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CONSTRUCTIVE INVARIANT THEORY FOR TORI

by David L. WEHLAU

Introduction.

Let $\rho : G \rightarrow GL(V)$ be a rational representation of a reductive algebraic group over the algebraically closed field \mathbf{k} . The action of G on V induces an action of G on $\mathbf{k}[V]$, the algebra of polynomial functions on V , via $(g \cdot f)(v) = f(\rho(g^{-1})v)$ for $g \in G$, $f \in \mathbf{k}[V]$ and $v \in V$. The functions which are fixed by this action form a finitely generated subalgebra, $\mathbf{k}[V]^G$, the ring of invariants. The problem of constructive invariant theory is to give an algorithm which in a finite number of steps will explicitly construct a minimal set of homogeneous generators for the \mathbf{k} -algebra, $\mathbf{k}[V]^G$.

Now if $\{f_1, \dots, f_p\}$ is such a set with $\deg f_1 \geq \deg f_2 \geq \dots \geq \deg f_p$ then although the f_i are not uniquely determined the p -tuple of degrees $(\deg f_1, \dots, \deg f_p)$ is unique. The number $N_{V,G} = \deg f_1$ is of special interest. It is the minimal integer N such that $\mathbf{k}[V]^G$ is generated by the subspace $\bigoplus_{m=0}^N \mathbf{k}[V]_m^G$ of invariants of degree at most N . Clearly an algorithm which constructs $\{f_1, \dots, f_p\}$ also produces $N_{V,G} = \max\{\deg f_i \mid 1 \leq i \leq p\}$. For many groups, G , (e.g. if $\text{char } \mathbf{k} = 0$ and G is reductive) the converse is also true : given $N_{V,G}$ there is a finite algorithm which constructs $\{f_1, \dots, f_p\}$ (cf. [K], [P]).

If G is a finite group and the characteristic of \mathbf{k} does not divide $|G|$, then by a celebrated theorem of Emmy Noether's, $N_{V,G} \leq |G|$ (see [N1]),

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[N2]). Recently Schmid has considered the question of whether this bound is sharp ([S]). She has shown that $N_{V,G} < |G|$ if G is not cyclic and has determined $N_{V,G}$ for various groups of small order including all abelian groups of order less than 30.

If G is semi-simple and the characteristic of \mathbf{k} is zero and the representation ρ is almost faithful, then Popov has given in [P] an upper bound for $N_{V,G}$. Following the methods of Popov, Kempf ([K]) derived an upper bound for $N_{V,G}$ in the case that G is a torus and the characteristic of \mathbf{k} is zero. Kempf also observed that these three bounds (for G finite, G semi-simple and G a torus) could be combined (by multiplying them) to obtain a bound for the general reductive group in characteristic zero.

The bounds for infinite groups are very large. In this paper we will consider the case $G = T$ is a torus and give better bounds for $N_{V,T}$. In addition we will construct certain distinguished elements of a minimal generating set for $\mathbf{k}[V]^T$.

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Diagonalization.

Let \mathbf{k} be an algebraically closed field of any characteristic. Let T be a torus, i.e., T is an algebraic group which is (abstractly) isomorphic to $(\mathbf{k}^*)^r$ and suppose that $\rho : T \rightarrow GL(V)$ is a rational representation of V . Let $X^*(T)$ denote the lattice of characters of T . Then $X^*(T)$ is (abstractly) isomorphic to \mathbb{Z}^r . From now on we will assume that we have chosen a fixed basis of V consisting of eigenvectors, $\{v_1, \dots, v_n\}$, and that $\{x_1, \dots, x_n\}$, is the corresponding dual basis of V^* . Furthermore we will denote the weight of v_i by ω_i . Then ρ induces an action of T on $V^* \subset \mathbf{k}[V]$ which in terms of weights is given by $t \cdot x_i = -\omega_i(t)x_i$. The action on all of $\mathbf{k}[V] \cong \mathbf{k}[x_1, \dots, x_n]$ is obtained from the action on V^* by the two requirements $t \cdot (fg) = (t \cdot f)(t \cdot g)$ and $t \cdot (f + g) = t \cdot f + t \cdot g$ for $t \in T$ and $f, g \in \mathbf{k}[V]$.

We will consider monomials $X^A = x_1^{a_1} x_2^{a_2} \dots x_n^{a_n}$ where $A = (a_1, \dots, a_n) \in \mathbb{N}^n$. Clearly T acts on X^A by $t \cdot X^A = \chi(t)X^A$ where χ is the character $\chi = -(a_1\omega_1 + \dots + a_n\omega_n)$. We will denote χ by $\text{wt}(X^A)$. The invariant monomials are in one-to-one correspondence with the semi-group, $S := \{A \in \mathbb{N}^n \mid X^A \in \mathbf{k}[V]^T\} = \{A \in \mathbb{N}^n \mid a_1\omega_1 + \dots + a_n\omega_n = \mathbf{o}\}$ where \mathbf{o} is the trivial character in $X^*(T)$. This semi-group was first studied

by Gordan. He used it to show that $\mathbf{k}[V]^T$ is a finitely generated algebra by showing that S is finitely generated as a semi-group (see [Go]).

Recall that a representation $\rho : G \rightarrow GL(V)$ is called *stable* if the union of the closed G -orbits in V contains an open dense subset of V . It is sufficient to consider only faithful stable torus representations, (cf. [W], Lemma 2). From now on we will suppose that ρ is both faithful and stable.

Kempf's bound.

Choosing an explicit isomorphism $\psi : T \rightarrow (k^*)^r$ induces an explicit isomorphism $\psi^* : X^*(T) \rightarrow \mathbb{Z}^r$. The isomorphism ψ is determined only up to $\text{Aut}(T) \cong GL(r, \mathbb{Z})$. Having fixed a choice for ψ we may write out the weights of V as r -tuples: $\omega_i = (\omega_{i,1}, \dots, \omega_{i,r}) \in \mathbb{Z}^r$ for $1 \leq i \leq n$. Then we may define $w := \max\{|\omega_{i,j}| : 1 \leq i \leq n, 1 \leq j \leq r\}$. Kempf showed in [K] that $N_{V,T} \leq nC(nr!w^r)$ where $C(m)$ is the least common multiple of the integers $1, 2, \dots, m$. This bound has the disadvantage of being dependent on w which depends on the choice of ψ .

Example 1. — Let $T \cong (k^*)^2$ and let V be the 4 dimensional representation of T with weights $(2, 2), (-1, 0), (0, -5)$ and $(2, -1)$. It is fairly simple, for example using the iterative method of the next section, to compute a homogeneous minimal system of generators for $\mathbf{k}[V]^T$. We find that $\mathbf{k}[V]^T = \mathbf{k}[X^{R_1}, X^{R_2}, X^A]$ where $R_1 = (5, 10, 2, 0), R_2 = (1, 6, 0, 2)$ and $A = (3, 8, 1, 1)$. Therefore $N_{V,T} = \text{deg } R_1 = 17$. Here $r = 2, n = 4$ and $w = 5$. Hence for this example Kempf's bound gives $N_{V,T} \leq 4C(4 \cdot 2! \cdot 5^2) = 4C(200) > 4(3 \times 10^{89}) > 10^{90}$.

An iterative method.

Consider first the case $r = 1$. Here the isomorphism of T with k^* is determined up to $GL(1, \mathbb{Z}) \cong \{\pm 1\}$ and thus w is completely determined in this case. Fixing one of the two choices $\psi : T \rightarrow k^*$ we may write the weights of V as integers: $\omega_1, \omega_2, \dots, \omega_n \in \mathbb{Z}$. Set $w_- := \min\{\omega_i | 1 \leq i \leq n\}$ and $w_+ := \max\{\omega_i | 1 \leq i \leq n\}$. Our assumptions that ρ is stable and faithful together imply that $w_- < 0$ and $w_+ > 0$.

THEOREM 1. — *Let V be a representation of k^* with weights $\omega_1 \geq \omega_2 \geq \dots \geq \omega_n$ and set $B := \omega_1 - \omega_n$. Then $N_{V,k^*} \leq B$.*

Proof. — Suppose $X^A \in \mathbf{k}[V]^T$ has degree d . We will construct a sequence of d monomials: h_1, h_2, \dots, h_d with $\omega_n \leq \text{wt}(h_i) \leq \omega_1 - 1$ for all $1 \leq i \leq d$ as follows. Choose j such that $\omega_j < 0$ and define $h_1 := x_j$. If $\text{wt}(h_m) \geq 0$ then we choose j such that x_j divides X^A/h_m and $\omega_j \leq 0$. Similarly if $\text{wt}(h_m) < 0$ then we choose j such that x_j divides X^A/h_m and $\omega_j > 0$. In either case we define $h_{m+1} := x_j h_m$. If $d > B$ then by the pigeon hole principle, two of the monomials have the same weight : $\text{wt}(h_i) = \text{wt}(h_j)$ where we may assume $i < j$. But then $h := h_j/h_i \in \mathbf{k}[V]^T$ divides X^A and so we see that X^A is not irreducible. \square

Remark 1. — If $\text{gcd}(\omega_1, \omega_n) = 1$ then the invariant $x_1^{-\omega_n} x_n^{\omega_1}$ is irreducible and has degree $B = N_{V, \mathbf{k}^*}$.

Remark 2. — Note that $w = \max\{\omega_1, -\omega_n\}$ and therefore $N_{V, \mathbf{k}^*} \leq 2w$.

THEOREM 2. — $N_{V, T} \leq (2w)^{2^r - 1}$

Proof. — We proceed by induction on r . The theorem is true for the case $r = 1$ by Remark 2. For higher values of r we consider the coordinate decomposition of T induced by the isomorphism ψ , i.e., $T \cong T_1 \times \dots \times T_r$ where $T_j \cong \mathbf{k}^*$ and the weight of x_i with respect to T_j is $\omega_{i,j}$. Set $T' = T_2 \times \dots \times T_r$ so that $T = T_1 \times T'$. By induction, there exist monomial generators h_1, \dots, h_p of $\mathbf{k}[V]^{T'}$ with $\text{deg } h_i \leq (2w)^{(2^{r-1}-1)}$ for all $1 \leq i \leq p$. Write $h_i = X^A$ and set $\nu_i := \text{wt}(h_i) \in X^*(T_1) \cong \mathbb{Z}$. Then $\nu_i = a_1 \omega_{1,1} + \dots + a_n \omega_{n,1}$. Hence $|\nu_i| \leq a_1 w + \dots + a_n w = (\text{deg } h_i) w \leq (2w)^{(2^{r-1}-1)} w$.

Let V_1 be a p dimensional \mathbf{k} -vector space and suppose that T_1 acts on V_1 by the weights $-\nu_1, \dots, -\nu_p$. Then we have a T_1 -equivariant surjection $\mathbf{k}[V_1] \twoheadrightarrow \mathbf{k}[V]^{T'} = \mathbf{k}[h_1, \dots, h_p]$. In particular we have the surjection $\mathbf{k}[V_1]^{T_1} \twoheadrightarrow (\mathbf{k}[V]^{T'})^{T_1} = \mathbf{k}[V]^T$. Hence $N_{V, T} \leq N_{V, T'} \cdot N_{V_1, T_1} \leq (2w)^{(2^{r-1}-1)} \cdot 2(2w)^{(2^{r-1}-1)} w = (2w)^{2^r - 1}$. \square

For the representation described in Example 1 (for which $N_{V, T} = 17$) this theorem gives the bound $N_{V, T} \leq 1000$. This is a better bound than Kempf's for this example but this is only because r is so small in the example. As a function of r the bound given in Theorem 2 grows much much faster than Kempf's bound. This new bound is, however, distinguished by the fact that it is independent of $n = \dim V$.

Geometric bounds.

In this section we will construct a set of distinguished monomials which is a subset of a minimal generating set for $\mathbf{k}[V]^T$. We begin with some notation and definitions. We will use \mathbf{o} to denote the origin in $X^*(T) \otimes \mathbb{Q} \cong \mathbb{Q}^n$. If $Z = (z_1, \dots, z_n) \in \mathbb{Q}^n$ define $\text{deg } Z := z_1 + \dots + z_n$. We also define $\text{supp}(Z) := \{i \mid 1 \leq i \leq n, z_i \neq 0\}$ and the length of Z , $\ell(Z) := \#\text{supp}(Z) - 1$. If $\{Z_1, \dots, Z_d\} \subset \mathbb{Q}^n$ then $\mathcal{H}(Z_1, \dots, Z_d)$ denotes the convex hull of the points Z_1, \dots, Z_d and $\mathcal{P}(Z_1, \dots, Z_d)$ denotes the convex set $\left\{ \sum_{i=1}^d \alpha_i Z_i \mid \alpha_i \in [0, 1] \text{ for } i = 1, \dots, d \right\}$. Notice that if $\{Z_1, \dots, Z_d\}$ is linearly independent then $\mathcal{P}(Z_1, \dots, Z_d)$ is a d -dimensional parallelepiped.

By a polytope we will mean a compact convex set having finitely many vertices. The vertices of a polytope P are characterized by the property that Y is a vertex of P if and only if the set $P \setminus \{Y\}$ is a convex set. A d dimensional polytope having $d + 1$ vertices is a simplex. We will often consider the case of a d dimensional polytope $P \subset \mathbb{Q}^m$ with $m \geq d$. In this case when we refer to the volume of P we mean the (positive) d dimensional volume of P obtained by considering P as a subset of the d dimensional affine space, \mathbb{A}^d , spanned by P . If we wish to consider the m dimensional volume of P (which is zero if $d < m$) we will write $\text{vol}_m(P)$. Similarly the relative interior of P refers to the interior of P defined by the subspace topology induced by $P \subset \mathbb{A}^d$.

The monomial generators of $\mathbf{k}[V]^T$ correspond to generators of the semi-group S . Gordan showed how to find the generators of S (see for example [O], Proposition 1.1 (ii)). Consider the pointed (half) cone $\Gamma \subset (\mathbb{Q}^+)^n$ generated by S : $\Gamma := (\mathbb{Q}^+ \cdot S)$ where $\mathbb{Q}^+ = \{q \in \mathbb{Q} \mid q \geq 0\}$. This cone, Γ , is just the set of solutions $(z_1, \dots, z_n) \in (\mathbb{Q}^+)^n$ of the system of equations :

$$(*) \quad z_1\omega_1 + \dots + z_n\omega_n = \mathbf{o}.$$

If \mathcal{L} is an extremal ray of Γ then $\mathcal{L} \cap S$ is a semigroup isomorphic to \mathbb{N} . Let $R_{\mathcal{L}}$ denote the unique generator of this semigroup. Write $\{R_1, \dots, R_s\} = \{R_{\mathcal{L}} \mid \mathcal{L} \text{ an extremal ray of } C\}$. The intersection $\mathcal{P}(R_1, \dots, R_s) \cap S$ is a finite generating set for S . Following Stanley ([St]), we call these R_j *completely fundamental generators* of S . These are characterized by the fact that if $mR_j = A + B$ for some $m \in \mathbb{N}$ and some $A, B \in S$ then $A = kR_j$ and $B = (m - k)R_j$ for some integer $k \leq m$ ([St], p. 36). The elements

X^{R_1}, \dots, X^{R_s} are the distinguished monomial generators we referred to earlier.

Now we are ready to begin our construction of the completely fundamental generators.

LEMMA 1. — *There exists $A \in S$ with $\text{supp}(A) = \Omega$ if and only if \mathfrak{o} lies in the relative interior of $\mathcal{H}(\omega_i \mid i \in \Omega)$.*

Proof. — Suppose $0 \neq A \in S$ and $\text{supp}(A) = \Omega$. Then we have $\mathfrak{o} = \sum_{i=1}^n a_i \omega_i = \sum_{i \in \Omega} a_i \omega_i = \sum_{i \in \Omega} (a_i / \text{deg } A) \omega_i$. Since $a_i \geq 0$ for all i and $\sum_{i \in \Omega} a_i = \text{deg } A$ we see that $\mathfrak{o} \in \mathcal{H}(\omega_i \mid i \in \Omega)$. Furthermore, since the coefficient $a_i / \text{deg } A$ is non-zero for each $i \in \Omega$, \mathfrak{o} is an interior point of $\mathcal{H}(\omega_i \mid i \in \Omega)$.

Conversely, suppose that \mathfrak{o} lies in the relative interior of $\mathcal{H}(\omega_i \mid i \in \Omega)$. Then there exist rational numbers p_i/q where $p_i, q \in \mathbb{N}$ with $1 \leq p_i \leq q$ such that $\sum_{i \in \Omega} (p_i/q) \omega_i = \mathfrak{o}$ and $\sum_{i \in \Omega} p_i/q = 1$. Hence if we define $p_i = 0$ if $i \notin \Omega$ we have $\sum_{i=1}^n p_i \omega_i = \mathfrak{o}$ and $A := (p_1, \dots, p_n) \in S$ with $\text{supp}(A) = \Omega$.

□

Define a partial order on $\Gamma \setminus \{\mathfrak{o}\}$ by inclusion of supports, i.e., if $Y_1, Y_2 \in \Gamma \setminus \{\mathfrak{o}\}$ with $\text{supp}(Y_1) \subseteq \text{supp}(Y_2)$ then $Y_1 \preceq Y_2$. Also given $Y \in \Gamma$, define $\sigma(Y) := \mathcal{H}(\omega_i \mid i \in \text{supp}(Y))$.

PROPOSITION 1. — *Let $\mathfrak{o} \neq Y \in S$ with $Y/m \notin S$ for all $m \geq 2$. Then the following are all equivalent :*

- (1) Y is minimal in Γ .
- (2) $\sigma(Y)$ is an $\ell(Y)$ dimensional simplex with \mathfrak{o} in its relative interior.
- (3) Y is a completely fundamental generator of S .

Proof. — The proof that (1) \implies (2) follows from Lemma 1. Let Y be an element of S which is minimal with respect to the partial order. Then by Lemma 1, \mathfrak{o} lies in the relative interior of $\sigma(Y)$. Therefore $\sigma(Y)$ is an $\ell(Y)$ dimensional simplex with \mathfrak{o} in its relative interior. For if this were not true, by Carathéodory's theorem (see for example [B], Corollary 2.4 or [O], Theorem A.3), we could find a proper subset $\Omega \subsetneq \text{supp}(Y)$ such that $\mathfrak{o} \in \mathcal{H}(\omega_i \mid i \in \Omega)$. But this would contradict the minimality of Y .

In particular, this implies that any proper subset of $\{\omega_i \mid i \in \text{supp}(Y)\}$ is linearly independent.

Now to see that (2) \implies (3), suppose (2) holds and that there exists $n \in \mathbb{N}$ and $A, B \in S$ with $nY = A + B$. Since $\sigma(Y)$ is a simplex, \mathbf{o} can be expressed *uniquely* as a convex linear combination of $\{\omega_i \mid i \in \text{supp}(Y)\}$:

$\sum_{i \in \text{supp}(Y)} \alpha_i \omega_i = \mathbf{o}$ where $\alpha_i \in [0, 1]$ and $\sum_{i \in \text{supp}(Y)} \alpha_i = 1$. Now $\sum_i a_i \omega_i = \mathbf{o}$ and $a_i = 0$ if $i \notin \text{supp}(Y)$. Hence, by the uniqueness, we have $a_i / \deg(A) = \alpha_i = y_i / \deg(Y)$. Therefore $A = (\deg A / \deg Y)Y$ from which it follows that Y is completely fundamental.

Finally, we prove that (3) \implies (1). Suppose Y is a completely fundamental generator of S and $Z \in \Gamma$ with $Z \preceq Y$. Clearly, clearing denominators, we may suppose that $Z \in S$. Since $Z \preceq Y$, for $m \in \mathbb{N}$ sufficiently large we have $my_i \geq z_i$ for all $1 \leq i \leq n$. Hence mY decomposes within S as $mY = Z + (mY - Z)$. Since Y is completely fundamental, this implies that $Z = kY$ for some $k \leq m$. Hence $\text{supp}(Y) = \text{supp}(Z)$ and $Y \preceq Z$. □

Thus to each minimal element Y of Γ we have an associated $\ell(Y)$ dimensional simplex, $\sigma(Y) := \mathcal{H}(\omega_i \mid i \in \text{supp}(Y))$. Given $\text{supp}(Y)$ we can recover Y since every point in a simplex can be written *uniquely* as a convex linear combination of the vertices of the simplex. Therefore the map $Y \mapsto \text{supp}(Y)$ is one-to-one. Moreover, if $Y \in \Gamma$ is minimal then $\{\omega_i \mid i \in \text{supp}(Y)\}$ is a minimal linearly dependent subset of $\{\omega_1, \dots, \omega_n\}$.

Note that the map $Y \mapsto \sigma(Y)$ is not necessarily one-to-one. More precisely, $\text{supp}(Y) \mapsto \sigma(Y)$ is one-to-one if and only if the weights of V are distinct. If V_1 and V_2 are two representations of T having the same weights (except for multiplicities) then clearly, $N_{V_1, T} = N_{V_2, T}$ and thus it would suffice to consider only representations whose weights were distinct.

THEOREM 3. — *If the R_j are ordered so that $\deg R_1 \geq \deg R_2 \geq \dots \geq \deg R_s$ then $N_{V, T} \leq \sum_{j=1}^{n-r} \deg R_j \leq (n-r) \deg R_1$.*

Proof. — Suppose $\mathbf{o} \neq A \in S$. By Carathéodory's theorem we may write

$$A = \alpha_1 R_{j_1} + \dots + \alpha_{n-r} R_{j_{n-r}}$$

where each $\alpha_j \geq 0$. If $\alpha_j > 1$ then we may decompose A within S as $A = (A - R_{j_i}) + R_{j_i}$. Hence if A is a generator of S then each $\alpha_i \leq 1$. But

then $\deg A = \alpha_1 \deg R_{j_1} + \dots + \alpha_{n-r} \deg R_{j_{n-r}} \leq \deg R_{j_1} + \dots + \deg R_{j_{n-r}} \leq \deg R_1 + \dots + \deg R_{n-r}$. \square

Remark 3. — Applying these two bounds to the representation of Example 1 we get $N_{V,T} \leq 17 + 9 = 26$ and $N_{V,T} \leq 2 \cdot 17 = 34$.

A theorem of Ewald and Wessels ([EW], Theorem 2) allows us to improve the preceding theorem. Specifically, (using the notation of Theorem 3) they show that if $\alpha_1 + \dots + \alpha_{n-r} > n - r - 1 \geq 1$ then A is decomposable within S . Thus we have the following corollary.

COROLLARY 1. — *If $n - r \geq 2$ then $N_{V,T} \leq (n - r - 1) \deg R_1$.*

Remark 4. — If we apply this result to Example 1 we find that $N_{V,T} \leq (4 - 2 - 1) \cdot 17 = 17$.

The following proposition shows how the completely fundamental solutions are distinguished among the elements of a monomial minimal generating set.

PROPOSITION 2 (Stanley [St], Theorem 3.7). — *Suppose $\{X^{A_1}, \dots, X^{A_q}\}$ is any minimal set of monomials such that $\mathbf{k}[V]^T$ is integral over $\mathbf{k}[X^{A_1}, \dots, X^{A_q}]$. Then $q = s$ and there exists a permutation π of $\{1, \dots, s\}$ such that $\text{supp}(R_j) = \text{supp}(A_{\pi(j)})$. In fact, there exist positive integers m_1, \dots, m_s such that $A_{\pi(j)} = m_j \cdot R_j$.*

Remark 5. — Kempf ([K]) also constructed the elements R_1, \dots, R_s . His method of construction is somewhat less direct than that which we will give in the next section and consequently the bound he gave for $\deg R_j$ is larger than the one we will give.

Computing the completely fundamental generators.

In this section we will give an algorithm for finding the completely fundamental generators. Suppose Ω is a minimal linearly dependent subset of $\{\omega_1, \dots, \omega_n\}$ with $\mathbf{o} \in \mathcal{H}(\omega \in \Omega)$. Then $\Omega = \{\omega_i \mid i \in \text{supp}(R_j)\}$ for some j . We want to compute R_j . Set $d := \ell(R_j) \leq r$. Then without loss of generality we may suppose that $\text{supp}(R_j) = \{1, 2, \dots, d + 1\}$. Consider the system of r linear equations in d unknowns :

$$(\dagger) \quad y_1\omega_1 + \dots + y_d\omega_d = -\omega_{d+1}.$$

These r equations impose only d conditions and so in order to solve this system we take the $r \times d$ matrix of rank d , $M := (\omega_1 \ \omega_2 \ \dots \ \omega_d)$ and choose a $d \times d$ non-singular submatrix M' . If M' consists of the rows j_1, \dots, j_d of M then the i th column of M' is $\omega'_i := (\omega_{i,j_1}, \dots, \omega_{i,j_d})$ for $1 \leq i \leq d$. Also define $\omega'_{d+1} := (\omega_{d+1,j_1}, \dots, \omega_{d+1,j_d})$. Then solving (†) is equivalent to solving

$$(††) \quad y_1 \omega'_1 + \dots + y_d \omega'_d = -\omega'_{d+1}.$$

But we may solve (††) by Cramer's rule :

$$y_1 = \frac{|\omega'_{d+1}, \omega'_2, \dots, \omega'_d|}{|\omega'_1, \omega'_2, \dots, \omega'_d|}, \quad \dots, \quad y_d = \frac{|\omega'_1, \dots, \omega'_{d-1}, \omega'_{d+1}|}{|\omega'_1, \omega'_2, \dots, \omega'_d|}.$$

Then if we define

$$\begin{aligned} q_i &= y_i |\omega'_1, \dots, \omega'_d| \\ &= |\omega'_1, \dots, \omega'_{i-1}, \omega'_{d+1}, \omega'_{i+1}, \dots, \omega'_d| \text{ for } 1 \leq i \leq d \\ \text{and } q_{d+1} &= -|\omega'_1, \omega'_2, \dots, \omega'_d| \end{aligned}$$

we have

$$q_1 \omega_1 + \dots + q_{d+1} \omega_{d+1} = \mathbf{o}$$

where each $q_i \in \mathbb{Z}$. This solution is unique up to scalar multiplication by an element of \mathbb{Q} . Since $\mathbf{o} \in \mathcal{H}(\omega_1, \dots, \omega_{d+1})$ all the q_i must have the same sign and, multiplying by -1 if necessary, we get each $q_i \in \mathbb{N}$. If we define $q_i = 0$ for all $i \notin \{1, \dots, d+1\} (= \text{supp}(R_j))$ and $Q_j := (q_1, \dots, q_n)$ then $R_j = Q_j/m$ where m is the greatest common divisor of the integers q_1, \dots, q_{d+1} .

Thus to construct $\{R_1, \dots, R_s\}$ we consider each minimal linearly dependent subset, Ω , of the weights $\{\omega_1, \dots, \omega_n\}$. For each such Ω we compute the determinants q_1, \dots, q_{d+1} . If any two of these determinants have opposite signs then Ω does not correspond to any invariant. If however, all the q_i have the same sign then $(q_1/m, \dots, q_n/m)$ is one of the completely fundamental generators.

Degrees as volumes.

In this section we will continue to study the fixed R_j of the previous section. We will obtain bounds on $\text{deg } R_j$ and thus on $N_{V,T}$ in terms of volumes of certain polytopes.

THEOREM 4. — *Let σ_j be the simplex $\sigma_j = \mathcal{H}(\omega_i \mid i \in \text{supp}(R_j))$. Then $\text{deg } R_j \leq d! \text{vol}(\sigma_j)$.*

Proof. — Let Δ denote the perpendicular (coordinate) projection :

$$\Delta : X^*(T) \otimes \mathbb{Q} \cong \mathbb{Q}^r \rightarrow \mathbb{Q}^d \text{ given by } \Delta(u_1, \dots, u_r) = (u_{j_1}, \dots, u_{j_d}).$$

Then $\Delta(\omega_i) = \omega'_i$. Define $\sigma'_j(i) := \mathcal{H}(\mathbf{o}, \omega_1, \dots, \omega_{i-1}, \omega_{i+1}, \dots, \omega_{d+1})$, $\sigma'_j := \Delta(\sigma_j)$ and $\sigma'_j(i) := \Delta(\sigma_j(i))$. Notice that q_i is the d dimensional volume of the parallelepiped $\mathcal{P}(\omega'_1, \dots, \omega'_{i-1}, \omega'_{i+1}, \dots, \omega'_{d+1})$. Hence $q_i = d! \text{vol}(\sigma'_j(i))$.

Now $\sigma'_j = \sigma'_j(1) \cup \dots \cup \sigma'_j(d+1)$ is a triangulation of σ'_j by d -simplices since \mathbf{o} lies in the relative interior of σ'_j . Thus $\text{deg } Q_j = q_1 + \dots + q_{d+1} = d! \text{vol}(\sigma'_j)$. Therefore $\text{deg } R_j \leq \text{deg } Q_j = d! \text{vol}(\sigma'_j) \leq d! \text{vol}(\sigma_j)$ where the last inequality follows for example from [Ga], (30) p. 253. □

Let $\mathcal{W} := \mathcal{H}(\omega_1, \dots, \omega_n)$, the convex hull of the weights in $X^*(T) \otimes \mathbb{Q} \cong \mathbb{Q}^r$.

THEOREM 5. — $\text{deg } R_j \leq r! \text{vol}(\mathcal{W})$.

Proof. — It is not true in general that $d! \text{vol}(\sigma'_j) \leq r! \text{vol}(\mathcal{W})$ when $d < r$. Hence to prove this theorem we consider a slightly different construction of R_j (when $d < r$). Recall that we have assumed that $\text{supp}(R_j) = \{1, \dots, d+1\}$. Without loss of generality we may assume that $\Sigma := \mathcal{H}(\omega_1, \dots, \omega_{d+1}, \dots, \omega_{r+1})$ is an r dimensional simplex. To construct R_j we solve the system of r linearly independent equations in r unknowns :

$$y_2\omega_2 + \dots + y_{r+1}\omega_{r+1} = -\omega_1.$$

As before we apply Cramer's rule to solve this system and so find $(a_1, \dots, a_{r+1}) \in \mathbb{N}^{r+1}$ with

$$a_1\omega_1 + \dots + a_{r+1}\omega_{r+1} = \mathbf{o}$$

$$\text{and } a_i = r! \text{vol}_r(\mathcal{H}(\mathbf{o}, \omega_1, \dots, \omega_{i-1}, \omega_{i+1}, \dots, \omega_{r+1})).$$

Again we set $a_{r+2} = \dots = a_n = 0$ and $A = (a_1, \dots, a_n)$. Notice that $a_{d+2} = \dots = a_{r+1} = 0$ and that A is a multiple of R_j . Hence $\text{deg } R_j \leq \text{deg } A = a_1 + \dots + a_n = r! \text{vol}(\Sigma) \leq r! \text{vol}(\mathcal{W})$. □

COROLLARY 2. — *If $n - r \geq 2$ then $N_{V,T} \leq (n - r - 1) r! \text{vol}(\mathcal{W})$. If $1 \leq n - r \leq 2$ then $N_{V,T} \leq r! \text{vol}(\mathcal{W})$.*

Remark 6. — This bound is invariant under the action of $\text{Aut}(T) \cong \text{GL}(r, \mathbb{Z})$ and thus is independent of the choice of ψ .

Remark 7. — For the representation of Example 1, \mathcal{W} is a quadrilateral of area $23/2$. Hence we get the bound $N_{V,T} \leq 2! \cdot (23/2) = 23$.

It seems likely that the factor $n - r - 1$ is unnecessary in the first statement of Corollary 2. I know of no examples of representations where $N_{V,T} > r! \text{vol}(\mathcal{W})$. Conversely for all values of n and r there exist faithful stable n dimensional representations, V , of $T \cong (\mathbf{k}^*)^r$ such that $N_{V,T} = r! \text{vol}(\mathcal{W})$ – for example this often occurs when \mathcal{W} is itself a simplex.

CONJECTURE. — *There is a (small) constant $c \in \mathbb{R}$ such that $N_{V,T} \leq cr! \text{vol}(\mathcal{W})$.*

Bounds in terms of w .

Next we bound $\text{deg } R_j$ in terms of $w := \max\{|\omega_{i,m}| : 1 \leq i \leq n, 1 \leq m \leq r\}$.

THEOREM 6. — $\text{deg } R_j \leq \lfloor w^d (d + 1)^{(d+1)/2} \rfloor$.

Proof. — We have $\text{deg } R_j \leq d! \text{vol}(\sigma'_j)$ where $\sigma'_j = \mathcal{H}(\omega'_1, \dots, \omega'_{d+1}) \subset [-w, w]^d \subset \mathbb{Q}^d$. Define $\tilde{\sigma}'_j := \mathcal{H}(\omega'_1/2w, \dots, \omega'_{d+1}/2w) + (1/2, \dots, 1/2)$. Then $\tilde{\sigma}'_j$ is a d dimensional simplex contained in $[0, 1]^d$ with $\text{vol}(\sigma'_j) = (2w)^d \text{vol}(\tilde{\sigma}'_j)$.

Thus we now seek to bound the value $B := \max\{\text{vol}(\tau) \mid \tau \subset [0, 1]^d \text{ is a } d \text{ dimensional simplex}\}$. By linear programming it is clear that the value B is attained by a simplex μ all of whose vertices are also vertices of the cube $[0, 1]^d$. Without loss of generality we may assume that $(0, \dots, 0)$ is one of the vertices of μ . Let ν_1, \dots, ν_d be the other vertices of μ . Then $\text{vol}(\mu) = |\det(M)|/d!$ where $M = (\nu_1 \dots \nu_d)$ is a $d \times d$ matrix all of whose entries are either 0 or 1. But then by a theorem of Ryser (see [R], Equation (11)) we have

$$|\det(M)| \leq 2 \left(\frac{\sqrt{d+1}}{2} \right)^{d+1}.$$

Thus we get the bound $\text{deg } R_j \leq w^d (d + 1)^{(d+1)/2} \leq w^r (r + 1)^{(r+1)/2}$. \square

COROLLARY 3. — *If $n - r \geq 2$ then $N_{V,T} \leq (n - r - 1) \lfloor w^r (r + 1)^{(r+1)/2} \rfloor$. If $1 \leq n - r \leq 2$ then $N_{V,T} \leq \lfloor w^r (r + 1)^{(r+1)/2} \rfloor$.*

Remark 8. — In Example 1 we had $n = 4$, $r = 2$ and $w = 5$. Thus Corollary 3 gives $N_{V,T} \leq \lfloor 5^2 \cdot (2+1)^{(2+1)/2} \rfloor = \lfloor 25 \cdot 3^{3/2} \rfloor = 129$.

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Note added in proof: The construction of the completely fundamental generators given here was also pointed out by B. Sturmfels in “Gröbner bases of toric varieties”, *Tôhoku Math. J.*, second series, vol. 43, no. 2 (1991).

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