0234, 0235, 0245\}$, i.e., $\alpha_4 = 0145$. Hence $145 \notin C$ and so $|C \cap V_1| = 4$ implies that $\{1, 125,$

$134, 12345\} \subseteq C$. Consequently we have

$$\{0,1,2,3,4,5,025,034,125,134,0123,0124,0135,0145,02345,12345\} \subseteq C.$$

So far $C \cap K_{0125} = \{3,4\}$ and $C \cap K_{0134} = \{2,5\}$. The only elements of K_{0125} which are available to be added to C are 235 and 245. Similarly the only elements of K_{0134} available are 234 and 345. Hence

$C' := \{0,1,2,3,4,5,025,034,125,134,234,235,245,345,0123,0124,0135,0145,02345, 12345\} \subseteq C$. Now C' is a maximal independent subset of $V(\mathcal{S}_5^*)$ and thus $C = C'$. Since the automorphism defined by $f(0) = 0123$, $f(1) = 345$, $f(2) = 134$, $f(3) = 025$, $f(4) = 02345$ and $f(5) = 0$ carries C_4 to C, the proof Claim 3 is complete. ▪

Step 2

Claims 1, 2 and 3 dispose of the possibility that $|C \cap X_4| \leq 4$. Hence we now assume that $C \cap X_4 = \{\alpha_1, \ldots, \alpha_n\}$, where $n \geq 5$. Since $X_4 \subseteq V_e$, we have $|C \cap V_e| \geq 5$. Consequently, we may assume that the following property, denoted by P, holds:

For every basis $\{v_0, \ldots, v_5\}$ contained in C, we have $|C \cap f(V_e)| \geq 5$, where f is the automorphism defined by $f(i) = v_i$ for $i = 0, 1, \ldots, 5$.

Now by an elementary counting argument we see that at least 4 of $\alpha_1, \ldots, \alpha_n$, (say $\alpha_1, \ldots, \alpha_4$), contain the same letter, say 0. We may assume w.l.o.g. that $\alpha_1 = 0123$ and $\alpha_2 = 0124$. Hence $C \cap \{012, 013, 014, 023, 024, 123, 124, 01234, 01235, 01245\} = \emptyset$. Also by a counting argument we see that at least one of α_3, α_4 must be of the form $0ijk$ where $i \in \{1,2\}$, say $\alpha_3 = 01jk \in C$ where $j \neq k \in \{2,3,4,5\}$. We distinguish 3 cases:

Case 1. $\alpha_3 = 0125$ (so, $\{0123, 0124, 0125\} \subseteq C$). Then $C \cap \{015, 025, 125\} = \emptyset$. First suppose that $0345 \in C$. Then $C \cap \{034, 035, 045, 01345, 02345, 1245, 1235, 1234\} = \emptyset$, and so $C \cap V_0 \subseteq \{0, 1345, 2345\}$, a contradiction. Similarly we can show that neither 1345 nor 2345 belongs to C. Now since α_4 is written with 0, and since there are automorphisms fixing the set $\{0,1,2,3,4,5,0123,0124,0125\}$ and permuting 0,1,2 on one hand and 3,4,5 on the other hand, we may assume that $\alpha_4 = 0134 \in C$. Hence $C \cap \{034, 134, 01345\} = \emptyset$. First we show that $C \cap X_5 = \emptyset$. Suppose $12345 \in C$, then $C \cap (V_0 \cap X_4) = \emptyset$. Moreover since $0123 + 12345 = 045 \notin C$ and $0124 + 12345 = 035 \notin C$, we get that $C \cap V_0 \subseteq \{0, 02345\}$, a contradiction. A similar argument holds if $02345 \in C$, this shows that $C \cap X_5 = \emptyset$. Consider the ordered basis $\mathcal{B} := \{0, 2, 4, 0124, 0125, 0134\}$ of $\mathbb{F}^6$ contained in C. If f is the automorphism that maps the ordered basis $\{0,1,2,3,4,5\}$ to $\mathcal{B}$, then $f(V_e) = V_1$. Applying property P we obtain that $|C \cap V_1| \geq 5$. Since 0 and 1 play symmetric roles, we also have $|C \cap V_0| \geq 5$. Now $C \cap X_3 \cap (V_0 \cup V_1) \subseteq \{035, 045, 135, 145\}$. Since $035 + 145 = 045 + 135 = 0134 \in C$, we have that $|C \cap X_3| \leq 2$. If both 035 and 045 belong to C, then $C \cap V_1$ must contain at least 4 words of length four (recall that $C \cap X_5 = \emptyset$). This is impossible since neither of 0345 and 2345 belongs to C. Hence at most one of 035 and 045 belongs to C. Consequently by the inequality $|C \cap V_0| \geq 5$, C must contain at least 3 words of length four written without 0. Since neither 1345 nor 2345 belongs to C, we deduce that $\{0234, 0235, 0245\} \subseteq C$. Again as 0 and 1 play symmetric roles, we have $\{1234, 1235, 1245\} \subseteq C$. This yields the contradiction $0235 + 1245 = 0134 \in C$.

Case 2. $\alpha_3 = 0135$ (so $\{0123, 0124, 0135\} \subseteq C$). Hence $C \cap \{015, 035, 135, 01345\} = \emptyset$, and so $C \cap X_3 \subseteq \{025, 034, 045, 125, 134, 145, 234, 235, 245, 345\}$. This gives $(C \cap V_i) \setminus X_4 \subseteq \{i, i25, i34, i45, i2345\}$, for $i = 0, 1$. Now C contains the basis $\{0, 2, 3, 0123, 0124, 0135\}$. Applying property P to this basis, we obtain $|C \cap V_1| \geq 5$. Again as 0 and 1 play symmetric roles we also have $|C \cap V_0| \geq 5$. Moreover as $025 + 145 = 125 + 045 = 0124 \in C$, we deduce that C cannot contain both 025 and 145, and also it cannot contain both 125 and 045. On the other hand $0124 + 0135 =$

$2345 \notin C$.

If $02345 \in C$, then $(C \cap V_1) \cap X_4 = \emptyset$ and so $C \cap V_1 \subseteq \{1, 125, 134, 145, 12345\}$, which gives the contradiction $145 + 02345 = 0123 \in C$. Similarly $12345 \notin C$ and therefore $C \cap X_5 = \emptyset$.

Let $\{k, \ell\} = \{0, 1\}$ and suppose that $k25 \in C$. Then as $\ell 45, k235, k245 \notin C$, we obtain that $C \cap V_\ell = \{\ell, \ell 25, \ell 34, k234, k345\} \Rightarrow (C \cap V_k) \setminus X_4 \subseteq \{k, k45\}$. This gives that $|C \cap V_k \cap X_4| \geq 3$, which is impossible as neither $\ell 235$ $(= 3 + \ell 25)$ nor $\ell 245$ $(= 4 + \ell 25)$ belong to C. Similarly we show that $k45 \notin C$. It follows that $(C \cap V_i) \setminus X_4 \subseteq \{i, i34\}$, for $i = 0, 1$.

Now $|C \cap V_i| \geq 5$ for $i = 0, 1$ imply that C must contain at least 3 elements of each of $\{i234, i235, i245, i345\}$, i$= 0, 1$. However C can contain at most one element from each of the sets $\{0234, 1245\}$, $\{0235, 1345\}$ and $\{0245, 1234\}$ because these sets have sums in C, so $|C \cap V_0| \geq 5$ and $|C \cap V_1| \geq 5$ cannot hold simultaneously, this contradiction settles this case. As 3 and 4 play symmetric roles we deduce that $\alpha_3 = 0145 \in C$ is impossible.

Case 3. $\alpha_3 = 0134$, (i.e., $\{0123, 0124, 0134\} \subseteq C$). Note that the set $\{0123, 0124, 0134\}$ is invariant under the action of the permutation group, $\Sigma_2 \times \Sigma_3$ (where Σ_2 acts on the set $\{0, 1\}$ and Σ_3 acts on the set $\{2, 3, 4\}$). We have two subcases:

Case 3. a. $\alpha_4 = 0235$. Here the permutation (12) applied to $\mathcal{S}_5^*$ transforms the set $\{0123, 0124, 0235\}$ to $\{0123, 0124, 0135\}$ and reduces this case to our Case 2. By symmetry the cases $\alpha_4 = 0245$ and $\alpha_4 = 0345$ are similar.

Case 3. b. $\alpha_4 = 0234$. Then $X := \{0123, 0124, 0134, 0234\} \subseteq C$. Note that the set X is symmetric with respect to $1, 2, 3$ and 4. Moreover the above cases show that C cannot contain any other word of length four written with the letter 0. However

$|C \cap X_4| \geq 5$, so at least one of 1234,1235,1245,1345 or 2345 belongs to C. By the symmetry of the set X all cases $ijk5 \in C$ are similar where $ijk \in \{123, 124, 134, 234\}$. If $1235 \in C$, then the permutation (021) applied on the set $\{0123, 0124, 1235\}$ gives $\{0123, 0124, 0135\} \subseteq C$, which is already settled. We are left with the case $1234 \in C$. Here $C \cap X_4 = \{0123, 0124, 0134, 0234, 1234\}$. This implies $C \cap X_5 = \emptyset$ and $C \cap X_3 \subseteq V_5$. Now $|C \cap V_0| \geq 4$ implies $|C \cap (V_0 \cap X_3)| \geq 2$. Note that the letters 0, 1, 2, 3 and 4 play symmetric roles. Therefore we may assume that $\{015, 025\} \subseteq C$ and so none of 135 $(= 025 + 0123)$, 235 $(= 015 + 0123)$ or 345 $(= 015 + 0134)$ belongs to C. Thus $C \cap V_3 \subseteq \{3, 035, 0124\}$, which contradicts $|C \cap V_3| \geq 4$. This disposes of the last case and the proof of Theorem 2 is complete. ∎

Note that $C_4 \cap V_i = \{i, i34, i35, i45\}$ for $i = 0, 1, 2$, and it is easy to see that no other element of V_i may be added to C_4 for $i = 0, 1, 2$. Thus we have:

Corollary 3 *$V(\mathcal{S}_5^*)$ has no independent subset C which meets every hyperplane of $\mathcal{S}_5^*$ in five or more points.* ∎

A combination of Lemma 1 and Theorem 2 gives the following consequence that will be used in the proof of Theorem 8:

Corollary 4 *If C is independent and satisfies $|C| \leq 19$, then there is a hyperplane K such that $|C \cap K| \leq 3$.*∎

Remark Theorem 2 shows that the size of an independent set that meets every hyperplane of $\mathcal{S}_5^*$ in at least four points is 20. Although we do not make use of the following result in this paper, it shows that if we require our independent set to meet every hyperplane in at least 3 points, then it must have at least 12 elements. We have:

Proposition 5 *If C is an independent subset of $V(\mathcal{S}_5^*)$ and $|C| \leq 11$, then there is a hyperplane K of $\mathcal{S}_5^*$ such that $|C \cap K| \leq 2$.*

Proof: Suppose for the sake of contradiction, that C meets every hyperplane in at least 3 points. As $|C| \leq 11$ we have by Theorem 2 that there is a hyperplane K such that $|C \cap K| = 3$. It is easy to see that w.l.o.g. we may assume that $C \cap K = \{0, 1, 2\}$. As the set C contains a basis, we may extend the set $\{0, 1, 2\}$ to a basis $\{0, 1, 2, 3, 4, 5\} \subseteq C$. Now there are 7 hyperplanes containing the set {0,1,2} namely $\langle 0, 1, 2, i, j\rangle$ where

$$(i, j) \in \{(3, 4), (3, 5), (4, 5), (3, 45), (4, 35), (5, 34), (34, 35)\}.$$

Since $\{0, \ldots, 5\} \subseteq C$ and $|C \cap K| = 3$ we deduce that $K = \langle 0, 1, 2, 35, 45\rangle$. Hence $C \cap \{012, 034, 134, 234, 035, 135, 235, 045, 145, 245, 0134, 0234, 1234, 0135, 0235, 1235, 0145, 0245, 1245, 01234, 01235, 01245\} = \emptyset$. Note that the letters 0,1,2 on one hand and 3,4,5 on the other hand play symmetric roles. First we show that $C \cap X_5 = \emptyset$. Indeed suppose $02345 \in C$, then $C \cap (V_1 \cap X_4) = \emptyset$ and $012345 \notin C$. Therefore C contains at least 3 words of length four since $|C \cap V_e| \geq 3$, and at least 2 more words (of length three or five) of V_1, contradicting $|C| \leq 11$. Similarly we can show that neither 12345 nor 01345 belongs to C. On the other hand the inequality $|C \cap V_e| \geq 3$ implies that C contains at least two elements of $Y_1 \cup Y_2$ where $Y_1 := \{0123, 0124, 0125\}$ and $Y_2 := \{0345, 1345, 2345\}$. We distinguish 3 cases:

Case 1. C contains at least one word from each of Y_1 and Y_2. By symmetry we may assume that both 0123 and 0345 belong to C, hence $C \cap \{013, 023, 123, 345\} = \emptyset$. Now $|C \cap V_3| \geq 3$ implies that $\{0124, 0125\} \subseteq C$. Moreover $|C \cap V_0| \geq 3$ implies that C must contain at least two more elements, contradicting $|C| \leq 11$.

Case 2. $C \cap X_4 \subseteq Y_1$. As C contains at least two words of length four, we may assume by symmetry that $\{0123, 0124\} \subseteq C$. Hence $C \cap \{013, 023, 123, 014, 024, 124\} = \emptyset$. Moreover since $C \cap Y_2 = \emptyset$ and $|C| \leq 11$, the inequalities $|C \cap V_i| \geq 3$ for $i = 0, 1, 2$ imply that $\{015, 025, 125\} \subseteq C$ and so $|C \cap V_3| = |C \cap V_4| = 2$, a contradiction.

Case 3 . $C \cap X_4 \subseteq Y_2$. As in Case 2, we may assume that $\{0345, 1345\} \subseteq C$. Then $345 \notin C$, and consequently $C \cap (V_i \cap V_j) = \emptyset$ for all $i \neq j \in \{3, 4, 5\}$ (indeed $C \cap Y_1 = C \cap X_5 = \emptyset$ and moreover $C \cap (V_i \cap V_j \cap X_3) = \emptyset$, for all $i \neq j \in \{3, 4, 5\}$). Now the inequalities $|C \cap V_j| \geq 3$, for $j = 3, 4, 5$ give the contradiction $|C| \geq |C \cap X_1| + \sum_{j=3}^{5} |(C \cap V_j) \setminus X_1| \geq 6 + 3 \cdot 2 = 12$. ∎

On the other hand the set

$$C_3 := \{0, 1, 2, 3, 4, 5, 025, 125, 0123, 0124, 2345, 01345\},$$

is independent, has size 12 and meets every hyperplane of $\mathcal{S}_5^*$ in at least 3 points.

Colouring Projective Spaces and the Main Result

Now we consider $\mathcal{S}_3^*$ and its colourings. Note that this is a 3-chromatic STS(15) (actually all STS(v) with v admissible, $v \leq 15$, are 3-chromatic [11]). The following result is due to Pelikán [12]:

Proposition 6 *In any 3-colouring of $\mathcal{S}_3^*$ all colour classes have the same size and every plane contains at most 3 points of the same colour.* ∎

It is mentioned in [2] that there exists a uniquely 3-colourable STS(33). The following result shows that $\mathcal{S}_3^*$ has a similar but weaker property that will be very useful for the proof of our main result. First note that from Proposition 6 we can easily deduce that no matter how we 3-colour $\mathcal{S}_3^*$, any colour class will consist of 4 linearly independent vectors and their sum. Let G denote the group $Aut(\mathcal{S}_3^*)$.

Proposition 7 *$\mathcal{S}_3^*$ is uniquely 3-colourable up to an automorphism.*

Proof: Let $\langle 0,1,2,3\rangle$ be a copy of $\mathcal{S}_3^*$ with $\mathbf{c} : V(\mathcal{S}_3^*) \to \{1,2,3\}$ a 3-colouring. Set $C_i := \mathbf{c}^{-1}(i),\ i = 1,2,3$. By the above remark, we may apply an element of G to ensure that $C_1 = \{0,1,2,3,0123\}$. Since there are 6 words of length two, one colour class, say C_2, must contain at least three of them. Call those three words of length two a,b and c. Since a,b and c are independent, they generate a plane $\mathcal{P}$. Now $\mathcal{P}$ contains only words of even length and by Proposition 6, $\mathcal{P} \cap C_2 = \{a,b,c\}$. This shows that for $i = 2,3$, C_i contains exactly three words of length two and two words of length three.

Since C_2 contains three words of length two, at least two of them must have a common letter. Put $\{i,j,k,s\} = \{0,1,2,3\}$ and let $ij,\ ik \in C_2$. Then $kj \notin C_2$ and moreover $is \in C_2$ would imply $\{jk, js, ks\} \subseteq C_3$, a contradiction. Thus C_2 must contain a subset of the form $\{ij, ik, js\}$. We want to show that given three words of length two in C_2, the remaining two words of length three in C_2 are uniquely determined. So let $\{ij, ik, js\} \subseteq C_2$ and suppose that C_2 contains the word jks. Then this would imply $ij+ik+js+jks = j \in C_2$, a contradiction. By a similar argument we have $iks \notin C_2$ and we are left with $\{ijk, ijs\} \subseteq C_2$. Hence $C_2 = \{ij, ik, js, ijk, ijs\}$. Since the group of permutations of $\{0,1,2,3\}$ is a subgroup of G the result is proved.

∎

Remark: From the above proof we may count the number of distinct colourings of $\mathcal{S}_3^*$. First we observe that there are $15 \cdot 14 \cdot 12 \cdot 8 = 20160$ different (ordered) bases of $\mathcal{S}_3^*$. Thus $|G| = 20160$. Now each possible C_1 contains $\binom{5}{4}$ bases of $\mathcal{S}_3^*$ and there are $4!$ orders in which we could arrange each such basis. Therefore there are $20160/(5 \cdot 24) = 168$ possible choices for C_1. Having fixed C_1 the there are $\binom{4}{2}$ ways to choose $\{i,j\}$ and 2 ways to order the set $\{k,s\}$. Hence there are $168 \cdot 6 \cdot 2 = 2016$

different colourings of $\mathcal{S}_3^*$.

Let $\mathbf{c} : V(\mathcal{S}_3^*) \to \{1,2,3\}$ be any one of these colourings and set $C_i := \mathbf{c}^{-1}(i)$ for $i = 1,2,3$. Consider the subgroup $A(\mathbf{c}) := \{\, \tau \in G : \mathbf{c} \circ \tau = \mathbf{c} \,\} = \{\, \tau \in G : \tau(C_i) = C_i \text{ for } i = 1,2,3 \,\}$. Since any two colourings, $\mathbf{c}_1$ and $\mathbf{c}_2$, lie on the same G-orbit, the corresponding subroups $A(\mathbf{c}_1)$ and $A(\mathbf{c}_2)$ are G-conjugate. In particular, all of these subgroups are isomorphic to a single abstract group which we will denote by A. Since the number of distinct colourings is $2016 = |G|/10$, we see that $|A| = 10$.

We will denote by $\mathbf{c}_0$ the colouring determined by $C_1 = \{0,1,2,3,0123\}$, $C_2 = \{01,12,23,012,123\}$, $C_3 = \{02,03,13,013,023\}$. Define $\alpha, \beta \in G$ by

$$\alpha : \begin{array}{cccc} 0 & 1 & 2 & 3 \\ \downarrow & \downarrow & \downarrow & \downarrow \\ 1 & 2 & 3 & 0123 \end{array} \qquad \text{and} \qquad \beta : \begin{array}{cccc} 0 & 1 & 2 & 3 \\ \downarrow & \downarrow & \downarrow & \downarrow \\ 3 & 2 & 1 & 0 \end{array}$$

Then α is a 5-cycle in $A(\mathbf{c}_0)$ and β is a 2-cycle in $A(\mathbf{c}_0)$. Since $\alpha\beta \neq \beta\alpha$ we see that $A \cong D_5$, the dihedral group of order 10.

If $\mathbf{c} : V(\mathcal{S}_3^*) \to \{1,2,3\}$ is a colouring then so is $\sigma \circ \mathbf{c}$ where $\sigma : \{1,2,3\} \to \{1,2,3\}$ is an element of Σ_3, the symmetric group on 3 letters. It is customary to identify the 6 different colourings that one obtains from $\mathbf{c}$ "by permuting the colours" in this manner. Let us denote by $[\mathbf{c}]$ the equivalence class of $\mathbf{c}$ under this relation. Define $B(\mathbf{c}) := \{\, \tau \in G : \mathbf{c} \circ \tau \in [\mathbf{c}] \,\} = \{\, \tau \in G : \exists \sigma \in \Sigma_3 \text{ such that } \tau(C_i) = C_{\sigma(i)} \text{ for } i = 1,2,3, \}$. As above we can show that all groups $B(\mathbf{c})$ are isomorphic to a single abstract group which we will denote by B. Moreover it is easy to see that A is a normal subgroup of B and $B/A \hookrightarrow \Sigma_3$. Now define $\gamma, \delta \in B(\mathbf{c_0})$ by

$$\gamma : \begin{array}{cccc} 0 & 1 & 2 & 3 \\ \downarrow & \downarrow & \downarrow & \downarrow \\ 23 & 123 & 12 & 012 \end{array} \qquad \text{and} \qquad \delta : \begin{array}{cccc} 0 & 1 & 2 & 3 \\ \downarrow & \downarrow & \downarrow & \downarrow \\ 0 & 2 & 0123 & 1 \end{array}$$

Then γ interchanges the sets C_1 and C_2 and stabilizes C_3 while δ interchanges the sets C_2 and C_3 and stabilizes C_1. Therefore $B(\mathbf{c}_0)/A(\mathbf{c}_0)$ is generated by $\gamma + A(\mathbf{c}_0)$ and $\delta + A(\mathbf{c}_0)$ and thus $B/A \cong B(\mathbf{c}_0)/A(\mathbf{c}_0) \cong \Sigma_3$.

We are now in a position to show our main result.

Theorem 8 *$\mathcal{S}_5^*$ is a 5-chromatic STS(63).*

Proof: From [14] we know that $\chi(\mathcal{S}_5^*)$ is either 4 or 5. Suppose that $\mathcal{S}_5^*$ is 4-colourable with colour classes C_1, C_2, C_3 and C_4. Then one of these colour classes, say C_4, has size at most 15. By Corollary 4, there is a hyperplane K such that $|C_4 \cap K| \leq 3$. The hyperplane K is generated by 5 independent points which we may denote by $0, 1, 2, 3, 4$. It is easy to see that there is a 3-flat, $H \subset K$ such that $C_4 \cap H = \emptyset$. Without loss of generality $H = \langle 0, 1, 2, 3 \rangle$. The 4-colouring of $\mathcal{S}_5^*$ induces a 3-colouring of H since $C_4 \cap H = \emptyset$. By Proposition 7, we may assume that $C_1 \cap H = \{0, 1, 2, 3, 0123\}$, $C_2 \cap H = \{01, 12, 23, 012, 123\}$ and $C_3 \cap H = \{02, 03, 13, 013, 023\}$. A computer program was used to check exhaustively that such a partition of the vertices of H together with the condition that $|C_4 \cap K| \leq 3$ cannot be extended to a 4-colouring of $\mathcal{S}_5^*$ and thus we have a proof of the theorem.

In fact the condition $|C_4 \cap K| \leq 3$ restricted the problem sufficiently to make it susceptable to the computer program.

We now sketch a method which allows one to complete the proof "by hand" without resorting to a computer.

This method relies on basic group theory to further restrict the problem. Since $\chi(\mathcal{S}_4^*) = 4$ ([12,14]), we have $C_4 \cap K \neq \emptyset$. Hence we may assume that $4 \in C_4 \cap K$ since there is an automorphism of K carrying any point x of $K \setminus H$ to 4 which fixes H pointwise. Now it is easy to check that (up to the action of the subgroup B studied above) there are only two colourings of K which satisfy $|C_4 \cap K| \leq 2$ and neither may be extended to a 4-colouring of $\mathcal{S}_5^*$. Hence we may assume $|C_4 \cap K|$ is exactly 3 and we may write $C_4 \cap K = \{4, y, z\} \subset K \setminus H$. Examining the action of B one finds that the B-orbit of $\{4, y, z\}$ contains either $\{4, 04, 14\}$ or $\{4, 04, 234\}$. These two possibilities represent the two cases: either the line $\{y + 4, z + 4, y + z\}$ is coloured with 2 colours or it is coloured with 3. It is possible (by considering cases) to determine all the 4-colourings of K that may be obtained from either of these two partial colourings. The first gives rise to 4 distinct colourings of K while the second yields 6 colourings of K. After some (uninstructive and fairly long) work we found that none of these colourings of K can be extended to a 4-colouring of $\mathcal{S}_5^*$. ▪

In [2] it is shown that for all $k \geq 3$, there is a least integer n_k such that for every admissible $v \geq n_k$, there exists a k-chromatic STS(v). In the same paper it was shown that $n_4 \leq 49$. From our next theorem, we can deduce an upper bound for n_5. However we need the following two constructions (described in [10]).

Bose's Construction. Let $(Q, \circ)$ be an idempotent commutative quasigroup of order $2u + 1$. Set $S := Q \times \{1, 2, 3\}$ and define a collection of triples T of S as follows:

(1) $\{(x, 1), (x, 2), (x, 3)\} \in T$ for every $x \in Q$; and

(2)if $x \neq y$ the three triples $\{(x, 1), (y, 1), (x \circ y, 2)\}$, $\{(x, 2), (y, 2), (x \circ y, 3)\}$ and $\{(x, 3), (y, 3), (x \circ y, 1)\} \in T$.

It is a routine matter to see that (S, T) is a 3-chromatic STS($6u + 3$).

Skolem's construction. A quasigroup $(Q, \circ)$ is called *half-idempotent* if $|Q| = 2u$, $Q = \{1, \ldots, 2u\}$ and

$$x \circ x = \begin{cases} x, & \text{if } x \leq u; \\ x - u, & \text{if } x > u. \end{cases}$$

Let $(Q, \circ)$ be a half-idempotent commutative quasigroup of order $2u$ and set $S := (Q \times \{1, 2, 3\}) \cup \{\infty\}$, where $\infty \notin Q \times \{1, 2, 3\}$. Define a collection, T, of triples of

S as follows:

(1) $\{(x,1),(x,2),(x,3)\} \in T$ for every $x \in Q,\ x \leq u$;

(2) for each $x > u$, the three triples $\{\infty,(x,1),(x-u,2)\},\ \{\infty,(x,2),(x-u,3)\}$ and $\{\infty,(x,3),(x-u,1)\} \in T$; and

(3) if $x \neq y$ the three triples $\{(x,1),(y,1),(x \circ y,2)\}$, $\{(x,2),(y,2),(x \circ y,3)\}$ and $\{(x,3),(y,3),(x \circ y,1)\}\ \in T$.

It is well known ([2]) that (S,T) is a 3-chromatic STS$(6u+1)$

Note that in both Bose's and Skolem's constructions, subquasigroups will always produce subsystems in the resulting STS.

Theorem 9 *Let $k \geq 5$ and $v \equiv 3 \pmod 6$. If there exists a k-chromatic STS(v), then there exists a k-chromatic STS(w) for all admissible $w \geq 2v+1$.*

Proof: The idea behind this proof comes from [2] and [10]. Let $k \geq 5$ and $\mathcal{R}$ be a k-chromatic STS(v) where $v = 6n+3$. Let $V(\mathcal{R}) = V_1 \cup V_2 \cup \ldots \cup V_k$ be a partition arising from a k-colouring of $\mathcal{R}$ where the positive integers $v_i := |V_i|$ $(i = 1,\ldots,k)$ satisfy $v_k \geq v_{k-1} \geq \ldots \geq v_1$. Then clearly $2n+1 > v_3 \geq v_2 \geq v_1$ and so we can find a partition of the set $V(\mathcal{R})$ into 3 subsets of equal size A_1, A_2 and A_3 such that $V_3 \subseteq A_3$, $V_2 \subseteq A_2$ and $V_1 \subseteq A_1$. Let $w \geq 2v+1$ be an admissible integer. We have two cases:

Case 1. $w \equiv 1 \pmod 6$. Then $w = 6u+1$ where $u \geq 2n+1$. There exists a half-idempotent commutative quasigroup $(Q,\circ)$ of order $2u$ containing an idempotent commutative quasigroup of order $2n+1$ [10]. Skolem's construction applied to such a quasigroup produces a STS$(6u+1)$ containing a subsystem of order $6n+3$. Unplug this subsystem of order $6n+3$ and replace it with a copy of $\mathcal{R}$ in such a way that $A_j \subseteq Q \times \{j\}$ for $j = 1,2,3$. We show that this gives a k-chromatic STS(w). Clearly it is at least k-chromatic. We show that it admits a k-colouring. Let $\mathbf{c}_1, \ldots, \mathbf{c}_k$ be the k different colours used to colour $\mathcal{R}$ where V_i is the colour class of $\mathbf{c}_i$ for $i = 1, \ldots, k$. Define $\mathbf{c} : (Q \times \{1,2,3\}) \cup \{\infty\} \to \{1, \ldots, k\}$ by $\mathbf{c}(x) := \mathbf{c}_i$ if $x \in V_i \subseteq V(\mathcal{R})$, $\mathbf{c}(x) := \mathbf{c}_j$ if $x \in Q \times \{j\} \setminus V(\mathcal{R})$, $j = 1, 2, 3$ and finally $\mathbf{c}(\infty)$ may be chosen to be any colour. It is straightforward to check that $\mathbf{c}$ is a k-colouring of our STS(w).

Case 2. $w \equiv 3 \pmod 6$. Then $w = 6u+3 \geq 2(6n+3)+1$ implies that $u \geq 2n+1$. By Cruse's Theorem [5] (see also [10]), there exists an idempotent commutative quasigroup $(Q, \circ)$ of odd order $2u + 1$ containing an idempotent commutative quasigroup of order $2n + 1$. Bose's construction applied to the quasigroup $(Q, \circ)$ produces a 3-chromatic STS($6u+3$) with a (3-chromatic) subsystem of order $6n+3$. Unplug this subsystem of order $6n + 3$ and replace it with a copy of $\mathcal{R}$ in the same fashion as for the case 1. Again it is easy to check that the resulting STS($6u + 3$) is k-chromatic. ▪

Corollary 10 *There exists a* 5*-chromatic STS(v) for every admissible integer* $v \geq 127$.

By Corollary 10 we have that $n_5 \leq 127$. We are not in a position to make any conjecture since constructing "small" 5-chromatic STS's is at present beyond our resources but it is our feeling that $n_5 \leq 63$.

Conclusion: Theorem 2 gives rise to the following general problem: Let C be an independent subset of $V(\mathcal{S}_n^*)$ and suppose that $|C \cap K| \geq t$ for all hyperplanes K. Let $\rho(n,t)$ be the minimal cardinality of such a set C. Find bounds on these numbers $\rho(n,t)$. From Theorem 2 and Proposition 6 we get that $\rho(5,4) = 20$ and $\rho(5,3) = 12$. A reasonable lower bound for $\rho(n,t)$ would be useful for determining $\chi(\mathcal{S}_n^*)$ for $n \geq 6$. We note that the analogous problem for affine spaces was solved recently by A. Bruen [4].

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