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Equidimensional Representations of 2-Simple Groups*

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1. INTRODUCTION

Let G be a complex reductive algebraic group and let $\rho: G \to GL(V)$ be a finite dimensional complex G-representation. There is a natural morphism $\pi_{G,V}: V \to V//G$, where $V//G$ denotes the set of closed orbits in V equipped with the structure of a complex variety. This morphism provides a natural quotient in the category of complex G-representations. A G-representation, V, is said to be *equidimensional* if all the fibres of $\pi_{G,V}$ have the same dimension. If $V//G$ is smooth then V is called *coregular.* A G-representation which is both equidimensional and coregular is said to be *cofree.*

In general, the G-module structure of the coordinate ring, $\mathbf{C}[V]$, of a G-representation is very complicated. However, representations which are coregular and/or equidimensional are much better understood. For example, if (V, G) is cofree then there exists a graded G-stable subspace S of $\mathbf{C}[V]$ such that $\mathbf{C}[V] \cong \mathbf{C}[V]^G \otimes S$. The G-module structure of S is described by a theorem due to Schwarz [Sch1, Prop. 4.6]. Here we give a generalization of this theorem to equidimensional representations (Proposition 2.1.1). Specializing this result to finite groups recovers a theorem of Stanley [St, Prop. 4.9].

In 1976, V. L. Popov conjectured that if G is a connected semi-simple group then every equidimensional G-representation is also coregular [P1]. Since then, this conjecture has been verified by Schwarz for simple groups [Sch2], by Littelmann for irreducible representations of semi-simple groups [Li1] and by the author for tori [Weh2]. (It has been further conjectured that the hypothesis that G be semi-simple may be weakened to G being reductive or even to G being any complex algebraic group.)

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TABLE I

	Spin$(m) \times$ **Spin**(n), $m \geqslant n$		
	Plain representation	Method	d
1	$\mathbf{C}^m \otimes \mathbf{C}^n$	Slice 6a.2	n

Here we prove the Popov Conjecture for a large class of the semi-simple groups having exactly two simple factors. The method of proof is via classification of the equidimensional representations of these groups: Let $G = G_1 \times G_2$ be a connected semi-simple group having two simple factors not both of which are isomorphic to special linear groups. If V is an equidimensional representation of G then either V decomposes as $(V, G) = (V_1, G_1) \oplus (V_2, G_2)$ or V is a subrepresentation of one of the representations listed in Tables I–XVIII. We show that all the representations we list are coregular. Hence our classification verifies the Popov Conjecture for groups of this form (Section 5).

Since every subrepresentation of an equidimensional (resp. coregular) representation is itself equidimensional (resp. coregular) [Weh3, Sch1], we need to find all the *maximally equidimensional* representations. That is, we must find all those representations (V, G) such that $(V \oplus W, G)$ is not equidimensional for all non-trivial G-representations W. Clearly adding a trivial representation to an equidimensional representation yields a new larger equidimensional representation. Thus we seek all the maximally equidimensional representations (V, G) with $V^G = \{0\}$. To verify the Popov Conjecture we must show that each of these representations is coregular.

TABLE II

	SP$(m) \times$ **SP**(n), $m \geqslant n$			
	Plain representation	Conditions	Method	d
1	$\mathbf{C}^m \otimes \mathbf{C}^n \oplus 2\mathbf{C}^m$	$m \geqslant n+2$	Slice 7b.2	$n+1$
2	$\overline{\mathbf{C}^m} \otimes \mathbf{C}^m \oplus 2\mathbf{C}^m$		Slice 7b.1	m
3	$\mathbf{C}^m \otimes \mathbf{C}^n \oplus \mathbf{C}^m \oplus \mathbf{C}^n$		Slice 7b.4	n
4	$\mathbf{C}^m \otimes \mathbf{C}^n \oplus 2\mathbf{C}^n$		Slice 7b.8	n
5	$\mathbf{C}^m \otimes \mathbf{C}^6 \oplus \varphi_2(6)$		Slice 7c.16	9
6	$\mathbf{C}^m \otimes \mathbf{C}^4 \oplus \mathbf{S}^2\mathbf{C}^4$		Slice 7c.18	6
7	$\mathbf{C}^m \otimes \mathbf{C}^4 \oplus 2\varphi_2(4)$		Slice 7c.20	7
8	$\mathbf{C}^m \otimes \mathbf{C}^4 \oplus \mathbf{C}^m \oplus \varphi_2(4)$		Restrict, 1-PSG	5
9	$\mathbf{C}^m \otimes \mathbf{C}^4 \oplus \varphi_2(4) \oplus \mathbf{C}^4$		Slice 7c.22	5
10	$\overline{\mathbf{C}^4} \otimes \mathbf{C}^4 \oplus \overline{\varphi_2(4)} \oplus \varphi_2(4)$		1-PSG	6

TABLE III

	$\mathbf{SP}(m)\times\mathbf{Spin}(n)$			
	Representation	Conditions	Method	d
1	$\mathbf{C}^m\otimes\mathbf{C}^n\oplus\mathbf{C}^m$	$m\geqslant n$	Slice 7a.2	n
2	$\mathbf{C}^m\otimes\mathbf{C}^n\oplus\mathbf{C}^m$	$m<n$, $(m,n)\neq(4,8)$	Slice 7a.1	m
3	$\mathbf{C}^m\otimes\mathbf{C}^n\oplus\mathbf{C}^n$	$m\geqslant n$	Slice 6b.1	n
4	$\mathbf{C}^m\otimes\mathbf{C}^n\oplus\mathbf{C}^n$	$m<n$	Slice 6b.8	$m+1$
5	$\mathbf{C}^m\otimes\sigma(7)$	$m\geqslant 8$	Slice 12a.9	7
6	$\mathbf{C}^6\otimes\sigma(7)$		Slice 3.8	6
7	$\mathbf{C}^m\otimes\mathbf{C}^8\oplus\sigma^+(8)$	$m\geqslant 8$	Slice 10.13	8
8	$\mathbf{C}^6\otimes\mathbf{C}^8\oplus\sigma^+(8)$		Slice 9c.9	7
9	$\mathbf{C}^m\otimes\mathbf{C}^6\oplus\sigma^+(6)$	$m\geqslant 6$	Slice 10.24	4
10	$\mathbf{C}^4\otimes\mathbf{C}^6\oplus\sigma^+(6)$		Slice 9e.12	3
11	$\mathbf{C}^m\otimes\mathbf{C}^5\oplus\sigma(5)$		Slice 10.33	4
12	$\mathbf{C}^4\otimes\mathbf{C}^n\oplus\varphi_2(4)$		Slice 7a.16	5
13	$\mathbf{C}^4\otimes\mathbf{C}^{12}\oplus\sigma^+(12)$		Strata	7
14	$\mathbf{C}^4\otimes\mathbf{C}^{10}\oplus\sigma^+(10)$		Ladder	4
15	$\mathbf{C}^4\otimes\mathbf{C}^9\oplus\sigma(9)$		Slice 9b.15	6
16	$\mathbf{C}^4\otimes\mathbf{C}^8\oplus\mathbf{C}^4\oplus\sigma^+(8)$		Slice 9c.24	6
17	$\mathbf{C}^4\otimes\mathbf{C}^8\oplus\varphi_2(4)\oplus\sigma^+(8)$		1-PSG	7
18	$\mathbf{C}^4\otimes\mathbf{C}^7\oplus\sigma(7)$		Slice 9d.9	5
19	$\mathbf{C}^4\otimes\sigma(7)\oplus\mathbf{C}^4$		Slice 3.16	5
20	$\mathbf{C}^4\otimes\sigma(7)\oplus\varphi_2(4)$		Slice 3.17	6

TABLE IV

	$\mathbf{SP}(4)\times\mathbf{G}_2$		
	Plain representation	Method	d
1	$\mathbf{C}^4\otimes\varphi_1(\mathbf{G}_2)$	Slice 3.18	4

TABLE V

	$\mathbf{H}\times\mathbf{SL}(n)$, $\mathbf{H}$ Simple			
	Representation	Conditions	Method	d
1	$W\otimes\mathbf{C}^n\oplus\rho(H)$	$\dim W=n$, $\rho(H)^H=\{0\}$, $\rho(H)$ maximally EQ	Ladder	$1+\dim\rho(H)//H$
2	$W\otimes\mathbf{C}^n\oplus\mathbf{C}^{n*}\oplus\rho(H)$	$\dim W=n$, W non-plain, $\rho(H)^H=\{0\}$, $(W\oplus\rho(H))$ maximally EQ	Strata	$1+\dim(W\oplus\rho(H))//H$

TABLE VI

(a) $\mathbf{SO}(m)\times\mathbf{SL}(n)$ $\quad V\supset S^2\mathbf{C}^{n^{(*)}}$

	Plain representation	Conditions	Method	d
1	$\mathbf{C}^m\otimes\mathbf{C}^n\oplus\mathbf{S}^2\mathbf{C}^n$	$m\leqslant n-1$	Expand	$m+1$
2	$\mathbf{C}^m\otimes\mathbf{C}^n\oplus\mathbf{S}^2\mathbf{C}^n$	$m\geqslant n$	Substrata	$n+1$
3	$\mathbf{C}^m\otimes\mathbf{C}^n\oplus\mathbf{S}^2\mathbf{C}^{n^*}$	$m\leqslant n-1$	Strata and induction	$m+1$
4	$\mathbf{C}^m\otimes\mathbf{C}^n\oplus\mathbf{S}^2\mathbf{C}^{n^*}$	$m\geqslant n$	Strata and induction	$n+1$

(b) $\mathbf{SO}(m)\times\mathbf{SL}(n)$ $\quad V\supset\bigwedge^2\mathbf{C}^{n^{(*)}}$

	Plain representation	Conditions	Method	d
1	$\mathbf{C}^m\otimes\mathbf{C}^n\oplus\bigwedge^2\mathbf{C}^n\oplus\mathbf{C}^m$	$m\leqslant n+1$, n even	Expand	$m+1$
2	$\mathbf{C}^m\otimes\mathbf{C}^n\oplus\bigwedge^2\mathbf{C}^n\oplus\mathbf{C}^m$	$m\geqslant n+2$	Substrata	$n+2$
3	$\mathbf{C}^m\otimes\mathbf{C}^n\oplus\bigwedge^2\mathbf{C}^n\oplus\mathbf{C}^n$	$m\leqslant n-1$	Compare 6a.1	$m+1$
4	$\mathbf{C}^m\otimes\mathbf{C}^n\oplus\bigwedge^2\mathbf{C}^n\oplus\mathbf{C}^n$	$m\geqslant n$	Compare 6a.2	$n+1$
5	$\mathbf{C}^m\otimes\mathbf{C}^n\oplus\bigwedge^2\mathbf{C}^n\oplus\mathbf{C}^{n^*}$	$m\leqslant n-1$, n even	Expand	$m+1$
6	$\mathbf{C}^m\otimes\mathbf{C}^n\oplus\bigwedge^2\mathbf{C}^n\oplus\mathbf{C}^{n^*}$	$m\geqslant n$, n even	Expand	$n+1$
7	$\mathbf{C}^m\otimes\mathbf{C}^n\oplus\bigwedge^2\mathbf{C}^{n^*}\oplus\mathbf{C}^m$	$m\leqslant n$, n even	Strata and induction	$m+1$
8	$\mathbf{C}^m\otimes\mathbf{C}^n\oplus\bigwedge^2\mathbf{C}^{n^*}\oplus\mathbf{C}^m$	$m\geqslant n+1$, n even	Strata and induction	$n+2$
9	$\mathbf{C}^m\otimes\mathbf{C}^n\oplus\bigwedge^2\mathbf{C}^{n^*}\oplus\mathbf{C}^m$	$m\leqslant n-1$, n odd	Strata	m
10	$\mathbf{C}^m\otimes\mathbf{C}^n\oplus\bigwedge^2\mathbf{C}^{n^*}\oplus\mathbf{C}^n$	$m\leqslant n-1$, n even	Slice 6b.9	$m+1$
11	$\mathbf{C}^m\otimes\mathbf{C}^n\oplus\bigwedge^2\mathbf{C}^{n^*}\oplus\mathbf{C}^n$	$m\geqslant n$, n even	Slice 6b.10	$n+1$
12	$\mathbf{C}^m\otimes\mathbf{C}^n\oplus\bigwedge^2\mathbf{C}^{n^*}\oplus\mathbf{C}^n$	$m\leqslant n-2$, n odd	Slice 6b.11	m
13	$\mathbf{C}^m\otimes\mathbf{C}^n\oplus\bigwedge^2\mathbf{C}^{n^*}\oplus\mathbf{C}^{n^*}$	$m\leqslant n-1$	Compare 6a.3	$m+1$
14	$\mathbf{C}^m\otimes\mathbf{C}^n\oplus\bigwedge^2\mathbf{C}^{n^*}\oplus\mathbf{C}^{n^*}$	$m\geqslant n$	Compare 6a.4	$n+1$

(c) $\mathbf{SO}(m)\times\mathbf{SL}(n)$, $m>n$ $\quad V\not\supset\mathbf{S}^2\mathbf{C}^{n^{(*)}}$, $V\not\supset\bigwedge^2\mathbf{C}^{n^{(*)}}$

	Plain representation	Conditions	Method	d
1	$\mathbf{C}^m\otimes\mathbf{C}^n\oplus r\mathbf{C}^{n^*}\oplus\mathbf{C}^m$	$r=\dfrac{n-2}{2}$, n even	Strata	$2+\binom{r+2}{2}$
2	$\mathbf{C}^m\otimes\mathbf{C}^n\oplus r\mathbf{C}^{n^*}\oplus\mathbf{C}^m$	$r=\dfrac{n-1}{2}$, $m\geqslant n+2$, n odd	Strata	$2+\binom{r+2}{2}$
3	$\mathbf{C}^{n+1}\otimes\mathbf{C}^n\oplus r\mathbf{C}^{n^*}\oplus\mathbf{C}^{n+1}$	$r=\dfrac{n-3}{2}$, n odd	Strata	$2+\binom{r+2}{2}$
4	$\mathbf{C}^m\otimes\mathbf{C}^n\oplus\mathbf{C}^n\oplus r\mathbf{C}^{n^*}$	$r=\left[\dfrac{n-1}{2}\right]$	Restrict	$1+\binom{r+2}{2}$
5	$\mathbf{C}^m\otimes\mathbf{C}^n\oplus r\mathbf{C}^{n^*}$	$r=n/2$, n even	Restrict	$1+\binom{r+1}{2}$
6	$\mathbf{C}^m\otimes\mathbf{C}^6\oplus\bigwedge^3\mathbf{C}^6$		Strata	6

Table continued

TABLE VI—*Continued*

	(d) $\mathbf{SO}(m)\times\mathbf{SL}(n)$, $m\leqslant n$	$V\not\supset \mathbf{S}^2\mathbf{C}^{n(*)}$, $V\not\supset\bigwedge^2\mathbf{C}^{n(*)}$		
	Plain representation	Conditions	Method	d
1	$k\mathbf{C}^m\otimes\mathbf{C}^n\oplus(n-km)\mathbf{C}^n\oplus b\mathbf{C}^{n*}\oplus c\mathbf{C}^m$	$2(kb+c)\leqslant m+1$ $(m,k,c)\neq(2b-1,1,0)$	Strata	$1+b(n-km)+\binom{kb+c+1}{2}$
2	$\mathbf{C}^{2b-1}\otimes\mathbf{C}^n\oplus(n-2b)\mathbf{C}^n\oplus b\mathbf{C}^{n*}$	$6\leqslant 2b\leqslant n$	Strata	$b(n-2b)+\binom{b+1}{2}$
3	$\overline{\mathbf{C}^n}\otimes\mathbf{C}^n\oplus\mathbf{C}^n\oplus r\mathbf{C}^{n*}$	$r=\left[\frac{n-1}{2}\right]$	Strata	$1+\binom{r+2}{2}$
4	$\mathbf{C}^m\otimes\mathbf{C}^n\oplus\bigwedge^2\mathbf{C}^m\oplus(n-m)\mathbf{C}^n\oplus\mathbf{C}^{n*}$		Strata	$n+1$
5	$\overline{\mathbf{C}^6}\otimes\mathbf{C}^6\oplus\bigwedge^3\mathbf{C}^6$		Strata	6
6	$\mathbf{C}^5\otimes\mathbf{C}^6\oplus\bigwedge^3\mathbf{C}^6$		Strata	5

TABLE VII

	(a) $\mathbf{SP}(m)\times\mathbf{SL}(n)$	$V\supset S^2\mathbf{C}^{n(*)}$		
	Plain representation	Conditions	Method	d
1	$\mathbf{C}^m\otimes\mathbf{C}^n\oplus\mathbf{S}^2\mathbf{C}^n\oplus\mathbf{C}^m$	$m\leqslant n-1$	Expand	$m+1$
2	$\mathbf{C}^m\otimes\mathbf{C}^n\oplus\mathbf{S}^2\mathbf{C}^n\oplus\mathbf{C}^m$	$m\geqslant n$	Substrata	$n+1$
3	$\mathbf{C}^m\otimes\mathbf{C}^n\oplus\mathbf{S}^2\mathbf{C}^n\oplus\mathbf{C}^n$	$m\leqslant n-2$	Expand	$m+2$
4	$\mathbf{C}^m\otimes\mathbf{C}^n\oplus\mathbf{S}^2\mathbf{C}^n\oplus\mathbf{C}^n$	$m\geqslant n-1$	Expand	$n+1$
5	$\mathbf{C}^m\otimes\mathbf{C}^n\oplus\mathbf{S}^2\mathbf{C}^n\oplus\mathbf{C}^{n*}$	$m\leqslant n-2$	Expand	$m+2$
6	$\mathbf{C}^m\otimes\mathbf{C}^n\oplus\mathbf{S}^2\mathbf{C}^n\oplus\mathbf{C}^{n*}$	$m\geqslant n$, n even	Expand	$n+1$
7	$\mathbf{C}^m\otimes\mathbf{C}^n\oplus\mathbf{S}^2\mathbf{C}^{n*}\oplus\mathbf{C}^m$	$m\leqslant n-1$	Slice 7a.9	$m+1$
8	$\mathbf{C}^m\otimes\mathbf{C}^n\oplus\mathbf{S}^2\mathbf{C}^{n*}\oplus\mathbf{C}^m$	$m\geqslant n$	Slice 7a.10	$n+1$
9	$\mathbf{C}^m\otimes\mathbf{C}^n\oplus\mathbf{S}^2\mathbf{C}^{n*}\oplus\mathbf{C}^n$	$m\leqslant n-2$	Strata and induction	$m+2$
10	$\mathbf{C}^m\otimes\mathbf{C}^n\oplus\mathbf{S}^2\mathbf{C}^{n*}\oplus\mathbf{C}^n$	$m\geqslant n-1$	Strata and induction	$n+1$
11	$\mathbf{C}^m\otimes\mathbf{C}^n\oplus\mathbf{S}^2\mathbf{C}^{n*}\oplus\mathbf{C}^{n*}$	$m\leqslant n-2$	Strata and induction	$m+2$
12	$\mathbf{C}^m\otimes\mathbf{C}^n\oplus\mathbf{S}^2\mathbf{C}^{n*}\oplus\mathbf{C}^{n*}$	$m\geqslant n$, n even	Strata and induction	$n+1$
13	$2\mathbf{C}^m\otimes\mathbf{C}^2\oplus\mathbf{S}^2\mathbf{C}^2$		Ladder	6
14	$\mathbf{C}^m\otimes\mathbf{C}^2\oplus\varphi_2(m)\oplus 2\mathbf{S}^2\mathbf{C}^2$		Ladder	$m+2$
15	$\mathbf{C}^m\otimes\mathbf{C}^2\oplus\varphi_2(m)\oplus\mathbf{S}^2\mathbf{C}^2\oplus\mathbf{C}^2$		Ladder	$m+1$
16	$\mathbf{C}^4\otimes\mathbf{C}^n\oplus\mathbf{S}^2\mathbf{C}^{n*}\oplus\varphi_2(4)$	$n\geqslant 4$	Ladder	6
17	$\mathbf{C}^6\otimes\mathbf{C}^2\oplus\varphi_3(6)\oplus\mathbf{S}^2\mathbf{C}^2$		Ladder	5

Table continued

TABLE VII—*Continued*

(b) $\mathbf{SP}(m) \times \mathbf{SL}(n)$ $V \supset \wedge^2 \mathbf{C}^n$

	Plain representation	Conditions	Method	d
1	$\mathbf{C}^m \otimes \mathbf{C}^n \oplus \wedge^2 \mathbf{C}^n \oplus 2\mathbf{C}^m$	$m \leqslant n$, n even	Expand	$m+1$
2	$\mathbf{C}^m \otimes \mathbf{C}^n \oplus \wedge^2 \mathbf{C}^n \oplus 2\mathbf{C}^m$	$m \geqslant n+2$	Substrata	$n+2$
3	$\mathbf{C}^m \otimes \mathbf{C}^n \oplus \wedge^2 \mathbf{C}^n \oplus \mathbf{C}^n \oplus \mathbf{C}^m$	$m \leqslant n-1$	Compare 7a.1	$m+1$
4	$\mathbf{C}^m \otimes \mathbf{C}^n \oplus \wedge^2 \mathbf{C}^n \oplus \mathbf{C}^n \oplus \mathbf{C}^m$	$m \geqslant n$	Compare 7a.2	$n+1$
5	$\mathbf{C}^m \otimes \mathbf{C}^n \oplus \wedge^2 \mathbf{C}^n \oplus \mathbf{C}^{n*} \oplus \mathbf{C}^m$	$m \leqslant n$, n even	Expand	$m+1$
6	$\mathbf{C}^m \otimes \mathbf{C}^n \oplus \wedge^2 \mathbf{C}^n \oplus \mathbf{C}^{n*} \oplus \mathbf{C}^m$	$m \geqslant n+2$, n even	Restrict	$n+1$
7	$\mathbf{C}^m \otimes \mathbf{C}^n \oplus \wedge^2 \mathbf{C}^n \oplus 2\mathbf{C}^n$	$m \leqslant n-2$	Compare 7a.3	$m+2$
8	$\mathbf{C}^m \otimes \mathbf{C}^n \oplus \wedge^2 \mathbf{C}^n \oplus 2\mathbf{C}^n$	$m \geqslant n-1$	Compare 7a.4	$n+1$
9	$\mathbf{C}^m \otimes \mathbf{C}^n \oplus \wedge^2 \mathbf{C}^n \oplus \mathbf{C}^n \oplus \mathbf{C}^{n*}$	$m \leqslant n-2$	Compare 7a.5	$m+2$
10	$\mathbf{C}^m \otimes \mathbf{C}^n \oplus \wedge^2 \mathbf{C}^n \oplus \mathbf{C}^n \oplus \mathbf{C}^{n*}$	$m \geqslant n$, n even	Compare 7a.6	$n+1$
11	$\mathbf{C}^m \otimes \mathbf{C}^n \oplus \wedge^2 \mathbf{C}^n \oplus 2\mathbf{C}^{n*}$	$m \leqslant n-2$, n even	Expand	$m+2$
12	$\mathbf{C}^m \otimes \mathbf{C}^n \oplus \wedge^2 \mathbf{C}^n \oplus 2\mathbf{C}^{n*}$	$m \geqslant n$, n even	Substrata	$n+1$
13	$\mathbf{C}^m \otimes \mathbf{C}^n \oplus \wedge^2 \mathbf{C}^n \oplus \mathbf{C}^{n*}$	$m \geqslant n-1$, n odd	Expand	$\frac{n-1}{2}$
14	$\mathbf{C}^m \otimes \mathbf{C}^4 \oplus 2 \wedge^2 \mathbf{C}^4$	$m \geqslant 4$	Strata	6
15	$\mathbf{C}^4 \otimes \mathbf{C}^n \oplus \varphi_2(4) \oplus \wedge^2 \mathbf{C}^n$	$n \geqslant 4$, n even	Strata	5
16	$\mathbf{C}^4 \otimes \mathbf{C}^n \oplus \varphi_2(4) \oplus \wedge^2 \mathbf{C}^n$	$n \geqslant 5$, n odd	Strata	2

(c) $\mathbf{SP}(m) \times \mathbf{SL}(n)$ $V \supset \wedge^2 \mathbf{C}^{n*}$

	Plain representation	Conditions	Method	d
1	$\mathbf{C}^m \otimes \mathbf{C}^n \oplus \wedge^2 \mathbf{C}^{n*} \oplus 2\mathbf{C}^m$	$m \leqslant n$, n even	Strata and induction	$m+1$
2	$\mathbf{C}^m \otimes \mathbf{C}^n \oplus \wedge^2 \mathbf{C}^{n*} \oplus 2\mathbf{C}^m$	$m \geqslant n+2$, n even	Strata and induction	$n+2$
3	$\mathbf{C}^m \otimes \mathbf{C}^n \oplus \wedge^2 \mathbf{C}^{n*} \oplus 2\mathbf{C}^m$	$m \leqslant n-1$, n odd	Strata	m
4	$\mathbf{C}^m \otimes \mathbf{C}^n \oplus \wedge^2 \mathbf{C}^{n*} \oplus \mathbf{C}^n \oplus \mathbf{C}^m$	$m \leqslant n-2$, n even	Strata and induction	$m+1$
5	$\mathbf{C}^m \otimes \mathbf{C}^n \oplus \wedge^2 \mathbf{C}^{n*} \oplus \mathbf{C}^n \oplus \mathbf{C}^m$	$m \geqslant n$, n even	Strata and induction	$n+1$
6	$\mathbf{C}^m \otimes \mathbf{C}^n \oplus \wedge^2 \mathbf{C}^{n*} \oplus \mathbf{C}^n \oplus \mathbf{C}^m$	$m \leqslant n-3$, n odd	Strata	m
7	$\mathbf{C}^m \otimes \mathbf{C}^n \oplus \wedge^2 \mathbf{C}^{n*} \oplus \mathbf{C}^{n*} \oplus \mathbf{C}^m$	$m \leqslant n-1$	Compare 7a.7	$m+1$
8	$\mathbf{C}^m \otimes \mathbf{C}^n \oplus \wedge^2 \mathbf{C}^{n*} \oplus \mathbf{C}^{n*} \oplus \mathbf{C}^m$	$m \geqslant n$	Compare 7a.8	$n+1$
9	$\mathbf{C}^m \otimes \mathbf{C}^n \oplus \wedge^2 \mathbf{C}^{n*} \oplus 2\mathbf{C}^n$	$m \leqslant n-2$, n even	Slice 7c.1	$m+2$
10	$\mathbf{C}^m \otimes \mathbf{C}^n \oplus \wedge^2 \mathbf{C}^{n*} \oplus 2\mathbf{C}^n$	$m \geqslant n$, n even	Slice 7c.2	$n+1$
11	$\mathbf{C}^m \otimes \mathbf{C}^n \oplus \wedge^2 \mathbf{C}^{n*} \oplus 2\mathbf{C}^n$	$m \leqslant n-3$, n odd	Slice 7c.3	$m+1$
12	$\mathbf{C}^m \otimes \mathbf{C}^n \oplus \wedge^2 \mathbf{C}^{n*} \oplus \mathbf{C}^n \oplus \mathbf{C}^{n*}$	$m \leqslant n-2$	Compare 7a.9	$m+2$
13	$\mathbf{C}^m \otimes \mathbf{C}^n \oplus \wedge^2 \mathbf{C}^{n*} \oplus \mathbf{C}^n \oplus \mathbf{C}^{n*}$	$m \geqslant n-1$	Compare 7a.10	$n+1$
14	$\mathbf{C}^m \otimes \mathbf{C}^n \oplus \wedge^2 \mathbf{C}^{n*} \oplus 2\mathbf{C}^{n*}$	$m \leqslant n-2$	Compare 7a.11	$m+2$
15	$\mathbf{C}^m \otimes \mathbf{C}^n \oplus \wedge^2 \mathbf{C}^{n*} \oplus 2\mathbf{C}^{n*}$	$m \geqslant n$, n even	Compare 7a.12	$n+1$

Table continued

TABLE VII—*Continued*

(c) $\mathbf{SP}(m)\times\mathbf{SL}(n)$ $\quad V\supset\bigwedge^2\mathbf{C}^{n^*}$

	Plain representation	Conditions	Method	d
16	$\mathbf{C}^6\otimes\mathbf{C}^n\oplus\bigwedge^2\mathbf{C}^{n^*}\oplus\varphi_2(6)$	$n\geqslant 6$, n even	Strata	10
17	$\mathbf{C}^6\otimes\mathbf{C}^n\oplus\bigwedge^2\mathbf{C}^{n^*}\oplus\varphi_2(6)$	$n\geqslant 7$, n odd	Strata	9
18	$\mathbf{C}^4\otimes\mathbf{C}^n\oplus\bigwedge^2\mathbf{C}^{n^*}\oplus\mathbf{S}^2\mathbf{C}^4$	$n\geqslant 6$, n even	Strata	7
19	$\mathbf{C}^4\otimes\mathbf{C}^n\oplus\bigwedge^2\mathbf{C}^{n^*}\oplus\mathbf{S}^2\mathbf{C}^4$	$n\geqslant 5$, n odd	Strata	6
20	$\mathbf{C}^4\otimes\mathbf{C}^n\oplus\bigwedge^2\mathbf{C}^{n^*}\oplus 2\varphi_2(4)$	$n\geqslant 6$, n even	Strata	8
21	$\mathbf{C}^4\otimes\mathbf{C}^n\otimes\bigwedge^2\mathbf{C}^{n^*}\oplus 2\varphi_2(4)$	$n\geqslant 5$, n odd	Strata	7
22	$\mathbf{C}^4\otimes\mathbf{C}^n\oplus\bigwedge^2\mathbf{C}^{n^*}\oplus\varphi_2(4)\oplus\mathbf{C}^4$	$n\geqslant 6$, n even	Strata	6
23	$\mathbf{C}^4\otimes\mathbf{C}^n\oplus\bigwedge^2\mathbf{C}^{n^*}\oplus\varphi_2(4)\oplus\mathbf{C}^4$	$n\geqslant 5$, n odd	Strata	5

(d) $\mathbf{SP}(m)\times\mathbf{SL}(n)$, $m>n$ $\quad V\not\supset\mathbf{S}^2\mathbf{C}^{n^{(*)}}$, $V\not\supset\bigwedge^2\mathbf{C}^{n^{(*)}}$

	Plain representation	Conditions	Method	d
1	$\mathbf{C}^m\otimes\mathbf{C}^n\oplus r\mathbf{C}^{n^*}$	$r=\dfrac{n+1}{2}$, n odd	Restrict	$\binom{r}{2}$
2	$\mathbf{C}^m\otimes\mathbf{C}^n\oplus r\mathbf{C}^{n^*}\oplus\mathbf{C}^m$	$r=n/2$, n even	Restrict	$1+\binom{r+1}{2}$
3	$\mathbf{C}^m\otimes\mathbf{C}^n\oplus\mathbf{C}^n\oplus r\mathbf{C}^{n^*}$	$r=n/2$, n even	Restrict	$1+\binom{r+1}{2}$
4	$\mathbf{C}^{n+1}\otimes\mathbf{C}^n\oplus r\mathbf{C}^{n^*}\oplus\mathbf{C}^{n+1}$	$r=\dfrac{n-1}{2}$, n odd	Strata	$1+\binom{r+1}{2}$
5	$\mathbf{C}^m\otimes\mathbf{C}^n\oplus r\mathbf{C}^{n^*}\oplus 2\mathbf{C}^m$	$r=\dfrac{n-2}{2}$, n even	Restrict, strata	$2+\binom{r+2}{2}$
6	$\mathbf{C}^{n+1}\otimes\mathbf{C}^n\oplus r\mathbf{C}^{n^*}\oplus 2\mathbf{C}^{n+1}$	$r=\dfrac{n-3}{2}$, n odd	Strata	$2+\binom{r+2}{2}$
7	$\mathbf{C}^m\otimes\mathbf{C}^n\oplus r\mathbf{C}^{n^*}\oplus 2\mathbf{C}^m$	$r=\dfrac{n-1}{2}$, $m\geqslant n+3$, n odd	Restrict, strata	$2+\binom{r+2}{2}$
8	$\mathbf{C}^m\otimes\mathbf{C}^n\oplus\mathbf{C}^n\oplus r\mathbf{C}^{n^*}\oplus\mathbf{C}^m$	$r=\left[\dfrac{n-1}{2}\right]$	Restrict, strata	$1+\binom{r+2}{2}$
9	$\mathbf{C}^m\otimes\mathbf{C}^n\oplus 2\mathbf{C}^n\oplus r\mathbf{C}^{n^*}$	$r=\left[\dfrac{n-1}{2}\right]$	Slice, strata	$1+\binom{r+2}{2}$
10	$\mathbf{C}^m\otimes\mathbf{C}^{m-1}\oplus\varphi_2(m)$	$m\neq 4$	Castle	$m/2-1$
11	$\mathbf{C}^m\otimes\mathbf{C}^6\oplus\bigwedge^3\mathbf{C}^6\oplus\mathbf{C}^m$		Strata	6
12	$\mathbf{C}^m\otimes\mathbf{C}^6\oplus\bigwedge^3\mathbf{C}^6\oplus\mathbf{C}^6$		Strata	6
13	$\mathbf{C}^m\otimes\mathbf{C}^4\oplus sl(4)$		Ladder	6
14	$\mathbf{C}^m\otimes\mathbf{C}^3\oplus sl(3)$		Ladder	3

Table continued

TABLE VII—*Continued*

	(d) $SP(m)\times SL(n)$, $m>n$	$V \not\supset S^2C^{n^{(*)}}$, $V \not\supset \wedge^2 C^{n^{(*)}}$		
	Plain representation	Conditions	Method	d
15	$2C^m\otimes C^2\oplus C^2$		Strata	5
16	$C^m\otimes C^2\oplus\varphi_2(m)\oplus S^4C^2$		Ladder	$m+1$
17	$C^m\otimes C^2\oplus\varphi_2(m)\oplus S^3C^2$		Ladder	m
18	$C^m\otimes C^2\oplus\varphi_2(m)\oplus 2C^2$		Ladder	m
19	$C^6\otimes C^5\oplus\varphi_3(6)\oplus C^{5^*}$		Strata	4
20	$C^6\otimes C^5\oplus\varphi_3(6)\oplus C^5$		Castle	4
21	$C^6\otimes C^4\oplus\varphi_3(6)$		Castle	3
22	$C^6\otimes C^3\oplus\varphi_3(6)\oplus C^3$		Strata	6
23	$C^6\otimes C^2\oplus\varphi_3(6)\oplus C^2$		Ladder	4
24	$C^6\otimes C^4\oplus\varphi_2(6)$		Castle	5
25	$C^6\otimes C^3\oplus\varphi_2(6)$		Strata	3
26	$C^4\otimes C^3\oplus S^2C^4$		Castle	4
27	$C^4\otimes C^2\oplus S^2C^4$		Strata	5
28	$C^4\otimes C^3\oplus 2\varphi_2(4)$		Castle	4
29	$C^4\otimes C^3\oplus\varphi_2(4)\oplus C^4$		Castle	3
30	$C^4\otimes C^3\oplus\varphi_2(4)\oplus C^{3^*}$		Strata	2
31	$C^4\otimes C^3\oplus\varphi_2(4)\oplus 2C^3$		1-PSG	5
32	$C^4\otimes C^2\oplus 2\varphi_2(4)$		1-PSG	6
33	$C^4\otimes C^2\oplus\varphi_2(4)\oplus C^4$		1-PSG	4

	(e) $SP(m)\times SL(n)$, $m\leqslant n$	$V \not\supset S^2C^{n^{(*)}}$, $\wedge^2 C^{n^{(*)}}$		
	Plain representation	Conditions	Method	d
1	$kC^m\otimes C^n\oplus(n-km)C^n\oplus bC^{n^*}\oplus cC^m$	$2(kb+c)\leqslant m+2$, $(m,k,c)\neq(2b-2,1,0)$	Strata	$b(n-km)+\binom{kb+c}{2}+1$
2	$C^{2b-2}\otimes C^n\oplus(n-2b+1)C^n\oplus bC^{n^*}$	$6\leqslant 2b\leqslant n+1$	Strata	$b(n-2b)+\binom{b+1}{2}$
3	$\overline{C^n}\otimes C^n\oplus C^n\oplus rC^{n^*}$	$r=n/2$, n even	Strata	$1+\binom{r+1}{2}$
4	$\overline{C^n}\otimes C^n\oplus C^n\oplus rC^{n^*}\oplus\overline{C^n}$	$r=\frac{n-2}{2}$, n even	Strata	$1+\binom{r+2}{2}$
5	$\overline{C^n}\otimes C^n\oplus 2C^n\oplus rC^{n^*}$	$r=\frac{n-2}{2}$, n even	Strata	$1+\binom{r+2}{2}$

Table continued

TABLE VII—*Continued*

	(e) $\mathbf{SP}(m)\times\mathbf{SL}(n)$, $m \leqslant n$ $\quad V \not\simeq \mathbf{S}^2\mathbf{C}^{n^{(*)}}, \bigwedge^2 \mathbf{C}^{n^{(*)}}$			
	Plain representation	Conditions	Method	d
6	$\mathbf{C}^{n-1}\otimes\mathbf{C}^n\oplus 2\mathbf{C}^n\oplus r\mathbf{C}^{n^*}$	$r=\frac{n-1}{2}$, n odd	Strata	$1+\binom{r+2}{2}$
7	$2\mathbf{C}^m\otimes\mathbf{C}^{2m}\oplus\mathbf{C}^{2m}$		Castle	2
8	$\mathbf{C}^m\otimes\mathbf{C}^n\oplus\mathbf{S}^2\mathbf{C}^m\oplus(n-m)\mathbf{C}^n \oplus\mathbf{C}^{n^*}$		Strata	$n+1$
9	$\mathbf{C}^m\otimes\mathbf{C}^n\oplus\varphi_2(m)\oplus(n-m)\mathbf{C}^n \oplus\mathbf{C}^{n^*}\oplus\mathbf{C}^m$		Strata	n
10	$\mathbf{C}^m\otimes\mathbf{C}^n\oplus\varphi_2(m)\oplus(n-m)\mathbf{C}^n \oplus 2\mathbf{C}^{n^*}$		Strata	$2n-m$
11	$2\mathbf{C}^m\otimes\mathbf{C}^n\oplus\varphi_2(m)\oplus(n-2m)\mathbf{C}^n \oplus\mathbf{C}^{n^*}$		Strata	$n-m$
12	$2\mathbf{C}^m\otimes\mathbf{C}^{2m-1}$	$m\neq 4$	Castle	1
13	$2\mathbf{C}^4\otimes\mathbf{C}^7\oplus\varphi_2(4)$		Castle	3
14	$\mathbf{C}^4\otimes\mathbf{C}^n\oplus\mathbf{C}^4\otimes\mathbf{C}^{n^*}$	$n\geqslant 5$	Strata	6
15	$\mathbf{C}^4\otimes\mathbf{C}^4\oplus\mathbf{C}^4\otimes\mathbf{C}^{4^*}$		Strata	7
16	$\mathbf{C}^6\otimes\mathbf{C}^n\oplus(n-6)\mathbf{C}^n\oplus 2\mathbf{C}^{n^*} \oplus\varphi_3(6)$		Strata	$2n-6$
17	$\mathbf{C}^6\otimes\mathbf{C}^n\oplus(n-6)\mathbf{C}^n\oplus\mathbf{C}^{n^*}\oplus\mathbf{C}^6 \oplus\varphi_3(6)$		Slice 7e.16	n
18	$\overline{\mathbf{C}^6}\otimes\mathbf{C}^6\oplus\mathbf{C}^6\oplus\overline{\varphi_3(6)}$		Castle	3
19	$\overline{\mathbf{C}^6}\otimes\mathbf{C}^6\oplus\bigwedge^3\mathbf{C}^6\oplus\mathbf{C}^6$		Strata	6
20	$\overline{\mathbf{C}^6}\otimes\mathbf{C}^6\oplus\bigwedge^3\mathbf{C}^6\oplus\overline{\mathbf{C}^6}$		Strata	6
21	$\mathbf{C}^4\otimes\mathbf{C}^7\oplus\bigwedge^3\mathbf{C}^7$		Strata	5
22	$\mathbf{C}^4\otimes\mathbf{C}^7\oplus\bigwedge^4\mathbf{C}^7$		Subdivide	5
23	$\mathbf{C}^4\otimes\mathbf{C}^6\oplus\bigwedge^3\mathbf{C}^6\oplus\mathbf{C}^6$		Strata	5
24	$\mathbf{C}^4\otimes\mathbf{C}^6\oplus\bigwedge^3\mathbf{C}^6\oplus\mathbf{C}^4$		Strata	4
25	$\mathbf{C}^4\otimes\mathbf{C}^6\oplus\bigwedge^3\mathbf{C}^6\oplus\varphi_2(4)$		Strata	4
26	$\overline{\mathbf{C}^4}\otimes\mathbf{C}^4\oplus sl(4)$		Strata	6

We will rely extensively on the techniques of [Sch1, Sch2, Li1] to do this. However, we will need to extend some of these methods to apply them to non-simple groups and/or to reducible representations. In addition, we will need to develop a few new methods to study some of our "more difficult" representations.

I thank G. Schwarz for all his help and encouragement with this paper.

TABLE VIII

(a) Exceptional × SL(n) dim $\varphi_1(G_1) > n$

	Plain representation	Method	d
1	$\varphi_1(\mathbf{E}_7)\otimes\mathbf{C}^{55}$	Castle	1
2	$\varphi_1(\mathbf{E}_7)\otimes\mathbf{C}^{2}$	Ladder	4
3	$\varphi_1(\mathbf{E}_6)\otimes\mathbf{C}^{26}\oplus\mathbf{C}^{26^*}$	Strata	3
4	$\varphi_1(\mathbf{E}_6)\otimes\mathbf{C}^{25}$	Castle	1
5	$\varphi_1(\mathbf{E}_6)\otimes\mathbf{C}^{24}$	Castle	3
6	$\varphi_1(\mathbf{E}_6)\otimes\mathbf{C}^{3}$	Ladder	3
7	$\varphi_1(\mathbf{E}_6)\otimes\mathbf{C}^{2}$	Ladder	1
8	$\varphi_1(\mathbf{F}_4)\otimes\mathbf{C}^{25}$	Castle	2
9	$\varphi_1(\mathbf{G}_2)\otimes\mathbf{C}^{6}\oplus\varphi_1(\mathbf{G}_2)$	Castle	3
10	$\varphi_1(\mathbf{G}_2)\otimes\mathbf{C}^{6}\oplus\mathbf{C}^{6}\oplus\mathbf{C}^{6^*}$	Slice 9d.3	5
11	$\varphi_1(\mathbf{G}_2)\otimes\mathbf{C}^{6}\oplus 2\mathbf{C}^{6^*}$	Slice 9d.4	5
12	$\varphi_1(\mathbf{G}_2)\otimes\mathbf{C}^{5}\oplus\varphi_1(\mathbf{G}_2)$	Castle	4
13	$\varphi_1(\mathbf{G}_2)\otimes\mathbf{C}^{5}\oplus\mathbf{C}^{5}$	Castle	3
14	$\varphi_1(\mathbf{G}_2)\otimes\mathbf{C}^{5}\oplus\mathbf{C}^{5^*}$	Slice 9d.7	3
15	$\varphi_1(\mathbf{G}_2)\otimes\mathbf{C}^{4}\oplus\bigwedge^2\mathbf{C}^{4}$	Slice 9d.9	5
16	$\varphi_1(\mathbf{G}_2)\otimes\mathbf{C}^{4}\oplus\mathbf{C}^{4}$	Slice 9d.10	4
17	$\varphi_1(\mathbf{G}_2)\otimes\mathbf{C}^{4}\oplus\mathbf{C}^{4^*}$	Slice 9d.11	4
18	$\varphi_1(\mathbf{G}_2)\otimes\mathbf{C}^{3}\oplus\mathbf{S}^2\mathbf{C}^{3^*}$	Ladder	5
19	$\varphi_1(\mathbf{G}_2)\otimes\mathbf{C}^{3}\oplus\mathbf{S}^2\mathbf{C}^{3}$	Ladder	5
20	$\varphi_1(\mathbf{G}_2)\otimes\mathbf{C}^{3}\oplus\mathbf{C}^{3}\oplus\mathbf{C}^{3^*}$	Ladder	5
21	$\varphi_1(\mathbf{G}_2)\otimes\mathbf{C}^{2}\oplus\varphi_1(\mathbf{G}_2)$	Ladder	4
22	$\varphi_1(\mathbf{G}_2)\otimes\mathbf{C}^{2}\oplus\mathbf{S}^2\mathbf{C}^{2}$	Ladder	3
23	$\varphi_1(\mathbf{G}_2)\otimes\mathbf{C}^{2}\oplus\mathbf{C}^{2}$	Ladder	2

(b) Exceptional × SL(n) $n \geqslant \dim\varphi_1(G_1)$

	Plain representation	Method	d
1	$2\varphi_1(\mathbf{E}_7)\otimes\mathbf{C}^{n}\oplus(n-112)\mathbf{C}^{n}\oplus\mathbf{C}^{n^*}$	Strata	$n-104$
2	$\varphi_1(\mathbf{E}_7)\otimes\mathbf{C}^{n}\oplus(n-56)\mathbf{C}^{n}\oplus\mathbf{C}^{n^*}\oplus\varphi_1(\mathbf{E}_7)$	Strata	$n-48$
3	$\varphi_1(\mathbf{E}_7)\otimes\mathbf{C}^{n}\oplus(n-56)\mathbf{C}^{n}\oplus 2\mathbf{C}^{n^*}$	Strata	$2n-104$
4	$\varphi_1(\mathbf{E}_7)\otimes\mathbf{C}^{56}\oplus\mathbf{C}^{56}$	Castle	2
5	$3\varphi_1(\mathbf{E}_6)\otimes\mathbf{C}^{n}\oplus(n-81)\mathbf{C}^{n}\oplus\mathbf{C}^{n^*}$	Strata	$n-69$
6	$2\varphi_1(\mathbf{E}_6)\otimes\mathbf{C}^{n}\oplus(n-54)\mathbf{C}^{n}\oplus\mathbf{C}^{n^*}\oplus\varphi_1(\mathbf{E}_6)$	Strata	$n-42$
7	$2\varphi_1(\mathbf{E}_6)\otimes\mathbf{C}^{n}\oplus(n-54)\mathbf{C}^{n}\oplus\mathbf{C}^{n^*}\oplus\varphi_1(\mathbf{E}_6)^*$	Strata	$n-42$
8	$\varphi_1(\mathbf{E}_6)\otimes\mathbf{C}^{n}\oplus(n-27)\mathbf{C}^{n}\oplus\mathbf{C}^{n^*}\oplus 2\varphi_1(\mathbf{E}_6)$	Strata	$n-15$
9	$\varphi_1(\mathbf{E}_6)\otimes\mathbf{C}^{n}\oplus(n-27)\mathbf{C}^{n}\oplus\mathbf{C}^{n^*}\oplus\varphi_1(\mathbf{E}_6)\oplus\varphi_1(\mathbf{E}_6)^*$	Strata	$n-15$
10	$\varphi_1(\mathbf{E}_6)\otimes\mathbf{C}^{n}\oplus(n-27)\mathbf{C}^{n}\oplus\mathbf{C}^{n^*}\oplus 2\varphi_1(\mathbf{E}_6)^*$	Strata	$n-15$

Table continued

TABLE VIII—*Continued*

	(b) Exceptional × $\mathbf{SL}(n)$ $\quad n \geqslant \dim \varphi_1(G_1)$		
	Plain representation	Method	d
11	$\varphi_1(\mathbf{E}_6)\otimes\mathbf{C}^n\oplus(n-27)\mathbf{C}^n\oplus 2\mathbf{C}^{n*}\oplus\varphi_1(\mathbf{E}_6)$	Strata	$2n-42$
12	$\varphi_1(\mathbf{E}_6)\otimes\mathbf{C}^n\oplus(n-27)\mathbf{C}^n\oplus 2\mathbf{C}^{n*}\oplus\varphi_1(\mathbf{E}_6)^*$	Strata	$2n-42$
13	$\varphi_1(\mathbf{E}_6)\otimes\mathbf{C}^n\oplus(n-27)\mathbf{C}^n\oplus 3\mathbf{C}^{n*}$	Strata	$3n-69$
14	$\varphi_1(\mathbf{E}_6)\otimes\mathbf{C}^{27}\oplus\mathbf{C}^{27}\oplus\varphi_1(\mathbf{E}_6)$	Castle	5
15	$\varphi_1(\mathbf{E}_6)\otimes\mathbf{C}^{27}\oplus\mathbf{C}^{27}\oplus\varphi_1(\mathbf{E}_6)^*$	Castle	5
16	$2\varphi_1(\mathbf{F}_4)\otimes\mathbf{C}^n\oplus(n-52)\mathbf{C}^n\oplus\mathbf{C}^{n*}$	Strata	$n-43$
17	$\varphi_1(\mathbf{F}_4)\otimes\mathbf{C}^n\oplus(n-26)\mathbf{C}^n\oplus\mathbf{C}^{n*}\oplus\varphi_1(\mathbf{F}_4)$	Strata	$n-17$
18	$\varphi_1(\mathbf{F}_4)\otimes\mathbf{C}^n\oplus(n-26)\mathbf{C}^n\oplus 2\mathbf{C}^{n*}$	Strata	$2n-43$
19	$3\varphi_1(\mathbf{G}_2)\otimes\mathbf{C}^n\oplus(n-21)\mathbf{C}^n\oplus\mathbf{C}^{n*}$	Strata	$n-13$
20	$2\varphi_1(\mathbf{G}_2)\otimes\mathbf{C}^n\oplus(n-14)\mathbf{C}^n\oplus\mathbf{C}^{n*}\oplus\varphi_1(\mathbf{G}_2)$	Strata	$n-6$
21	$\varphi_1(\mathbf{G}_2)\otimes\mathbf{C}^n\oplus(n-7)\mathbf{C}^n\oplus\mathbf{C}^{n*}\oplus 2\varphi_1(\mathbf{G}_2)$	Strata	$n+1$
22	$\varphi_1(\mathbf{G}_2)\otimes\mathbf{C}^n\oplus(n-7)\mathbf{C}^n\oplus 2\mathbf{C}^{n*}\oplus\varphi_1(\mathbf{G}_2)$	Strata	$2n-6$
23	$\varphi_1(\mathbf{G}_2)\otimes\mathbf{C}^n\oplus(n-7)\mathbf{C}^n\oplus 3\mathbf{C}^{n*}$	Strata	$3n-13$
24	$\varphi_1(\mathbf{G}_2)\otimes\mathbf{C}^7\oplus\mathbf{C}^7\oplus\mathbf{C}^{7*}$	Strata	4
25	$\varphi_1(\mathbf{G}_2)\otimes\mathbf{C}^7\oplus\mathbf{C}^7\oplus\varphi_1(\mathbf{G}_2)$	Strata	4

TABLE IX

	(a) $\mathbf{Spin}(m)\times\mathbf{SL}(n)$ $\quad V\supset\sigma(m),\ \ m\geqslant 11;\ m>n$		
	Plain representation	Method	d
1	$\mathbf{C}^{14}\otimes\mathbf{C}^{13}\oplus\sigma^+(14)$	Castle	3
2	$\mathbf{C}^{14}\otimes\mathbf{C}^{12}\oplus\sigma^+(14)$	Castle	4
3	$\mathbf{C}^{14}\otimes\mathbf{C}^{3}\oplus\sigma^+(14)$	Strata	7
4	$\mathbf{C}^{14}\otimes\mathbf{C}^{2}\oplus\sigma^+(14)$	Ladder	4
5	$\mathbf{C}^{12}\otimes\mathbf{C}^{11}\oplus\sigma^+(12)\oplus\mathbf{C}^{11}$	Castle	4
6	$\mathbf{C}^{12}\otimes\mathbf{C}^{11}\oplus\sigma^+(12)\oplus\mathbf{C}^{11*}$	Strata	4
7	$\mathbf{C}^{12}\otimes\mathbf{C}^{10}\oplus\sigma^+(12)\oplus\mathbf{C}^{10}$	Castle	5
8	$\mathbf{C}^{12}\otimes\mathbf{C}^{10}\oplus\sigma^+(12)\oplus\mathbf{C}^{10*}$	Expand	5
9	$\mathbf{C}^{12}\otimes\mathbf{C}^{9}\oplus\sigma^+(12)\oplus\mathbf{C}^{9}$	Castle	6
10	$\mathbf{C}^{12}\otimes\mathbf{C}^{9}\oplus\sigma^+(12)\oplus\mathbf{C}^{9*}$	Expand	6
11	$\mathbf{C}^{12}\otimes\mathbf{C}^{8}\oplus\sigma^+(12)\oplus\mathbf{C}^{8}$	Castle	7
12	$\mathbf{C}^{12}\otimes\mathbf{C}^{8}\oplus\sigma^+(12)\oplus\mathbf{C}^{8*}$	Expand	7
13	$\mathbf{C}^{12}\otimes\mathbf{C}^{7}\oplus\sigma^+(12)$	Castle	4
14	$\mathbf{C}^{12}\otimes\mathbf{C}^{6}\oplus\sigma^+(12)$	Strata	6
15	$\mathbf{C}^{12}\otimes\mathbf{C}^{5}\oplus\sigma^+(12)\oplus\mathbf{C}^{5}$	Strata	7
16	$\mathbf{C}^{12}\otimes\mathbf{C}^{4}\oplus\sigma^+(12)\oplus\mathbf{C}^{4}$	Strata	6
17	$\mathbf{C}^{12}\otimes\mathbf{C}^{4}\oplus\sigma^+(12)\oplus\mathbf{C}^{4*}$	Strata	6
18	$\mathbf{C}^{12}\otimes\mathbf{C}^{3}\oplus\sigma^+(12)\oplus\mathbf{C}^{3}$	Ladder	5

Table continued

TABLE IX—*Continued*

(a) $\mathbf{Spin}(m)\times\mathbf{SL}(n)$

$V\supset\sigma(m)$, $m\geqslant 11$; $m>n$

	Plain representation	Method	d
19	$\mathbf{C}^{12}\otimes\mathbf{C}^{2}\oplus\sigma^{+}(12)\oplus\mathbf{S}^{2}\mathbf{C}^{2}$	Ladder	5
20	$\mathbf{C}^{12}\otimes\mathbf{C}^{2}\oplus\sigma^{+}(12)\oplus\mathbf{C}^{2}$	Ladder	4
21	$\mathbf{C}^{11}\otimes\mathbf{C}^{10}\oplus\sigma(11)$	Castle	3
22	$\mathbf{C}^{11}\otimes\mathbf{C}^{9}\oplus\sigma(11)$	Castle	4
23	$\mathbf{C}^{11}\otimes\mathbf{C}^{8}\oplus\sigma(11)$	Castle	5
24	$\mathbf{C}^{11}\otimes\mathbf{C}^{7}\oplus\sigma(11)$	Castle	6
25	$\mathbf{C}^{11}\otimes\mathbf{C}^{4}\oplus\sigma(11)$	Strata	6
26	$\mathbf{C}^{11}\otimes\mathbf{C}^{3}\oplus\sigma(11)$	Ladder	5
27	$\mathbf{C}^{11}\otimes\mathbf{C}^{2}\oplus\sigma(11)$	Ladder	4

(b) $\mathbf{Spin}(m)\times\mathbf{SL}(n)$

$V\supset\sigma(m)$, $m=9,\ 10$; $m>n$

	Plain representation	Method	d
1	$\mathbf{C}^{10}\otimes\mathbf{C}^{9}\oplus\sigma^{+}(10)\oplus 3\mathbf{C}^{9*}$	Strata	11
2	$\mathbf{C}^{10}\otimes\mathbf{C}^{8}\oplus\sigma^{+}(10)\oplus 3\mathbf{C}^{8*}$	Expand	12
3	$\mathbf{C}^{10}\otimes\mathbf{C}^{7}\oplus\sigma^{+}(10)\oplus 2\mathbf{C}^{7*}$	Expand	8
4	$\mathbf{C}^{10}\otimes\mathbf{C}^{6}\oplus\sigma^{+}(10)\oplus\mathbf{C}^{6*}$	Strata	5
5	$\mathbf{C}^{10}\otimes\mathbf{C}^{5}\oplus\sigma^{+}(10)\oplus\bigwedge^{2}\mathbf{C}^{5}$	Strata	7
6	$\mathbf{C}^{10}\otimes\mathbf{C}^{5}\oplus\sigma^{+}(10)\oplus 2\mathbf{C}^{5*}$	Strata	8
7	$\mathbf{C}^{10}\otimes\mathbf{C}^{4}\oplus\sigma^{+}(10)\oplus\bigwedge^{2}\mathbf{C}^{4}$	Ladder	5
8	$\mathbf{C}^{10}\otimes\mathbf{C}^{4}\oplus\sigma^{+}(10)\oplus\mathbf{C}^{4*}$	Ladder	4
9	$\mathbf{C}^{10}\otimes\mathbf{C}^{3}\oplus\sigma^{+}(10)\oplus\mathbf{C}^{3*}$	Ladder	4
10	$\mathbf{C}^{10}\otimes\mathbf{C}^{2}\oplus\sigma^{+}(10)$	Ladder	2
11	$\mathbf{C}^{9}\otimes\mathbf{C}^{8}\oplus\sigma(9)\oplus 2\mathbf{C}^{8*}$	Strata	8
12	$\mathbf{C}^{9}\otimes\mathbf{C}^{7}\oplus\sigma(9)\oplus 2\mathbf{C}^{7*}$	Castle	9
13	$\mathbf{C}^{9}\otimes\mathbf{C}^{6}\oplus\sigma(9)\oplus\mathbf{C}^{6*}$	Castle	6
14	$\mathbf{C}^{9}\otimes\mathbf{C}^{5}\oplus\sigma(9)\oplus\mathbf{C}^{5*}$	Strata	7
15	$\mathbf{C}^{9}\otimes\mathbf{C}^{4}\oplus\sigma(9)\oplus\bigwedge^{2}\mathbf{C}^{4}$	Strata	7
16	$\mathbf{C}^{9}\otimes\mathbf{C}^{4}\oplus\sigma(9)\oplus\mathbf{C}^{4*}$	Strata	6
17	$\mathbf{C}^{9}\otimes\mathbf{C}^{3}\oplus\sigma(9)\oplus\mathbf{C}^{3*}$	Ladder	5
18	$\mathbf{C}^{9}\otimes\mathbf{C}^{2}\oplus\sigma(9)$	Ladder	3

(c) $\mathbf{Spin}(8)\times\mathbf{SL}(n)$

$V\supset\sigma^{+}(8)$, $8>n$

	Plain representation	Method	d
1	$\mathbf{C}^{8}\otimes\mathbf{C}^{7}\oplus 2\sigma^{+}(8)\oplus\mathbf{C}^{7}$	Castle	6
2	$\mathbf{C}^{8}\otimes\mathbf{C}^{7}\oplus 2\sigma^{+}(8)\oplus\mathbf{C}^{7*}$	Strata	6
3	$\mathbf{C}^{8}\otimes\mathbf{C}^{7}\oplus\sigma^{+}(8)\oplus\sigma^{-}(8)\oplus\mathbf{C}^{7*}$	Strata	6

Table continued

TABLE IX—*Continued*

(c) **Spin**(8) × **SL**(n)

$V \supset \sigma^+(8)$, $8 > n$

	Plain representation	Method	d
4	$\mathbf{C}^8 \otimes \mathbf{C}^7 \oplus \sigma^+(8) \oplus \mathbf{C}^8 \oplus \mathbf{C}^{7*}$	Strata	6
5	$\mathbf{C}^8 \otimes \mathbf{C}^7 \oplus \sigma^+(8) \oplus \mathbf{C}^7 \oplus 2\mathbf{C}^{7*}$	Castle	9
6	$\mathbf{C}^8 \otimes \mathbf{C}^6 \oplus 2\sigma^+(8) \oplus \mathbf{C}^6$	Castle	7
7	$\mathbf{C}^8 \otimes \mathbf{C}^6 \oplus 2\sigma^+(8) \oplus \mathbf{C}^{6*}$	Castle	7
8	$\mathbf{C}^8 \otimes \mathbf{C}^6 \oplus \sigma^+(8) \oplus \sigma^-(8) \oplus \mathbf{C}^{6*}$	Slice 9b.12	7
9	$\mathbf{C}^8 \otimes \mathbf{C}^6 \oplus \sigma^+(8) \oplus \wedge^2 \mathbf{C}^6$	Restrict	8
10	$\mathbf{C}^8 \otimes \mathbf{C}^6 \oplus \sigma^+(8) \oplus \wedge^2 \mathbf{C}^{6*}$	Restrict	8
11	$\mathbf{C}^8 \otimes \mathbf{C}^6 \oplus \sigma^+(8) \oplus \mathbf{C}^8 \oplus \mathbf{C}^{6*}$	Slice 9c.5	7
12	$\mathbf{C}^8 \otimes \mathbf{C}^6 \oplus \sigma^+(8) \oplus \mathbf{C}^6 \oplus \mathbf{C}^{6*}$	Castle	6
13	$\mathbf{C}^8 \otimes \mathbf{C}^6 \oplus \sigma^+(8) \oplus 2\mathbf{C}^{6*}$	Castle	6
14	$\mathbf{C}^8 \otimes \mathbf{C}^5 \oplus 2\sigma^+(8)$	Castle	5
15	$\mathbf{C}^8 \otimes \mathbf{C}^5 \oplus \sigma^+(8) \oplus \sigma^-(8)$	Castle	5
16	$\mathbf{C}^8 \otimes \mathbf{C}^5 \oplus \sigma^+(8) \oplus \mathbf{C}^8$	Castle	5
17	$\mathbf{C}^8 \otimes \mathbf{C}^5 \oplus \sigma^+(8) \oplus \wedge^2 \mathbf{C}^{5*}$	Strata	6
18	$\mathbf{C}^8 \otimes \mathbf{C}^5 \oplus \sigma^+(8) \oplus \mathbf{C}^5 \oplus \mathbf{C}^{5*}$	Strata	7
19	$\mathbf{C}^8 \otimes \mathbf{C}^4 \oplus 2\sigma^+(8)$	Strata	7
20	$\mathbf{C}^8 \otimes \mathbf{C}^4 \oplus \sigma^+(8) \oplus \sigma^-(8)$	Slice 9b.14	6
21	$\mathbf{C}^8 \otimes \mathbf{C}^4 \oplus \sigma^+(8) \oplus \mathbf{C}^8$	Slice 9c.18	6
22	$\mathbf{C}^8 \otimes \mathbf{C}^4 \oplus \sigma^+(8) \oplus \mathbf{S}^2\mathbf{C}^4$	Strata	7
23	$\mathbf{C}^8 \otimes \mathbf{C}^4 \oplus \sigma^+(8) \oplus \mathbf{S}^2\mathbf{C}^{4*}$	Strata	7
24	$\mathbf{C}^8 \otimes \mathbf{C}^4 \oplus \sigma^+(8) \oplus \wedge^2 \mathbf{C}^4 \oplus \mathbf{C}^4$	Strata	7
25	$\mathbf{C}^8 \otimes \mathbf{C}^4 \oplus \sigma^+(8) \oplus \wedge^2 \mathbf{C}^4 \oplus \mathbf{C}^{4*}$	Strata	7
26	$\mathbf{C}^8 \otimes \mathbf{C}^4 \oplus \sigma^+(8) \oplus \mathbf{C}^4 \oplus \mathbf{C}^{4*}$	Strata	6
27	$\mathbf{C}^8 \otimes \mathbf{C}^4 \oplus \sigma^+(8) \oplus 2\mathbf{C}^{4*}$	Strata	6
28	$\mathbf{C}^8 \otimes \mathbf{C}^3 \oplus 2\sigma^+(8) \oplus \mathbf{C}^3$	Strata	7
29	$\mathbf{C}^8 \otimes \mathbf{C}^3 \oplus \sigma^+(8) \oplus \sigma^-(8) \oplus \mathbf{C}^{3*}$	Strata	7
30	$\mathbf{C}^8 \otimes \mathbf{C}^3 \oplus \sigma^+(8) \oplus \mathbf{C}^8 \oplus \mathbf{C}^{3*}$	Strata	7
31	$\mathbf{C}^8 \otimes \mathbf{C}^3 \oplus \sigma^+(8) \oplus \mathbf{S}^2\mathbf{C}^3$	Ladder	5
32	$\mathbf{C}^8 \otimes \mathbf{C}^3 \oplus \sigma^+(8) \oplus \mathbf{S}^2\mathbf{C}^{3*}$	Ladder	5
33	$\mathbf{C}^8 \otimes \mathbf{C}^3 \oplus \sigma^+(8) \oplus \mathbf{C}^3 \oplus \mathbf{C}^{3*}$	Ladder	5
34	$\mathbf{C}^8 \otimes \mathbf{C}^2 \oplus \sigma^+(8) \oplus \sigma^-(8)$	Ladder	4
35	$\mathbf{C}^8 \otimes \mathbf{C}^2 \oplus \sigma^+(8) \oplus \mathbf{C}^8$	Ladder	4
36	$\mathbf{C}^8 \otimes \mathbf{C}^2 \oplus 3\sigma^+(8) \oplus \mathbf{S}^2\mathbf{C}^2$	Strata	12
37	$\mathbf{C}^8 \otimes \mathbf{C}^2 \oplus 3\sigma^+(8) \oplus \mathbf{C}^2$	Strata	11

Table continued

TABLE IX—*Continued*

(d) $\mathbf{Spin}(7)\times\mathbf{SL}(n)$

$V\supset\sigma(7)$, $7>n$

	Plain representation	Method	d
1	$\mathbf{C}^7\otimes\mathbf{C}^6\oplus 2\sigma(7)$	Castle	5
2	$\mathbf{C}^7\otimes\mathbf{C}^6\oplus\sigma(7)\oplus\mathbf{C}^7$	Castle	4
3	$\mathbf{C}^7\otimes\mathbf{C}^6\oplus\sigma(7)\oplus\mathbf{C}^6\oplus\mathbf{C}^{6*}$	Slice 9c.11	6
4	$\mathbf{C}^7\otimes\mathbf{C}^6\oplus\sigma(7)\oplus 2\mathbf{C}^{6*}$	Strata	6
5	$\mathbf{C}^7\otimes\mathbf{C}^5\oplus 2\sigma(7)$	Castle	6
6	$\mathbf{C}^7\otimes\mathbf{C}^5\oplus\sigma(7)\oplus\mathbf{C}^7$	Castle	5
7	$\mathbf{C}^7\otimes\mathbf{C}^5\oplus\sigma(7)\oplus\mathbf{C}^{5*}$	Slice 9c.13	4
8	$\mathbf{C}^7\otimes\mathbf{C}^5\oplus\sigma(7)\oplus\mathbf{C}^5$	Castle	4
9	$\mathbf{C}^7\otimes\mathbf{C}^4\oplus\sigma(7)\oplus\wedge^2\mathbf{C}^4$	Strata	6
10	$\mathbf{C}^7\otimes\mathbf{C}^4\oplus\sigma(7)\oplus\mathbf{C}^4$	Strata	5
11	$\mathbf{C}^7\otimes\mathbf{C}^4\oplus\sigma(7)\oplus\mathbf{C}^{4*}$	Strata	5
12	$\mathbf{C}^7\otimes\mathbf{C}^3\oplus\sigma(7)\oplus\mathbf{S}^2\mathbf{C}^3$	Ladder	6
13	$\mathbf{C}^7\otimes\mathbf{C}^3\oplus\sigma(7)\oplus\mathbf{S}^2\mathbf{C}^{3*}$	Ladder	6
14	$\mathbf{C}^7\otimes\mathbf{C}^3\oplus\sigma(7)\oplus\mathbf{C}^3\oplus\mathbf{C}^{3*}$	Ladder	6
15	$\mathbf{C}^7\otimes\mathbf{C}^2\oplus 2\sigma(7)$	Strata	6
16	$\mathbf{C}^7\otimes\mathbf{C}^2\oplus\sigma(7)\oplus\mathbf{C}^7$	Ladder	5
17	$\mathbf{C}^7\otimes\mathbf{C}^2\oplus\sigma(7)\oplus\mathbf{S}^2\mathbf{C}^2$	Ladder	4
18	$\mathbf{C}^7\otimes\mathbf{C}^2\oplus\sigma(7)\oplus\mathbf{C}^2$	Ladder	3

(e) $\mathbf{Spin}(m)\times\mathbf{SL}(n)$

$V\supset\sigma(m)$ or $\sigma^{\pm}(6)^2$, $m=5, 6;\ m>n$

	Plain representation	Method	d
1	$\mathbf{C}^6\otimes\mathbf{C}^5\oplus\sigma^{+}(6)^2$	Castle	3
2	$\mathbf{C}^6\otimes\mathbf{C}^5\oplus 2\sigma^{+}(6)\oplus\sigma^{-}(6)$	Castle	4
3	$\mathbf{C}^6\otimes\mathbf{C}^5\oplus 2\sigma^{+}(6)\oplus\mathbf{C}^{5*}$	Strata	4
4	$\mathbf{C}^6\otimes\mathbf{C}^5\oplus\sigma^{+}(6)\oplus\sigma^{-}(6)\oplus\mathbf{C}^{5*}$	Strata	4
5	$\mathbf{C}^6\otimes\mathbf{C}^5\oplus\sigma^{+}(6)\oplus\sigma^{-}(6)\oplus\mathbf{C}^5$	Castle	4
6	$\mathbf{C}^6\otimes\mathbf{C}^5\oplus\sigma^{+}(6)\oplus\mathbf{C}^6$	Castle	3
7	$\mathbf{C}^6\otimes\mathbf{C}^5\oplus\sigma^{+}(6)\oplus\mathbf{C}^5\oplus\mathbf{C}^{5*}$	Castle	5
8	$\mathbf{C}^6\otimes\mathbf{C}^4\oplus\sigma^{+}(6)^2$	Castle	4
9	$\mathbf{C}^6\otimes\mathbf{C}^4\oplus 2\sigma^{+}(6)$	Castle	2
10	$\mathbf{C}^6\otimes\mathbf{C}^4\oplus\sigma^{+}(6)\oplus\sigma^{-}(6)$	Castle	3
11	$\mathbf{C}^6\otimes\mathbf{C}^4\oplus\sigma^{+}(6)\oplus\mathbf{C}^6$	Castle	4
12	$\mathbf{C}^6\otimes\mathbf{C}^4\oplus\sigma^{+}(6)\oplus\wedge^2\mathbf{C}^4$	Strata	4
13	$\mathbf{C}^6\otimes\mathbf{C}^4\oplus\sigma^{+}(6)\oplus\mathbf{C}^4$	Strata	3
14	$\mathbf{C}^6\otimes\mathbf{C}^4\oplus\sigma^{+}(6)\oplus\mathbf{C}^{4*}$	Strata	3

Table continued

TABLE IX—*Continued*

	(e) $\mathbf{Spin}(m) \times \mathbf{SL}(n)$ $V \supset \sigma(m)$ or $\sigma^+(6)^2$, $m = 5, 6$; $m > n$		
	Plain representation	Method	d
15	$\mathbf{C}^6 \otimes \mathbf{C}^3 \oplus \sigma^+(6) \oplus \sigma^-(6)$	Slice 9d.11	4
16	$\mathbf{C}^6 \otimes \mathbf{C}^3 \oplus \sigma^+(6) \oplus \mathbf{C}^6$	Strata	5
17	$\mathbf{C}^6 \otimes \mathbf{C}^3 \oplus \sigma^+(6) \oplus S^2\mathbf{C}^3$	Strata	5
18	$\mathbf{C}^6 \otimes \mathbf{C}^3 \oplus \sigma^+(6) \oplus S^2\mathbf{C}^{3*}$	Strata	5
19	$\mathbf{C}^6 \otimes \mathbf{C}^3 \oplus \sigma^+(6) \oplus \mathbf{C}^3 \oplus \mathbf{C}^{3*}$	Strata	5
20	$\mathbf{C}^6 \otimes \mathbf{C}^2 \oplus \sigma^+(6)^2$	Strata	4
21	$\mathbf{C}^6 \otimes \mathbf{C}^2 \oplus \sigma^+(6) \oplus \mathbf{C}^6$	Strata	4
22	$\mathbf{C}^6 \otimes \mathbf{C}^2 \oplus 2\sigma^+(6)$	Ladder	2
23	$\mathbf{C}^6 \otimes \mathbf{C}^2 \oplus \sigma^+(6) \oplus \sigma^-(6) \oplus S^2\mathbf{C}^2$	Ladder	5
24	$\mathbf{C}^6 \otimes \mathbf{C}^2 \oplus \sigma^+(6) \oplus \sigma^-(6) \oplus \mathbf{C}^2$	Ladder	4
25	$\mathbf{C}^5 \otimes \mathbf{C}^4 \oplus 2\sigma(5)$	Castle	3
26	$\mathbf{C}^5 \otimes \mathbf{C}^4 \oplus \sigma(5) \oplus \mathbf{C}^4$	Castle	3
27	$\mathbf{C}^5 \otimes \mathbf{C}^4 \oplus \sigma(5) \oplus \mathbf{C}^{4*}$	Strata	3
28	$\mathbf{C}^5 \otimes \mathbf{C}^3 \oplus \sigma(5) \oplus \mathbf{C}^3$	Strata	4
29	$\mathbf{C}^5 \otimes \mathbf{C}^3 \oplus \sigma(5) \oplus \mathbf{C}^{3*}$	Strata	4
30	$\mathbf{C}^5 \otimes \mathbf{C}^2 \oplus \sigma(5) \oplus S^2\mathbf{C}^2$	Ladder	4
31	$\mathbf{C}^5 \otimes \mathbf{C}^2 \oplus \sigma(5) \oplus \mathbf{C}^2$	Ladder	3

2. Statement of Results

Our main result is:

Classification. Let $G = G_1 \times G_2$ be a simply-connected connected semi-simple group having exactly two simple factors and not of the form $G \cong \mathbf{SL}(m) \times \mathbf{SL}(n)$. Suppose (V, G) is an equidimensional representation with $V^G = \{0\}$. Then either V decomposes as a sum of equidimensional representations of simple groups: $(V, G) = (V_1, G_1) \oplus (V_2, G_2)$ or V is (up to an outer automorphism of G) a subrepresentation of one of the representations listed in Tables I–XVIII. Each representation listed is maximally equidimensional. Furthermore, all of the representations listed are coregular and hence also cofree. In particular, the Popov Conjecture is true for all such groups, G.

Remark 2.0.1. We do in fact list some representations of some groups of the form $\mathbf{SL}(m) \times \mathbf{SL}(n)$. Specifically, we list all maximally equidimensional representations of the groups $\mathbf{SL}(m) \times \mathbf{Spin}(6) \cong \mathbf{SL}(m) \times \mathbf{SL}(4)$ which are not of the form $V \cong k_1 \mathbf{C}^m \otimes \mathbf{C}^4 \oplus k_2 \mathbf{C}^m \otimes \mathbf{C}^{4*} \oplus k_3 \mathbf{C}^{m*} \otimes \mathbf{C}^4 \oplus k_4 \mathbf{C}^{m*} \otimes \mathbf{C}^{4*} \oplus \rho_1(\mathbf{SL}(m)) \oplus \rho_2(\mathbf{SL}(4))$.

TABLE X

	$\mathbf{Spin}(m)\times\mathbf{SL}(n),\ n\geqslant m$	$V\supset\sigma(m)$		
	Plain representation	Conditions	Method	d
1	$k\mathbf{C}^{14}\otimes\mathbf{C}^{n}\oplus\sigma^{+}(14)\oplus(n-14k)\mathbf{C}^{n}\oplus b\mathbf{C}^{n^*}\oplus c\mathbf{C}^{14}$	$kb+c=2$	Strata	$b(n-14k)+8$
2	$k\mathbf{C}^{13}\otimes\mathbf{C}^{n}\oplus\sigma(13)\oplus(n-13k)\mathbf{C}^{n}\oplus b\mathbf{C}^{n^*}\oplus c\mathbf{C}^{13}$	$kb+c=2$	Strata	$b(n-13k)+13$
3	$\mathbf{C}^{12}\otimes\mathbf{C}^{n}\oplus 2\sigma^{+}(12)\oplus(n-12)\mathbf{C}^{n}\oplus\mathbf{C}^{n^*}$		Strata	$n-1$
4	$\mathbf{C}^{12}\otimes\mathbf{C}^{n}\oplus\sigma^{+}(12)\oplus\sigma^{-}(12)\oplus(n-12)\mathbf{C}^{n}\oplus\mathbf{C}^{n^*}$		Strata	$n-1$
5	$k\mathbf{C}^{12}\otimes\mathbf{C}^{n}\oplus\sigma^{+}(12)\oplus(n-12k)\mathbf{C}^{n}\oplus b\mathbf{C}^{n^*}\oplus c\mathbf{C}^{12}$	$kb+c=3$	Strata	$b(n-12k)+11$
6	$\overline{\mathbf{C}^{12}}\otimes\mathbf{C}^{12}\oplus\overline{\sigma^{+}(12)}\oplus\mathbf{C}^{12}$		Castle	3
7	$k\mathbf{C}^{11}\otimes\mathbf{C}^{n}\oplus\sigma(11)\oplus(n-11k)\mathbf{C}^{n}\oplus b\mathbf{C}^{n^*}\oplus c\mathbf{C}^{11}$	$kb+c=3$	Strata	$b(n-11k)+14$
8	$k\mathbf{C}^{10}\otimes\mathbf{C}^{n}\oplus 2\sigma^{+}(10)\oplus(n-10k)\mathbf{C}^{n}\oplus b\mathbf{C}^{n^*}\oplus c\mathbf{C}^{10}$	$kb+c=2$	Strata	$b(n-10k)+11$
9	$k\mathbf{C}^{10}\otimes\mathbf{C}^{n}\oplus\sigma^{+}(10)\oplus\sigma^{-}(10)\oplus(n-10k)\mathbf{C}^{n}\oplus b\mathbf{C}^{n^*}\oplus c\mathbf{C}^{10}$	$kb+c=2$	Strata	$b(n-10k)+11$
10	$k\mathbf{C}^{10}\otimes\mathbf{C}^{n}\oplus\sigma^{+}(10)\oplus(n-10k)\mathbf{C}^{n}\oplus b\mathbf{C}^{n^*}\oplus c\mathbf{C}^{10}$	$kb+c=4$	Strata	$b(n-10k)+15$
11	$\mathbf{C}^{9}\otimes\mathbf{C}^{n}\oplus 2\sigma(9)\oplus(n-9)\mathbf{C}^{n}\oplus\mathbf{C}^{n^*}$		Strata	n
12	$k\mathbf{C}^{9}\otimes\mathbf{C}^{n}\oplus\sigma(9)\oplus(n-9k)\mathbf{C}^{n}\oplus b\mathbf{C}^{n^*}\oplus c\mathbf{C}^{9}$	$kb+c=3$	Strata	$b(n-9k)+11$
13	$\mathbf{C}^{8}\otimes\mathbf{C}^{n}\oplus\sigma^{+}(8)\oplus\bigwedge^{2}\mathbf{C}^{n}$	n even	Expand	9
14	$\mathbf{C}^{8}\otimes\mathbf{C}^{n}\oplus\sigma^{+}(8)\oplus\bigwedge^{2}\mathbf{C}^{n^*}$	$n=2r\geqslant 10$	Strata	9
15	$\mathbf{C}^{8}\otimes\mathbf{C}^{n}\oplus\sigma^{+}(8)\oplus\bigwedge^{2}\mathbf{C}^{n^*}$	n odd	Strata	8
16	$k\mathbf{C}^{8}\otimes\mathbf{C}^{n}\oplus 2\sigma^{+}(8)\oplus(n-8k)\mathbf{C}^{n}\oplus b\mathbf{C}^{n^*}\oplus c\mathbf{C}^{8}$	$kb+c=2$	Strata	$b(n-8k)+8$

17	$k\mathbf{C}^8 \otimes \mathbf{C}^n \oplus \sigma^+(8) \oplus \sigma^-(8) \oplus (n-8k)\mathbf{C}^n \oplus b\mathbf{C}^{n*} \oplus c\mathbf{C}^8$	$kb + c = 2$	Strata	$b(n-8k)+8$
18	$\underline{k\mathbf{C}}^8 \otimes \mathbf{C}^n \underline{\oplus \sigma^+}(8) \oplus (n-8k)\mathbf{C}^n \oplus b\mathbf{C}^{n*} \oplus c\mathbf{C}^8$	$kb + c = 3$	Strata	$b(n-8k)+8$
19	$\overline{\mathbf{C}^8} \otimes \mathbf{C}^8 \oplus \overline{\sigma^+(8)} \oplus \mathbf{C}^8 \oplus 2\mathbf{C}^{8*}$		Strata	8
20	$\overline{\mathbf{C}^8} \otimes \mathbf{C}^8 \oplus \overline{2\sigma^+(8)} \oplus \mathbf{C}^8$		Strata	5
21	$\mathbf{C}^7 \otimes \mathbf{C}^n \oplus 2\sigma(7) \oplus (n-7)\mathbf{C}^n \oplus \mathbf{C}^{n*}$		Strata	$n-1$
22	$\underline{k\mathbf{C}}^7 \otimes \mathbf{C}^n \underline{\oplus \sigma(7)} \oplus (n-7k)\mathbf{C}^n \oplus b\mathbf{C}^{n*} \oplus c\mathbf{C}^7$	$kb + c = 3$	Strata	$b(n-7k)+9$
23	$\overline{\mathbf{C}^7} \otimes \mathbf{C}^7 \oplus \overline{\sigma(7)} \oplus \mathbf{C}^7 \oplus \mathbf{C}^{7*}$		Castle	5
24	$\mathbf{C}^6 \otimes \mathbf{C}^n \oplus \sigma^+(6) \oplus \bigwedge^2 \mathbf{C}^{n*}$	$n = 2r \geqslant 8$	Strata	5
25	$\mathbf{C}^6 \otimes \mathbf{C}^n \oplus \sigma^+(6) \oplus \bigwedge^2 \mathbf{C}^{n*}$	n odd	Strata	4
26	$\mathbf{C}^6 \otimes \mathbf{C}^n \oplus \sigma^+(6) \oplus \bigwedge^2 \mathbf{C}^{n}$	n even	Expand	5
27	$k\mathbf{C}^6 \otimes \mathbf{C}^n \oplus 2\sigma^+(6) \oplus (n-6k)\mathbf{C}^n \oplus b\mathbf{C}^{n*} \oplus c\mathbf{C}^6$	$kb + c = 2$	Strata	$b(n-6k)+6$
28	$k\mathbf{C}^6 \otimes \mathbf{C}^n \oplus \sigma^+(6) \oplus \sigma^-(6) \oplus (n-6k)\mathbf{C}^n \oplus b\mathbf{C}^{n*} \oplus c\mathbf{C}^6$	$kb + c = 2$	Strata	$b(n-6k)+6$
29	$\mathbf{C}^6 \otimes \mathbf{C}^n \oplus \sigma^+(6)^2 \oplus \sigma^+(6) \oplus (n-6)\mathbf{C}^n \oplus \mathbf{C}^{n*}$		Strata	n
30	$\underline{\mathbf{C}}^6 \otimes \mathbf{C}^n \oplus \sigma^+(6)^2 \oplus \underline{\sigma^-(6)} \oplus (n-6)\mathbf{C}^n \oplus \mathbf{C}^{n*}$		Strata	n
31	$\overline{\mathbf{C}^6} \otimes \mathbf{C}^6 \oplus \overline{\sigma^+(6)} \oplus \sigma^-(6) \oplus \mathbf{C}^6$		Castle	3
32	$\overline{\mathbf{C}^6} \otimes \mathbf{C}^6 \oplus \overline{\sigma^+(6)} \oplus \mathbf{C}^6 \oplus \mathbf{C}^{6*}$		Strata	4
33	$\mathbf{C}^5 \otimes \mathbf{C}^n \oplus \sigma(5) \oplus \bigwedge^2 \mathbf{C}^{n*}$	n even	Strata	5
34	$\mathbf{C}^5 \otimes \mathbf{C}^n \oplus \sigma(5) \oplus \bigwedge^2 \mathbf{C}^{n*}$	$n = 2r+1 \geqslant 7$	Strata	4
35	$\mathbf{C}^5 \otimes \mathbf{C}^n \oplus 2\sigma(5) \oplus (n-5)\mathbf{C}^n \oplus \mathbf{C}^{n*}$		Strata	$n-1$
36	$\underline{k\mathbf{C}}^5 \otimes \mathbf{C}^n \underline{\oplus \sigma(5)} \oplus (n-5k)\mathbf{C}^n \oplus b\mathbf{C}^{n*} \oplus c\mathbf{C}^5$	$kb + c = 2$	Strata	$b(n-5k)+5$
37	$\overline{\mathbf{C}^5} \otimes \mathbf{C}^5 \oplus \overline{\sigma(5)} \oplus \mathbf{C}^5$		Castle	2

TABLE XI

	$\mathbf{SP}(6)\times\mathbf{SL}(n)$ — $V\supset\varphi_2(\mathbf{SP}(6))\otimes\mathbf{C}^n$, $n\geqslant\dim\varphi_2(\mathbf{SP}(6))=14$		
	Representation	Method	d
1	$\varphi_2(\mathbf{SP}(6))\otimes\mathbf{C}^n\oplus(n-14)\mathbf{C}^n\oplus 2\mathbf{C}^{n*}$	Strata	$2n-19$

TABLE XII

(a) $\mathbf{Spin}(m)\times\mathbf{SL}(n)$, $n\geqslant\dim\sigma(m)$ $V\supset\sigma(m)\otimes\mathbf{C}^n$

	Representation	Conditions	Method	d
1	$\sigma^+(12)\otimes\mathbf{C}^n\oplus(n-32)\mathbf{C}^n\oplus 2\mathbf{C}^{n*}\oplus\mathbf{C}^{12}$		Strata	$2n-53$
2	$\sigma(11)\otimes\mathbf{C}^n\oplus(n-32)\mathbf{C}^n\oplus 2\mathbf{C}^{n*}$		Strata	$2n-54$
3	$\sigma^+(10)\otimes\mathbf{C}^n\oplus(n-16)\mathbf{C}^n\oplus 2\mathbf{C}^{n*}\oplus 2\mathbf{C}^{10}$		Strata	$2n-21$
4	$\sigma^+(10)\otimes\mathbf{C}^n\oplus(n-16)\mathbf{C}^n\oplus 2\mathbf{C}^{n*}\oplus\sigma^-(10)$		Strata	$2n-25$
5	$\sigma(9)\otimes\mathbf{C}^n\oplus(n-16)\mathbf{C}^n\oplus 2\mathbf{C}^{n*}\oplus\mathbf{C}^9$		Strata	$2n-23$
6	$\sigma(7)\otimes\mathbf{C}^n\oplus(n-8)\mathbf{C}^n\oplus 2\mathbf{C}^{n*}\oplus\mathbf{C}^7$		Strata	$2n-10$
7	$\sigma(7)\otimes\mathbf{C}^n\oplus(n-8)\mathbf{C}^n\oplus 2\mathbf{C}^{n*}\oplus\sigma(7)$		Strata	$2n-9$
8	$\sigma(7)\otimes\mathbf{C}^n\oplus(n-8)\mathbf{C}^n\oplus 3\mathbf{C}^{n*}$		Strata	$3n-17$
9	$\sigma(7)\otimes\mathbf{C}^n\oplus\bigwedge^2\mathbf{C}^n$	n even	Slice 10.13	8
10	$\sigma(7)\otimes\mathbf{C}^n\oplus\bigwedge^2\mathbf{C}^{n*}$	$n=2r\geqslant 10$	Strata	8
11	$\sigma(7)\otimes\mathbf{C}^n\oplus\bigwedge^2\mathbf{C}^{n*}$	$n=2r+1\geqslant 9$	Strata	7

(b) $\mathbf{Spin}(m)\times\mathbf{SL}(n)$

$V\supset\sigma(m)\otimes\mathbf{C}^n$ $\dim\sigma(m)-2\geqslant n$, $m\geqslant 8$

	Representation	Method	d
1	$\sigma^+(12)\otimes\mathbf{C}^2\oplus\mathbf{C}^{12}$	Ladder	7
2	$\sigma(11)\otimes\mathbf{C}^2$	Ladder	6
3	$\sigma^+(10)\otimes\mathbf{C}^{14}\oplus\mathbf{C}^{14}$	Castle	2
4	$\sigma^+(10)\otimes\mathbf{C}^{14}\oplus\mathbf{C}^{14*}$	Castle	2
5	$\sigma^+(10)\otimes\mathbf{C}^{14}\oplus\mathbf{C}^{10}$	Castle	3
6	$\sigma^+(10)\otimes\mathbf{C}^{14}\oplus\sigma^+(10)$	Castle	3
7	$\sigma^+(10)\otimes\mathbf{C}^{14}\oplus\sigma^-(10)$	Castle	3
8	$\sigma^+(10)\otimes\mathbf{C}^{13}$	Castle	1
9	$\sigma^+(10)\otimes\mathbf{C}^{12}$	Castle	4
10	$\sigma^+(10)\otimes\mathbf{C}^4$	Θ-rep	4

Table continued

TABLE XII—*Continued*

(b) $\mathbf{Spin}(m) \times \mathbf{SL}(n)$

$V \supset \sigma(m) \otimes \mathbf{C}^n$ dim $\sigma(m) - 2 \geqslant n$, $m \geqslant 8$

	Representation	Method	*d*
11	$\sigma^+(10) \otimes \mathbf{C}^3 \oplus \mathbf{S}^2\mathbf{C}^3$	Ladder	4
12	$\sigma^+(10) \otimes \mathbf{C}^3 \oplus \mathbf{S}^2\mathbf{C}^{3*}$	Ladder	4
13	$\sigma^+(10) \otimes \mathbf{C}^3 \oplus \mathbf{C}^3 \oplus \mathbf{C}^{3*}$	Ladder	4
14	$\sigma^+(10) \otimes \mathbf{C}^2 \oplus \sigma^+(10)$	Strata	3
15	$\sigma^+(10) \otimes \mathbf{C}^2 \oplus \sigma^-(10)$	Ladder	3
16	$\sigma^+(10) \otimes \mathbf{C}^2 \oplus 2\mathbf{C}^{10}$	Ladder	7
17	$\sigma^+(10) \otimes \mathbf{C}^2 \oplus \mathbf{C}^{10} \oplus \mathbf{S}^2\mathbf{C}^2$	Ladder	5
18	$\sigma^+(10) \otimes \mathbf{C}^2 \oplus \mathbf{C}^{10} \oplus \mathbf{C}^2$	Ladder	4
19	$\sigma^+(10) \otimes \mathbf{C}^2 \oplus \mathbf{S}^4\mathbf{C}^2$	Ladder	3
20	$\sigma^+(10) \otimes \mathbf{C}^2 \oplus \mathbf{S}^3\mathbf{C}^2$	Ladder	2
21	$\sigma^+(10) \otimes \mathbf{C}^2 \oplus 2\mathbf{S}^2\mathbf{C}^2$	Ladder	4
22	$\sigma^+(10) \otimes \mathbf{C}^2 \oplus \mathbf{S}^2\mathbf{C}^2 \oplus \mathbf{C}^2$	Ladder	3
23	$\sigma^+(10) \otimes \mathbf{C}^2 \oplus 2\mathbf{C}^2$	Ladder	2
24	$\sigma(9) \otimes \mathbf{C}^{14}$	Castle	2
25	$\sigma(9) \otimes \mathbf{C}^{13}$	Castle	4
26	$\sigma(9) \otimes \mathbf{C}^3$	Strata	4
27	$\sigma(9) \otimes \mathbf{C}^2 \oplus \mathbf{C}^9$	Ladder	5
28	$\sigma(9) \otimes \mathbf{C}^2 \oplus \mathbf{S}^2\mathbf{C}^2$	Ladder	4
29	$\sigma(9) \otimes \mathbf{C}^2 \oplus \mathbf{C}^2$	Ladder	3
30	$\mathbf{C}^8 \otimes \mathbf{C}^2 \oplus \sigma^+(8) \otimes \mathbf{C}^2$	Ladder	4

(c) $\mathbf{Spin}(m) \times \mathbf{SL}(n)$, $m = 6, 7$

$V \supset W \otimes \mathbf{C}^n$, dim $W - 2 \geqslant n$, $W = \sigma(7)$, $\sigma^+(6)^2$, or $\sigma^+(6) \oplus \mathbf{C}^6$

	Representation	Method	*d*
1	$\sigma(7) \otimes \mathbf{C}^6 \oplus \mathbf{C}^7 \oplus \mathbf{C}^6$	Slice 9c.6	6
2	$\sigma(7) \otimes \mathbf{C}^6 \oplus \sigma(7) \oplus \mathbf{C}^{6*}$	Slice 9c.11	6
3	$\sigma(7) \otimes \mathbf{C}^6 \oplus \mathbf{C}^7 \oplus \mathbf{C}^{6*}$	Slice 9c.7	5
4	$\sigma(7) \otimes \mathbf{C}^6 \oplus \bigwedge^2 \mathbf{C}^6$	Slice 9c.9	7
5	$\sigma(7) \otimes \mathbf{C}^6 \oplus \bigwedge^2 \mathbf{C}^{6*}$	Slice 9c.10	7
6	$\sigma(7) \otimes \mathbf{C}^6 \oplus \mathbf{C}^6 \oplus \mathbf{C}^{6*}$	Slice 9c.12	5
7	$\sigma(7) \otimes \mathbf{C}^6 \oplus 2\mathbf{C}^{6*}$	Slice 9c.13	5
8	$\sigma(7) \otimes \mathbf{C}^5 \oplus \sigma(7)$	Castle	4
9	$\sigma(7) \otimes \mathbf{C}^5 \oplus \mathbf{C}^7$	Castle	3
10	$\sigma(7) \otimes \mathbf{C}^5 \oplus \bigwedge^2 \mathbf{C}^{5*}$	Strata	5
11	$\sigma(7) \otimes \mathbf{C}^5 \oplus \mathbf{C}^5 \oplus \mathbf{C}^{5*}$	Strata	6
12	$\sigma(7) \otimes \mathbf{C}^4 \oplus \sigma(7)$	Strata	5
13	$\sigma(7) \otimes \mathbf{C}^4 \oplus \mathbf{C}^7$	Strata	5

Table continued

TABLE XII—*Continued*

	(c) **Spin**$(m) \times$ **SL**(n), $m = 6, 7$ $V \supset W \otimes \mathbf{C}^n$, dim $W - 2 \geq n$, $W = \sigma(7)$, $\sigma^+(6)^2$, or $\sigma^+(6) \oplus \mathbf{C}^6$		
	Representation	Method	d
14	$\sigma(7) \otimes \mathbf{C}^4 \oplus \mathbf{S}^2\mathbf{C}^4$	Strata	6
15	$\sigma(7) \otimes \mathbf{C}^4 \oplus \mathbf{S}^2\mathbf{C}^{4*}$	Strata	6
16	$\sigma(7) \otimes \mathbf{C}^4 \oplus \bigwedge^2 \mathbf{C}^4 \oplus \mathbf{C}^4$	Strata	6
17	$\sigma(7) \otimes \mathbf{C}^4 \oplus \bigwedge^2 \mathbf{C}^4 \oplus \mathbf{C}^{4*}$	Strata	6
18	$\sigma(7) \otimes \mathbf{C}^4 \oplus \mathbf{C}^4 \oplus \mathbf{C}^{4*}$	Strata	5
19	$\sigma(7) \otimes \mathbf{C}^4 \oplus 2\mathbf{C}^{4*}$	Strata	5
20	$\sigma(7) \otimes \mathbf{C}^3 \oplus \sigma(7) \oplus \mathbf{C}^{3*}$	Strata	6
21	$\sigma(7) \otimes \mathbf{C}^3 \oplus \mathbf{C}^7 \oplus \mathbf{C}^3$	Strata	5
22	$\sigma(7) \otimes \mathbf{C}^3 \oplus \mathbf{S}^2\mathbf{C}^3$	Ladder	4
23	$\sigma(7) \otimes \mathbf{C}^3 \oplus \mathbf{S}^2\mathbf{C}^{3*}$	Ladder	4
24	$\sigma(7) \otimes \mathbf{C}^3 \oplus \mathbf{C}^3 \oplus \mathbf{C}^{3*}$	Ladder	4
25	$\sigma(7) \otimes \mathbf{C}^2 \oplus \sigma(7)$	Ladder	3
26	$\sigma(7) \otimes \mathbf{C}^2 \oplus 2\mathbf{C}^7 \oplus \mathbf{S}^2\mathbf{C}^2$	Strata	9
27	$\sigma(7) \otimes \mathbf{C}^2 \oplus 2\mathbf{C}^7 \oplus \mathbf{C}^2$	Strata	8
28	$\sigma^+(6)^2 \otimes \mathbf{C}^8$	Castle	2
29	$\sigma^+(6) \otimes \mathbf{C}^8 \oplus \mathbf{C}^6 \otimes \mathbf{C}^8$	Castle	2
30	$\sigma^+(6)^2 \otimes \mathbf{C}^2$	Ladder	2
31	$\sigma^+(6) \otimes \mathbf{C}^2 \oplus \mathbf{C}^6 \otimes \mathbf{C}^2$	Ladder	2

TABLE XIII

	SP$(6) \times$ **SL**(2)		
	Non-Plain representation	Method	d
1	$\varphi_3(6) \otimes \mathbf{C}^2$	Strata	4

TABLE XIV

	(a) **Spin**$(m) \times$ **SL**(n), $m \geq 8$ $V \supset W \otimes \mathbf{C}^n$, dim $W = n - 1$, W Non-Plain		
	Representation	Method	d
1	$\sigma^+(14) \otimes \mathbf{C}^{64} \oplus \mathbf{C}^{64}$	Castle	2
2	$\sigma^+(14) \otimes \mathbf{C}^{63} \oplus \mathbf{C}^{14}$	Castle	3
3	$\sigma(13) \otimes \mathbf{C}^{63}$	Castle	2

Table continued

TABLE XIV—*Continued*

(a) **Spin**$(m) \times$ **SL**(n), $m \geqslant 8$

$V \supset W \otimes \mathbf{C}^n$, dim $W = n-1$, W Non-Plain

	Representation	Method	d
4	$\sigma^+(12) \otimes \mathbf{C}^{43} \oplus \mathbf{C}^{12} \otimes \mathbf{C}^{43}$	Castle	2
5	$\sigma^+(12) \otimes \mathbf{C}^{32} \oplus \mathbf{C}^{32} \oplus \mathbf{C}^{12}$	Castle	3
6	$\sigma^+(12) \otimes \mathbf{C}^{31} \oplus \mathbf{C}^{31*}$	Strata	3
7	$\sigma^+(12) \otimes \mathbf{C}^{31} \oplus \mathbf{C}^{12}$	Castle	2
8	$\sigma(11) \otimes \mathbf{C}^{32} \oplus \mathbf{C}^{32}$	Castle	2
9	$\sigma(11) \otimes \mathbf{C}^{31} \oplus \mathbf{C}^{11}$	Castle	3
10	$2\sigma^+(10) \otimes \mathbf{C}^{32} \oplus \mathbf{C}^{32}$	Castle	2
11	$2\sigma^+(10) \otimes \mathbf{C}^{31}$	Castle	1
12	$\sigma^+(10) \otimes \mathbf{C}^{31} \oplus \sigma^-(10) \otimes \mathbf{C}^{31}$	Castle	2
13	$\sigma^+(10) \otimes \mathbf{C}^{25} \oplus \mathbf{C}^{10} \otimes \mathbf{C}^{25}$	Castle	2
14	$\sigma^+(10) \otimes \mathbf{C}^{17} \oplus 2\mathbf{C}^{17}$	Castle	2
15	$\sigma^+(10) \otimes \mathbf{C}^{16} \oplus 2\mathbf{C}^{16}$	Castle	2
16	$\sigma^+(10) \otimes \mathbf{C}^{16} \oplus \mathbf{C}^{16} \oplus \mathbf{C}^{16*}$	Strata	3
17	$\sigma^+(10) \otimes \mathbf{C}^{16} \oplus \mathbf{C}^{16} \oplus \sigma^-(10)$	Castle	2
18	$\sigma^+(10) \otimes \mathbf{C}^{16} \oplus \mathbf{C}^{16} \oplus \mathbf{C}^{10}$	Castle	3
19	$\sigma^+(10) \otimes \mathbf{C}^{15} \oplus \mathbf{C}^{15} \oplus \mathbf{C}^{10}$	Castle	4
20	$\sigma^+(10) \otimes \mathbf{C}^{15} \oplus \mathbf{C}^{15*} \oplus \mathbf{C}^{10}$	Strata	4
21	$\sigma^+(10) \otimes \mathbf{C}^{15} \oplus \sigma^+(10)$	Castle	2
22	$\sigma^+(10) \otimes \mathbf{C}^{15} \oplus \sigma^-(10)$	Castle	1
23	$\sigma^+(10) \otimes \mathbf{C}^{15} \oplus 2\mathbf{C}^{10}$	Castle	5
24	$\sigma(9) \otimes \mathbf{C}^{16} \oplus \mathbf{C}^{16}$	Castle	2
25	$\sigma(9) \otimes \mathbf{C}^{15} \oplus \mathbf{C}^{15}$	Castle	2
26	$\sigma(9) \otimes \mathbf{C}^{15} \oplus \mathbf{C}^{15*}$	Strata	3
27	$\sigma(9) \otimes \mathbf{C}^{15} \oplus \mathbf{C}^{9}$	Castle	3
28	$\sigma^+(8) \otimes \mathbf{C}^{15} \oplus \mathbf{C}^{8} \otimes \mathbf{C}^{15}$	Castle	2

(b) **Spin**$(m) \times$ **SL**(n), $6 \leqslant m \leqslant 7$

$V \supset W \otimes \mathbf{C}^n$, dim $W = n-1$, W Non-plain

	Representation	Method	d
1	$\sigma(7) \otimes \mathbf{C}^{14} \oplus \mathbf{C}^{7} \otimes \mathbf{C}^{14}$	Castle	2
2	$\sigma(7) \otimes \mathbf{C}^{8} \oplus \mathbf{C}^{8} \oplus 2\mathbf{C}^{8*}$	Strata	7
3	$\sigma(7) \otimes \mathbf{C}^{8} \oplus \mathbf{C}^{8} \oplus 2\mathbf{C}^{7}$	Castle	5

Table continued

TABLE XIV—*Continued*

	(b) $\mathbf{Spin}(m)\times\mathbf{SL}(n)$, $6\leqslant m\leqslant 7$ $V\supset W\otimes\mathbf{C}^n$, dim $W=n-1$, W Non-Plain Representation	Method	d
4	$\overline{\sigma(7)}\otimes\mathbf{C}^7\oplus\overline{\sigma(7)}\oplus\mathbf{C}^{7*}$	Strata	5
5	$\overline{\sigma(7)}\otimes\mathbf{C}^7\oplus 3\overline{\mathbf{C}^7}$	Castle	8
6	$\overline{\sigma(7)}\otimes\mathbf{C}^7\oplus\overline{\mathbf{C}^7}\oplus\mathbf{C}^7$	Castle	4
7	$\overline{\sigma(7)}\otimes\mathbf{C}^7\oplus\overline{\mathbf{C}^7}\oplus\mathbf{C}^{7*}$	Strata	4
8	$\overline{\sigma(7)}\otimes\mathbf{C}^7\oplus\mathbf{C}^7\oplus 2\mathbf{C}^{7*}$	Slice 9c.5	8
9	$\sigma^+(6)^2\otimes\mathbf{C}^{13}\oplus\sigma^+(6)\otimes\mathbf{C}^{13}$	Castle	2
10	$\sigma^+(6)^2\otimes\mathbf{C}^{13}\oplus\sigma\;\;(6)\otimes\mathbf{C}^{13}$	Castle	2
11	$\sigma^+(6)\otimes\mathbf{C}^{13}\oplus\sigma\;\;(6)\otimes\mathbf{C}^{13}\oplus\mathbf{C}^6\otimes\mathbf{C}^{13}$	Castle	2
12	$2\sigma^+(6)\otimes\mathbf{C}^{13}\oplus\mathbf{C}^6\otimes\mathbf{C}^{13}$	Castle	2
13	$\sigma^+(6)^2\otimes\mathbf{C}^{10}\oplus\mathbf{C}^{10}$	Castle	2
14	$\sigma^+(6)\otimes\mathbf{C}^{10}\oplus\mathbf{C}^6\otimes\mathbf{C}^{10}\oplus\mathbf{C}^{10}$	Castle	2
15	$\sigma^+(6)^2\otimes\mathbf{C}^9\oplus\sigma^+(6)$	Castle	2
16	$\sigma^+(6)^2\otimes\mathbf{C}^9\oplus\sigma^-(6)$	Castle	2
17	$\sigma^+(6)\otimes\mathbf{C}^9\oplus\mathbf{C}^6\otimes\mathbf{C}^9\oplus\sigma^+(6)$	Castle	2
18	$\sigma^+(6)\otimes\mathbf{C}^9\oplus\mathbf{C}^6\otimes\mathbf{C}^9\oplus\sigma^-(6)$	Castle	2
19	$\sigma^+(6)\otimes\mathbf{C}^9\oplus\mathbf{C}^6\otimes\mathbf{C}^9\oplus\mathbf{C}^6$	Castle	3
20	$\sigma^+(6)\otimes\mathbf{S}^4\mathbf{C}^2$	Castle	2
21	$\sigma^+(6)\otimes\mathbf{S}^3\mathbf{C}^2\oplus\sigma^+(6)$	Castle	2

TABLE XV

	$\mathbf{SP}(m)\times\mathbf{SL}(n)$ $V\supset W\otimes\mathbf{C}^n$, dim $W=n-1$, W Non-plain Representation	Method	d
1	$\varphi_3(6)\otimes\mathbf{C}^{19}\oplus\mathbf{C}^6\otimes\mathbf{C}^{19}$	Castle	2
2	$\varphi_2(6)\otimes\mathbf{C}^{19}\oplus\mathbf{C}^6\otimes\mathbf{C}^{19}$	Castle	2
3	$\varphi_3(6)\otimes\mathbf{C}^{14}\oplus\mathbf{C}^{14}$	Castle	2
4	$\varphi_3(6)\otimes\mathbf{C}^{13}\oplus\mathbf{C}^6$	Castle	2
5	$\varphi_2(6)\otimes\mathbf{C}^{13}\oplus\mathbf{C}^6$	Castle	2
6	$\mathbf{S}^2\mathbf{C}^4\otimes\mathbf{C}^9$	Castle	2
7	$\varphi_2(4)\otimes\mathbf{C}^9\oplus\mathbf{C}^4\otimes\mathbf{C}^9\oplus\mathbf{C}^9$	Castle	2
8	$\varphi_2(4)\otimes\mathbf{C}^8\oplus\mathbf{C}^4\otimes\mathbf{C}^8$	Castle	1

TABLE XVI

	$G_2 \times SL(13)$		
	$V \supset W \otimes C^n$, dim $W = n-1$, W Non-plain		
	Representation	Method	d
1	$\varphi_2(G_2) \otimes C^{13}$	Castle	2

Remark 2.0.2. Determining the maximally equidimensional representations of the group $SL(m) \times SL(n)$ is more difficult because the representations $(pC^q, SL(q))$ for $p < q$ have no invariants and so are trivially equidimensional. Adding a single irreducible representation W to $(kC^m \otimes C^n, SL(m) \times SL(n))$, where $km < n$ can create quite complex representations which may or may not be equidimensional.

2.1. *A Structure Theorem*

Suppose that (V, G) is an equidimensional representation of a reductive complex algebraic group and let $f_1, f_2, ..., f_d$ be a homogeneous system of parameters for $C[V]^G$. (See [Ma, Chaps. 5 and 6] for the definitions of a homogeneous system of parameters, a regular sequence and a Cohen–Macaulay ring.) Let S be a graded G-stable subspace such that $C[V] = \mathscr{I} \oplus S$, where $\mathscr{I}$ is the ideal of $C[V]$ generated by $C[f_1, f_2, ..., f_d]_+$ and $C[f_1, f_2, ..., f_d]_+$ denotes the set of polynomials in the d variables $f_1, ..., f_d$ having zero constant term. (When we say that S is graded we mean that if $s = s_1 + s_2 \in S$ with the s_i homogeneous then the $s_i \in S$.) If (V, G) is cofree then $C[V] \cong C[f_1, ..., f_d] \otimes S$ [Ko].

TABLE XVII

	$H \times SL(n)$			
	$V \supset \varphi_1(H) \otimes W$	W Non-Plain		
	Representation	Conditions	Method	d
1	$\varphi_1(SP(m)) \otimes S^3C^2$		Strata	3
2	$\varphi_1(SP(m)) \otimes S^2C^2 \oplus \varphi_1(SP(m))$		Strata	3
3	$\varphi_1(SP(m)) \otimes S^2C^2 \oplus S^2C^2$		Strata	3
4	$\varphi_1(SP(m)) \otimes S^2C^2 \oplus C^2$		Ladder	2
5	$\varphi_1(SP(4)) \otimes S^2C^2 \oplus \varphi_2(SP(4))$		Strata	4
6	$\varphi_1(SO(m)) \otimes S^2C^2$	$m \geqslant 8$ or $m = 5$	Strata	3
7	$\varphi_1(Spin(7)) \otimes S^2C^2 \oplus \sigma(7)$		Ladder	5
8	$\varphi_1(Spin(6)) \otimes S^2C^2 \oplus \sigma^+(6)$		Strata	4
9	$\varphi_1(G_2) \otimes S^2C^2$		Ladder	4

TABLE XVIII

	H × SL(2)		
	$V \supset W_1 \otimes W_2$, W_1, W_2 Non-Plain		
	Representation	Method	d
1	$\sigma(7) \otimes \mathbf{S}^2\mathbf{C}^2$	Ladder	3
2	$\sigma^+(10) \otimes \mathbf{S}^2\mathbf{C}^2$	Strata	3

Given a representation M of G and an irreducible G-representation ρ, we denote by $M_{(\rho)}$ the isotypic component of M of type ρ. Thus we have the (unique) decomposition of M into its isotypic components: $M = \bigoplus_\rho M_{(\rho)}$.

If (V, G) is cofree and the f_i are chosen so that $\mathbf{C}[V]^G = \mathbf{C}[f_1, f_2, ..., f_d]$ then a formula for the multiplicity of any isotypic component in S is given in [Sch1, Prop. 4.6]. A similar theorem is proven by combinatorial methods, in [St, Prop. 4.9] for representations of finite groups. We will show that this structure theorem is valid for equidimensional representations of any complex reductive group. In particular the two theorems mentioned above are specializations of our Proposition 2.1.1.

Since $\mathbf{C}[V]^G$ is Cohen–Macaulay [HR], there is a positive integer t and homogeneous elements $\eta_1, ..., \eta_t$ of $\mathbf{C}[V]^G$ such that

$$\mathbf{C}[V]^G = \bigoplus_{i=1}^{t} \mathbf{C}[f_1, ..., f_d] \cdot \eta_i.$$

Let H be a principal isotropy group of (V, G) and (N, H) the principal slice representation. Write $N = V_1 \oplus \theta$, where θ is a trivial H-representation and $V_1^H = \{0\}$.

Proposition 2.1.1. *There are isomorphisms of the G-representations,*

$$S \cong t \text{ copies of } \bigoplus_\rho (\mathbf{C}[G *_H V_1])_{(\rho)}$$

$$= t \text{ copies of } \bigoplus_\rho (\dim(\mathbf{C}[V_1] \otimes W_\rho^*)^H) W_\rho.$$

In particular, if (V, G) is stable then S is isomorphic to the direct sum of t copies of the G-representation $\mathbf{C}[G/H]$.

Proof. Let $m(\rho)$ denote the multiplicity of the irreducible representation W_ρ in S. Then, since $\mathbf{C}[V] \cong \mathbf{C}[f_1, ..., f_d] \otimes S$, $m(\rho)$ is also the multiplicity of W_ρ in $\mathbf{C}[F]$ for any fibre, F, of $f = (f_1, ..., f_d)$: $V \to \mathbf{C}^d$. Now $\mathbf{C}^d$ contains an open dense subset U such that $f^{-1}(\xi)$ is exactly t principal fibres for all $\xi \in U$. But all principal fibres are isomorphic to $G *_H V_1$ by the slice theorem of Luna (see Section 4.2). Therefore $m(\rho) W_\rho = S_{(\rho)} = (\bigoplus_{i=1}^{t} \mathbf{C}[G *_H V_1])_{(\rho)} = t$ copies of $(\mathbf{C}[G *_H V_1])_{(\rho)}$.

As in the proof of [Sch1, Prop. 4.6], we may now use a version of Frobenius reciprocity to show that the multiplicity of ρ in $\mathbf{C}[G *_H V_1]$ is equal to $\dim(\mathbf{C}[V_1] \otimes W_\rho^*)^H$ and from this the result follows. ∎

3. The Tables

We use the notation of [Sch2] (which follows the numbering of [E]) to denote the irreducible representations of simple groups. However, we denote the spinor representations slightly differently. The spinor representations of $\mathbf{Spin}(2n)$ denoted there by φ_{n-1} and φ_n are indicated here by $\sigma^-(2n)$ and $\sigma^+(2n)$, respectively. Moreover we use $\sigma^{\pm}(2n)$ or $\sigma(2n)$ to indicate either of these two representations. The spinor representation of $\mathbf{Spin}(2n+1)$ denoted by φ_n is denote here by $\sigma(2n+1)$. Often we will write $\mathbf{C}^n$ for the $\varphi_1(H)$, where H is one of the classical groups and $n = \dim \varphi_1(H)$. Similarly we sometimes use $S^r\mathbf{C}^n$ and $\bigwedge^r \mathbf{C}^n$ instead of the φ_r-notation. We also use $sl(n)$ to denote the adjoint representation of $\mathbf{SL}(n)$. The symbol $\rho(G)$ is sometimes used to indicate a representation of G where we wish to emphasize the group. Furthermore we use φ^* to indicate the representation dual to φ. The notation $\varphi^{(*)}$ will be used to indicate either φ or φ^*. We will write θ_p to denote the p-dimensional trivial representation and θ to denote a trivial representation without specifying the dimension. The symbol $\mathbf{G}_2$ indicates the rank two exceptional simple group while G_2 stands for any complex reductive algebraic (usually simple) group.

We will sometimes use overbars (or tildes or hats) to indicate which group is acting on a representation. For example, if $G = \mathbf{Spin}(m) \times \mathbf{SL}(m)$ we write $(V, G) = (\overline{\sigma(m) \otimes \mathbf{C}^m \oplus \bigwedge^2 \mathbf{C}^m} \oplus \mathbf{C}^m, \overline{\mathbf{Spin}(m)} \times \mathbf{SL}(m))$ to indicate that $\mathbf{Spin}(m)$ is acting on $\sigma(m)$ and $\bigwedge^2 \mathbf{C}^m$ while $\mathbf{SL}(m)$ acts on the two copies of $\mathbf{C}^m$.

When we refer to $\mathbf{Spin}(m)$ we assume that $m \geqslant 5$. Similarly $m \geqslant 4$ for $\mathbf{SP}(m)$ and $n \geqslant 2$ for $\mathbf{SL}(n)$. Furthermore, we assume that $n \geqslant 4$ for table entries involving $(\bigwedge^2 \mathbf{C}^n, \mathbf{SL}(n))$ and that $n \geqslant 5$ for those involving $(\bigwedge^2 \mathbf{C}^{n*}, \mathbf{SL}(n))$. If an entry contains $(n-r)W$ as a sub-representation then n must be greater than or equal to r.

We shall need the following definition:

Definition. Let H be a simple group. We call a representation (W, H) *plain* if W is (up to an outer automorphism of H) of the form: $W \cong k_1 \varphi_1 \oplus k_2 \varphi_1^* \oplus \theta_r$. Let $G = G_1 \times G_2$, where G_1 and G_2 are both simple.

A representation (V, G) of G is called *plain* if (up to an outer automorphism of G) V is of the form: $V \cong k_1\varphi_1(G_1) \otimes \varphi_1(G_2) \oplus k_2\varphi_1(G_1) \otimes \varphi_1^*(G_2) \oplus k_3\varphi_1^*(G_1) \otimes \varphi_1(G_2) \oplus k_4\varphi_1^*(G_1) \otimes \varphi_1^*(G_2) \oplus \rho_1(G_1) \oplus \rho_2(G_2)$.

The isomorphisms $\mathbf{Spin}(5) \cong \mathbf{SP}(4)$ and $\mathbf{Spin}(6) \cong \mathbf{SL}(4)$ cause some question as to which of the tables should contain certain representations. For plain representations this ambiguity is settled by listing a representation in the form which exhibits its plainness. Thus for example the plain representation $(2\varphi_2(5) \otimes \mathbf{C}^7 \oplus \mathbf{C}^5, \mathbf{Spin}(5) \times \mathbf{SL}(7)) \cong (2\mathbf{C}^4 \otimes \mathbf{C}^7 \oplus \varphi_2(4), \mathbf{SP}(4) \times \mathbf{SL}(7))$ is listed as a representation of $\mathbf{SP}(4) \times \mathbf{SL}(7)$.

In Tables I–X we list all the equidimensional plain representations. Tables XI–XVIII contain the non-plain representations. Note also that Table V contains some non-plain representations.

For each representation, (V, G), in the tables we give in the column headed "d," the dimension of $V//G$. We indicate for each representation the principal methods we used to prove that it is equidimensional. Most of these methods are described in Section 4. Kempf has shown that every representation of a connected semi-simple group with $\dim V//G \leqslant 2$ is cofree [Ke] and thus for representations, V, in the tables with $\dim V//G \leqslant 2$ we list in the method column a method for computing $\dim V//G$.

Similarly, it is known that every *theta representation* is cofree [V]. Thus the theta representation $(\sigma^+(10) \otimes \mathbf{C}^4, \mathbf{Spin}(10) \times \mathbf{SL}(4))$ is cofree.

4. Techniques

In this section we describe the methods we use to determine whether a representation (V, G) is equidimensional or coregular. We begin with some preliminaries. Proofs of most of these results may be found in [Kr].

Let X be a factorial affine complex variety on which the reductive complex algebraic group G acts. Then we say X is a (factorial) G-variety and write (X, G). The ring $C[X]^G$ of G-invariant functions on X is the coordinate ring of the quotient variety $X//G$. If $X//G$ is smooth then (X, G) is *coregular*. If all the fibres of the quotient map $\pi_{G,X}: X \to X//G$ have the same dimension (which is then necessarily $\dim X - \dim X//G$) then (X, G) is *equidimensional*. If (X, G) is both coregular and equidimensional then it is *cofree*.

Let $x \in X$. The set of subgroups of G conjugate to G_x is called the *isotropy class* of G_x and is denoted by (G_x). We say that a closed orbit Gx is an *orbit of type* (K) if $G_x \in (K)$. If the orbit of x is a closed subset of X then (G_x) is called a *closed isotropy class* and in this case the *closed isotropy* *group* G_x is reductive (Matsushima's Theorem). A representation can have only finitely many closed isotropy classes. There is always a dense, open subset U of X such that the isotropy groups, G_u, for $u \in U$, all lie in the same isotropy class. This class is called the *generic isotropy class* and its elements are called *generic isotropy groups*.

If (G_{x_1}) and (G_{x_2}) are isotropy classes and some conjugate of G_{x_1} is properly contained in G_{x_2} then (G_{x_1}) is said to be *less than* (G_{x_2}) and we write $(G_{x_1}) < (G_{x_2})$. We consider this partial order restricted to the set of closed isotropy classes. With respect to this order, there is a unique absolute minimum closed isotropy class, the *principal isotropy class*, whose elements are called *principal isotropy groups*. A closed orbit of X whose isotropy class is principal is called a *principal orbit*. The union of the principal orbits is an open, dense subset of the set of points lying on closed orbits. In case the generic and principal isotropy groups coincide the representation is called *stable*. A representation (V, G) is stable if and only if V contains an open dense subset consisting entirely of closed orbits.

The closed isotropy classes of (X, G) give a stratification of $X//G$ as follows: If K is a closed isotropy group of (X, G), let $X^{(K)}$ denote the union of all orbits of type K and define $X^{\langle K\rangle} := X^{(K)} \cap X^K$. Now let $(X//G)_{(K)}$ denote the subset $\pi_{G,X}(X^{\langle K\rangle})$ of $X//G$. Then

$$X//G = \bigsqcup_{(K)} (X//G)_{(K)}, \tag{1}$$

where the (disjoint) union is taken over all the closed isotropy classes, (K) of (X, G). The fibres of $\pi_{G,X}$ above $(X//G)$ are all isomorphic to one another.

The strata of $X//G$ are related by (see [Sch3])

$$\overline{(X//G)_{(K)}} = \bigsqcup_{(L) \geqslant (K)} (V//G)_{(L)}.$$

Furthermore, the natural morphism,

$$\eta\colon (X^K)//(N_G K) \to \overline{(X//G)_{(K)}}$$

is the normalization [L2]. The fibres of the morphism $X^{\langle K\rangle} \to (X//G)_{(K)}$ are isomorphic to $(N_G K)/K$.

Suppose now that G is a semi-simple group. Then, by [LV], $\dim X//G = \dim X - \dim(\text{largest orbit in } X)$. Hence,

$$\dim X//G = \dim V - \dim G + \dim H, \tag{2}$$

where H is a generic isotropy group of (X, G).

4.1. *The Hilbert–Mumford Criterion*

The *null fibre* or *null cone*, $Z_G(V) := \pi_{G,V}^{-1}(\pi_{G,V}(0))$ plays a pivotal role in the study of representations. Since $\pi_{G,V}$ is a morphism there is a dense subset U of $V//G$ such that $\dim \pi_{G,V}^{-1}(u) = \dim V - \dim V//G$ for all $u \in U$, and there are no fibres with smaller dimension [Kr, AI.3.3]. Conversely, there are no fibres of $\pi_{G,V}$ having dimension greater than that of $Z_G(V)$ [Kr, II.4.2]. Therefore (V, G) is equidimensional if and only if the following equation is satisfied:

$$\dim V - \dim Z_G(V) = \dim V//G. \tag{3}$$

A *one parameter subgroup* (1-PSG) λ of G is a morphism of algebraic groups $\lambda\colon \mathbf{C}^* \to G$. If there is a one parameter subgroup λ such that

$$\lim_{t \to 0} \lambda(t) \cdot v = w \qquad \text{(classical topology)}$$

then, clearly, $f(v) = f(w)$ for all $f \in \mathbf{C}[V]^G$. Since the invariants separate closed G-stable sets, this implies that $\overline{G \cdot v} \subseteq \pi_{G,V}^{-1}(\pi_{G,V}(w))$. The converse of this result is known as the *Hilbert–Mumford Criterion.* The Hilbert–Mumford Criterion states that $w \in \overline{Gv}$ if and only if there exists a λ with $\lim_{t \to 0} \lambda(t) \cdot v \in \overline{Gw}$. This amazing result allows us to study the null cone without having to use the algebra of invariants.

For each one parameter subgroup λ, we define the set $Z_\lambda(V) := \{v \in V \mid \lim_{t \to 0} \lambda(t) \cdot v = 0\}$. We may conjugate λ so that its image lies in a fixed maximal torus T of G. Then the Hilbert–Mumford criterion asserts that $Z_G(V)$ is the union of the sets $G \cdot Z_\lambda(V)$ as λ varies through all one parameter subgroups of T. By considering the T-weight space decomposition of V we see that there are only finitely many sets $Z_\lambda(V)$ and thus we may write $Z_G(V) = \bigcup_{i=1}^{r} G \cdot Z_{\lambda_i}(V)$. If $V^T = V^{\lambda(\mathbf{C}^*)}$ then λ is said to be *generic*. Clearly we may assume that each of the $\lambda_1, \ldots, \lambda_r$ is generic.

There is a maximal subgroup P_λ of G which preserves $Z_\lambda(V)$. P_λ is parabolic and contains the parabolic subgroup $P(\lambda)$ defined by Mumford in [MumF]. Therefore,

$$\dim G \cdot Z_\lambda(V) \leqslant \dim Z_\lambda(V) + \dim G - \dim P_\lambda \tag{4}$$

$$\leqslant \dim Z_\lambda(V) + \dim G - \dim B, \tag{5}$$

where B is a Borel subgroup of G.

Schwarz [Sch2, Prop. 2.10] shows that if V is a self-dual representation and λ is generic then $\dim Z_\lambda(V) = \frac{1}{2}(\dim V - \dim V^T)$ and hence for self-dual representations

$$\dim Z_G(V) \leqslant \tfrac{1}{2}(\dim V + \dim G - \operatorname{rk} G - \dim V^T). \tag{6}$$

Now let $\mathscr{P}_\lambda$ and $\mathscr{G}$ be the Lie algebras of P_λ and G, respectively. Write $\mathscr{G} = \mathscr{P}_\lambda \oplus \mathscr{U}_\lambda$, where $\mathscr{U}_\lambda$ is the subalgebra of $\mathscr{G}$ spanned by the roots not contained in $\mathscr{P}_\lambda$. Let U_λ be the connected subgroup of G whose Lie algebra is $\mathscr{U}_\lambda$. Then [Sch2, Section 2]

$$\dim G \cdot Z_\lambda(V) = \dim Z_\lambda(V) + \sup_{v \in Z_\lambda(V)} \dim \frac{\mathscr{U}_\lambda v + Z_\lambda(V)}{Z_\lambda(V)}. \tag{7}$$

This allows us to compute the dimension of the null cone using only $\lambda_1, ..., \lambda_r$ and the actions of $\mathscr{U}_{\lambda_1}, ..., \mathscr{U}_{\lambda_r}$. For many representations, especially when G is simple, r is small enough and the actions simple enough to make this computation practical.

EXAMPLE 4.1.1. We use this method to compute the dimension of $Z_G(V)$ for $(V, G) = (p\mathbf{C}^n \oplus q\mathbf{C}^{n*}, \mathbf{SL}(n))$ for some small values of p and q which we will need. If $p + q \leqslant n - 2$ then (V, G) is equidimensional and then all the fibres of the quotient map have the same dimension, $pn + qn - pq$. If $(V, G) = (p\mathbf{C}^n \oplus q\mathbf{C}^{n*}, \mathbf{SL}(n))$ and $p \geqslant q$, then

$$\operatorname{codim} Z_{\mathbf{SL}(n)}(V) = \begin{cases} 0, & \text{if } q = 0 \text{ and } p < n; \\ p - n + 1, & \text{if } q = 0 \text{ and } p \geqslant n; \\ pq, & \text{if } 1 \leqslant q \leqslant p \leqslant n \text{ and } a \leqslant 1; \\ pq - \left\lceil \dfrac{a^2}{4} \right\rceil, & \text{if } 1 \leqslant q \leqslant p \leqslant n \text{ and } a \geqslant 0, \end{cases} \tag{8}$$

where $a := p + q - n$.

EXAMPLE 4.1.2. We may also compute codim $Z_G(V)$ for the representations $(V, G) = (p\mathbf{C}^{2r}, \mathbf{SP}(2r))$ by considering one parameter subgroups. Here we find

$$\operatorname{codim} Z_{\mathbf{SP}(2r)}(p\mathbf{C}^{2r}) = \begin{cases} p(p-1)/2, & \text{if } p \leqslant r + 1; \\ r(2p - r - 1)/2, & \text{if } p \geqslant r + 1. \end{cases} \tag{9}$$

EXAMPLE 4.1.3. If $(V, G) = (p\mathbf{C}^m, \mathbf{SO}(m))$ then

$$\operatorname{codim} Z_{\mathbf{SO}(m)}(p\mathbf{C}^m) = \begin{cases} p(p+1)/2, & \text{if } p \leqslant r; \\ r(2p - r + 1)/2, & \text{if } p \geqslant r, \end{cases} \tag{10}$$

where $r := \lceil (m+1)/2 \rceil$.

The following two examples illustrate how to use one parameter subgroups to determine equidimensionality.

EXAMPLE 4.1.4. Take $(V, G) = (2\overline{\mathbf{C}^4} \otimes \mathbf{C}^4, \overline{\mathbf{SP}(4)} \times \mathbf{SL}(4))$. Then $\dim V//G \geqslant \dim V - \dim G = 32 - 25 = 7$. Now consider $\lambda: \mathbf{C}^* \to \{e\} \times \mathbf{SL}(4) \subset \overline{\mathbf{SP}(4)} \times \mathbf{SL}(4)$ with weights 1, 1, 1, and -3 in $\mathbf{C}^4$. Then $\dim Z_\lambda(V) = 24$. Set $P := (P_\lambda \cap \mathbf{SL}(4))$, $U := U_\lambda \cap \mathbf{SL}(4)$, and $\mathscr{U} :=$ the Lie algebra of U. Then $\dim P = 12$. For almost all vectors, v, in $Z_\lambda(V)$ we have $\dim((\mathscr{U}v + Z_\lambda(V))/Z_\lambda(V)) = 3$. Hence $\dim G \cdot Z_\lambda(V) \geqslant \dim \mathbf{SL}(4) \cdot Z_\lambda(V) = 24 + 3 = 27$. Hence $\operatorname{codim} Z_G(V) \leqslant 5 < 7 = \dim V//G$, and thus (V, G) is not equidimensional.

EXAMPLE 4.1.5. Consider $(V, G) = (\mathbf{C}^4 \otimes \sigma(7) \oplus \mathbf{C}^4, \mathbf{SP}(4) \times \mathbf{Spin}(7))$ and $(V', G) = (V \oplus \mathbf{C}^7, G)$. Computing the invariants (or more easily, using the slice theorem of Luna described below) we find that $\dim V//G = 5$ and $\dim V'//G = 12$. Then the inequality (6) shows that (V, G) is equidimensional. Moreover, this shows that there is a 1-PSG λ and an element $v \in Z_\lambda(V)$ such that $\dim((\mathscr{U}_\lambda v + Z_\lambda(V))/Z_\lambda(V)) = \dim \mathscr{U}_\lambda = 13$. Now using the same λ and v we obtain $\operatorname{codim}_{V'} G \cdot Z_\lambda(V') = \operatorname{codim}_V G \cdot Z_G(V) + 4 = 9$. Hence (V', G) is not equidimensional.

4.2. *The Slice Theorem of Luna*

Now we give a brief description of the Luna Slice Theorem. This description is tailored to our needs and is not the most general version of the Luna Slice Theorem. Proofs of the results we state may be found in [Kr].

Let G_v be a closed isotropy group of the representation (V, G). Then the *slice representation* at v or the *slice representation* of G_v is the representation of G_v on $S = \mathscr{T}_v V/\mathscr{T}_v(Gv)$. Since $Gv \cong G/G_v$, $\mathscr{T}_v(Gv) \cong \mathscr{T}_e G/\mathscr{T}_e(G_v) \cong \operatorname{Ad}(G)/\operatorname{Ad}(G_v)$. Also $\mathscr{T}_v V \cong V$ and thus we may compute (S, G_v) using the formula

$$(S \oplus \operatorname{Ad}(G), G_v) \cong (V \oplus \operatorname{Ad}(G_v), G_v).$$

The slice representation (S, G_v) inherits many of the properties of the original representation (V, G). For example, the generic and principal isotropy groups of (S, G_v) are generic and principal isotropy groups of (V, G), respectively. Also, $\dim S//G_v = \dim V//G$. If G is semi-simple, then it is not true, in general, that G_v is also. However, Eq. (2) does hold for the slice representation (S, G_v). Furthermore, if (V, G) is cofree (resp. coregular, equidimensional) then (S, G_v) is cofree (resp. coregular, equidimensional).

If (S, K) is a slice representation of (V, G) then the fibres of $\pi_{G,V}$ above $(V//G)_{(K)}$ are all isomorphic to the twisted product, $G *_K Z_K(S)$. In particular, their dimension is $\dim G - \dim K + \dim Z_K(S)$.

Since $\dim V//G = \dim S//G_v$ the Luna slice theorem is often applied one or more times to determine $\dim V//G$. In addition, it is often the easiest way to prove that a representation is or is not equidimensional or coregular as is illustrated by the following example.

EXAMPLE 4.2.1. Let $(\underline{V, G}) = (\overline{\mathbf{C}^m} \otimes \mathbf{C}^m \oplus S^2\mathbf{C}^{m^{(*)}} \oplus \overline{\mathbf{C}^m}, \overline{\mathbf{SO}(m)} \times \mathbf{SL}(m))$. The subrepresentation $(\overline{\mathbf{C}^m} \otimes \mathbf{C}^m, \overline{\mathbf{SO}(m)} \times \mathbf{SL}(m))$ has principal isotropy group $\mathbf{SO}(m)$ embedded diagonally in $\overline{\mathbf{SO}(m)} \times \mathbf{SL}(m)$. The associated slice representation of V is $(S^2\mathbf{C}^m \oplus \mathbf{C}^m \oplus \theta_1, \mathbf{SO}(m))$ which is neither equidimensional [Sch2] nor coregular [Sch1]. Therefore, (V, G) is also neither equidimensional nor coregular.

4.3. *Estimate*

Recall that $\dim V//G = \dim V - \dim G + \dim H$, where H is a generic isotropy group of (V, G). Thus $\dim V//G \geqslant \dim V - \dim G$. Hence in order to show that a representation (V, G) is not equidimensional it suffices to show that $\dim Z_G(V) > \dim G$.

4.4. *Comparison*

The action of 1-PSG's on the representations $W_1 := (S^2\mathbf{C}^n, \mathbf{SL}(n))$ and $W_2 := (\bigwedge^2 \mathbf{C}^n \oplus \mathbf{C}^n, \mathbf{SL}(n))$ are closely related. Let λ be a 1-PSG of $\mathbf{SL}(n)$ with weights $a_1, ..., a_n$ on $\mathbf{C}^n$. Then the λ weights of W_1 are $a_i + a_j$ for $1 \leqslant i < j \leqslant n$ and $2a_i$ for $1 \leqslant i \leqslant n$. The λ weights of W_2 are $a_i + a_j$ for $1 \leqslant i < j \leqslant n$ and a_i for $1 \leqslant i \leqslant n$. Hence $\dim Z_\lambda(W_1) = \dim Z_\lambda(W_2)$. Moreover if we denote by U_λ^k the unipotent group not preserving $Z_\lambda(W_k)$ then $U_\lambda^1 \supseteq U_\lambda^2$. Hence $\dim \mathbf{SL}(n) \cdot Z_\lambda(W_1) \geqslant \dim \mathbf{SL}(n) \cdot Z_\lambda(W_2)$. Now consider $(V_k, G) := (\mathbf{C}^m \otimes \mathbf{C}^n \oplus W_k, \mathbf{SO}(m) \times \mathbf{SL}(n))$, where $n \leqslant m$. Using the Luna slice theorem we find that $\dim V_k//G = n + 1$. Furthermore, we were able to show that V_1 was equidimensional. Hence every 1-PSG λ of G satisfies $\dim G \cdot Z_\lambda(V_1) \geqslant n + 1$. By the above it then follows that $\dim G \cdot Z_\lambda(V_2) \geqslant n + 1$ and therefore V_2 is also an equidimensional representation.

Remark 4.4.1. Note that $(\bigwedge^2 \mathbf{C}^4 \otimes \mathbf{C}^3 \oplus S^2\mathbf{C}^4, \mathbf{SL}(4) \times \mathbf{SL}(3))$ fails to be equidimensional while $(\bigwedge^2 \mathbf{C}^4 \otimes \mathbf{C}^3 \oplus \bigwedge^2 \mathbf{C}^4 \oplus \mathbf{C}^4, \mathbf{SL}(4) \times \mathbf{SL}(3)) \cong (\mathbf{C}^6 \otimes \mathbf{C}^3 \oplus \mathbf{C}^6 \oplus \sigma^+(6), \mathbf{Spin}(6) \times \mathbf{SL}(3))$ is cofree.

4.5. *Finite Covering Groups*

In order to simplify many of the computations we take advantage of the following result: Suppose that (V, G) and (V, G') are representations with $G^0 = (G')^0$ and $(V, G^0) = (V, (G')^0)$. Then (V, G) is equidimensional if and only if (V, G') is equidimensional. In particular this allows us to pass from (V, G) to (V, G^0). Note, however, that it is not true that (V, G) is coregular if and only if (V, G') is. See Examples 4.6.6, 4.6.7, 4.10.2, and 4.13.1 for examples of how useful this method can be.

4.6. *Ladders*

LEMMA 4.6.1. *Let V be a representation of the group G, where G decomposes as $G = G_1 \times G_2$. Set $X = V//G_1$. Then*

(1) *(V, G) is coregular $\Leftrightarrow$ (X, G_2) is coregular.*

(2) *If both (V, G_1) and (X, G_2) are equidimensional then (V, G) is equidimensional.*

(3) *If (X, G_2) is not equidimensional then (V, G) is not equidimensional.*

The proof of this lemma is easy and is left to the reader.

DEFINITION. We call the sequence: V, $V//G_1$, $V//G$ a ladder or a ladder of (V, G).

In order to understand the G_2-variety $X = V//G_1$, recall that X is defined to be the variety whose coordinate ring is $\mathbf{C}[V]^{G_1}$. Thus we must understand how the G_1-invariant functions in $\mathbf{C}[V]$ transform under the action of G_2. Descriptions of how various invariants transform may be found in [Sch1, Sch5].

EXAMPLE 4.6.2. Here we give a simple example showing the structure of $V//G_1$ as a G_2-representation. Take $(V, G) = (\overline{\mathbf{C}^n} \otimes \mathbf{C}^n, \overline{\mathbf{SL}(n)} \times \mathbf{SL}(n))$. Then $(V, \overline{\mathbf{SL}(n)}) \cong (n\overline{\mathbf{C}^n}, \overline{\mathbf{SL}(n)})$. Hence $\mathbf{C}[V]^{\overline{\mathbf{SL}(n)}} = \mathbf{C}[f]$, where f is the determinant $f\colon V = n\overline{\mathbf{C}^n} \to \mathbf{C}$. Now if we consider f as a function on $(V, \mathbf{SL}(n)) \cong n\mathbf{C}^n$ we see that again f is the determinant. Thus f is an invariant of both groups and so $(V//\overline{\mathbf{SL}(n)}, \mathbf{SL}(n)) \cong \theta_1$.

Remark 4.6.3. Note that in the above example, since $(V//\overline{\mathbf{SL}(n)}, \mathbf{SL}(n))$ is a one dimensional representation of the semi-simple group $\mathbf{SL}(n)$, it can only be the irreducible trivial representation.

EXAMPLE 4.6.4. Here we give a (slightly) more complicated example illustrating the G_2-representation $V//G_1$. Let $(V, G) = (\mathbf{C}^m \otimes \mathbf{C}^n, \mathbf{SO}(m) \times \mathbf{SL}(n))$, where $m > n$. By classical invariant theory the $\mathbf{SO}(m)$ invariants of V are generated by the $\binom{n+1}{2}$ inner products among the n copies of $\mathbf{C}^m$. Viewing these inner products as functions on $m\mathbf{C}^n$ we see that they transform under the action of $\mathbf{SL}(n)$ as the representation $(\mathbf{S}^2\mathbf{C}^n, \mathbf{SL}(n))$.

EXAMPLE 4.6.5. Next we show how to use a ladder to determine whether a representation is coregular and/or equidimensional. Take $(V, G) = (\mathbf{C}^m \otimes \mathbf{C}^n \oplus \bigwedge^2 \mathbf{C}^{n*} \oplus \mathbf{C}^m, \mathbf{SO}(m) \times \mathbf{SL}(n))$, where $m > n + 1$. Then $X = V//\mathbf{SO}(m) \cong \mathbf{S}^2\mathbf{C}^n \oplus \bigwedge^2 \mathbf{C}^{n*} \oplus \mathbf{C}^n \oplus \theta_1$. Hence $(X, \mathbf{SL}(n))$ and thus also

(V, G) is neither coregular nor equidimensional if n is odd. If n is even then X and hence also V is coregular [Sch2]. If $m > 2n-1$ and n is even then both $(V, \mathbf{SO}(m))$ and $(X, \mathbf{SL}(n))$ are cofree [Sch2] and thus (V, G) is also cofree.

EXAMPLE 4.6.6. Here we give an example where replacing the group with a finite extension makes the computations easier. If $(V, G) = (\mathbf{C}^m \otimes \mathbf{C}^m \oplus \mathbf{C}^m, \mathbf{SO}(m) \times \mathbf{SO}(m))$, then $(V, \mathbf{SO}(m))$ is not coregular. However, $(V, \mathbf{O}(m))$ is coregular and $(V//\mathbf{O}(m), \mathbf{SO}(m)) \cong (S^2\mathbf{C}^m \oplus \mathbf{C}^m, \mathbf{SO}(m))$ is not equidimensional [Sch2]. Hence (V, G) is not equidimensional.

EXAMPLE 4.6.7. If $(V, G) = (\mathbf{C}^n \otimes \mathbf{C}^{n+1} \oplus \bigwedge^2 \mathbf{C}^{n+1} \oplus \mathbf{C}^{n+1*}, \mathbf{SO}(n) \times \mathbf{SL}(n+1))$ then both $(V, \mathbf{SL}(n))$ and $(V, \mathbf{SO}(n))$ fail to be coregular. While $(V, \mathbf{O}(n))$ is also not coregular, $Y := V//\mathbf{O}(n)$ is the hypersurface in $W = S^2\mathbf{C}^{n+1} \oplus \bigwedge^2 \mathbf{C}^{n+1} \oplus \mathbf{C}^{n+1*}$ given by $V//\mathbf{O}(n) = \{(\omega, v, x) \in W \mid \operatorname{rank} \omega \leqslant n\}$. Hence it is the hypersurface in W cut out by the discriminant $f: S^2\mathbf{C}^{n+1} \to \mathbf{C}$. Now $(W, \mathbf{SL}(n+1))$ is coregular and we may choose a minimal generating set $\{f_0, f_1, ..., f_{n+1}\}$ for $\mathbf{C}[W]^{\mathbf{SL}(n+1)}$ with $f_0 = f$. Then $\mathbf{C}[Y]^{\mathbf{SL}(n+1)} = \mathbf{C}[f_1, ..., f_{n+1}]$. Hence $(Y, \mathbf{SL}(n+1))$ is coregular but this does not imply that (V, G) is coregular since we replaced $\mathbf{SO}(n)$ by $\mathbf{O}(n)$. To prove the coregularity of (V, G) we shall need the method described in Section 4.13. However, $Z_{\mathbf{SL}(n+1)}(Y)$ is the subvariety of Y cut out by the functions $f_1, f_2, ..., f_{n+1}$. Thus $Z_{\mathbf{SL}(n+1)}(Y) = Z_{\mathbf{SL}(n+1)}(W)$, and $\dim Y - \dim Y//\mathbf{SL}(n+1) = \dim W - \dim W//\mathbf{SL}(n+1)$. Therefore, $(Y, \mathbf{SL}(n+1))$ is equidimensional if and only if $(W, \mathbf{SL}(n+1))$ is. In particular Y, and hence V, is not equidimensional if n is odd [Sch2].

Often (V, G_1) fails to be coregular. In this case we often had to compute the generators of $\mathbf{C}[V]^{G_1}$ rather than using the tables in [Sch1]. Classical invariant theory or the methods of [Sch4] or [Sch5] were sufficient for many representations. The following example illustrates the methods of [Sch5] as well as some useful other results.

EXAMPLE 4.6.8. Take $(W, G) := (\mathbf{C}^{10} \otimes \mathbf{C}^6 \oplus \sigma^+(10), \mathbf{Spin}(10) \times \mathbf{SL}(6))$, $V_1 := W \oplus \mathbf{C}^{6*}$ and $V_2 := W \oplus 2\mathbf{C}^{6*}$. Then neither $(V_b, \mathbf{Spin}(10))$ nor $(V_b, \mathbf{SL}(6))$ is coregular for $b = 1, 2$. The method described at the end of Section 4.8 would allow us to study V_1 but it is of no use for V_2. Instead we apply the methods of [Sch5]. A set of generators for $\mathbf{C}[5\mathbf{C}^{10} \oplus \sigma^+(10)]^{\mathbf{Spin}(10)}$ is given in [Sch1]. We may polarize these generators to obtain 33 invariants $f_1, f_2, ..., f_{33}$ which are part of a minimal generating set for $\mathbf{C}[W]^{\mathbf{Spin}(10)}$. For the definition of polarization we refer the reader to [W, Chap. 1]. We need to determine if some new types of generators other than these 33 occur in $\mathbf{C}[W]^{\mathbf{Spin}(10)}$ as a result of the additional copy of $\mathbf{C}^{10}$.

Suppose that f is such a new generator. By [Sch5, Theorem 1.22] if we restrict f to (say the last) two copies of $\mathbf{C}^{10}$ in W it must then transform as a highest weight vector for the representation $\bigwedge^2 \mathbf{C}^{10}$. Hence, f restricted to $4\mathbf{C}^{10} \oplus \sigma^+(10)$ must transform under the action of $\mathbf{Spin}(10)$ as a highest weight vector of $\bigwedge^2 \mathbf{C}^{10*} \cong \bigwedge^2 \mathbf{C}^{10}$. Therefore we are reduced to finding the relevant copies of $\bigwedge^2 \mathbf{C}^{10}$ in $\mathbf{C}[4\mathbf{C}^{10} \oplus \sigma^+(10)]$. A subrepresentation of $\mathbf{C}[V]$ is *relevant* if it does not lie in $\mathbf{C}[V]^G_+ \cdot \mathbf{C}[V] \subset \mathbf{C}[V]$.

Since $(4\mathbf{C}^{10} \oplus \sigma^+(10), \mathbf{Spin}(10))$ is cofree we may use Proposition 2.1.1 or [Sch1, Prop. 4.6] to compute that $\mathbf{C}[4\mathbf{C}^{10} \oplus \sigma^+(10)]$ contains 18 relevant copies of $\bigwedge^2 \mathbf{C}^{10}$. Using the methods described in [Li2] we may explicitly identify these 18 copies and show that the corresponding 18 generators, f, all lie in $\mathbf{C}[f_1, f_2, ..., f_{33}]$. Thus $\mathbf{C}[W]^{\mathbf{Spin}(10)} = \mathbf{C}[f_1, f_2, ..., f_{33}]$.

Now the free $\mathbf{SL}(6)$-module spanned by $f_1, f_2, ..., f_{33}$ is isomorphic to $X := \mathbf{S}^2\mathbf{C}^6 \oplus \mathbf{C}^6 \oplus \mathbf{C}^{6*}$. Using the Luna slice theorem we may compute that $Y := W//\mathbf{Spin}(10)$ is a 31 dimensional variety. By [P2, p. 519, Remark] this implies that Y is a complete intersection, i.e., Y is cut out of X by two functions, h_1 and h_2, which must then be $\mathbf{SL}(6)$-invariants. Hence, $Y \oplus b\mathbf{C}^{6*}$ is an equidimensional $\mathbf{SL}(6)$-variety if and only if $X \oplus b\mathbf{C}^{6*}$ is an equidimensional $\mathbf{SL}(6)$-representation and this last condition is fulfilled precisely when $b \leqslant 1$. In particular V_2 is not equidimensional. Furthermore, we may now apply the method of strata described in the next section to see that V_1 is equidimensional.

To prove that V_1 is coregular we need to show that $Y \oplus \mathbf{C}^{6*}$ is coregular. This is equivalent to showing that $\{h_1, h_2\}$ is part of a minimal generating set for $\mathbf{C}[X \oplus \mathbf{C}^{6*}]^{\mathbf{SL}(6)}$. Since $h_1, h_2 \in \mathbf{C}[X]^{\mathbf{SL}(6)}$ we see that (V_1, G) is coregular if and only if $\{h_1, h_2\}$ is part of a minimal generating set for $\mathbf{C}[X]^{\mathbf{SL}(6)}$. Hence, V_1 is coregular if and only if X is coregular if and only if (W, G) is coregular. Finally the fact that (W, G) is coregular is an easy consequence of the method of castling described in Section 4.8. Note that the above argument also proves that (V_2, G) is coregular.

Sometimes neither the techniques of classical invariant theory nor the methods of [Sch1, Sch5] were not sufficient to compute the generators of $\mathbf{C}[V]^{G_1}$ and we had to resort to the following two lemmas:

LEMMA 4.6.9 [Sch1, Lemma 3.13]. *Suppose that (V_1, G) is stable and that there exists $f \in \mathbf{C}[V_1]^G$ which generates the ideal in $\mathbf{C}[V_1]^G$ vanishing on the non-principal orbits. Let H be a principal isotropy group of (V_1, G) and set $N := N_G H$. Then for any representation (W, G) the restriction mapping induces an isomorphism,* $\mathbf{C}[V_1 \oplus W]^G_f \cong \mathbf{C}[V_1^H \oplus W]^N_f$.

LEMMA 4.6.10 (Compare [Sch3, 15.11]). *Let B be a finitely generated algebra over $\mathbf{C}$ which is a domain and let A be the subalgebra generated by the non-zero elements $f_1, ..., f_{d+1}$. Suppose that $A_{f_1} = B_{f_1}$, $A_{f_2} = B_{f_2}$, $\dim B = d$, and that (f_1, f_2) is a partial regular sequence in A. Then $A = B$.*

Proof. Clearly $\dim A = \dim A_{f_1} = \dim B_{f_1} = d$. Therefore, either A is the algebra of a hypersurface or A is a regular ring. In either case A is Cohen–Macaulay [Ma, Section 16]. Thus f_1 and f_2 must be relatively prime in A since they form a regular sequence in A. Let $b \in B$ and write $b = a_1/f_1^i = a_2/f_2^j$, where $a_1, a_2 \in A$ and $i, j \in \mathbf{N}$. Then $f_1^i a_2 = f_2^j a_1$ and hence f_1^i divides a_1 in A. Therefore $b \in A$. ▮

An example of a representation where we made use of these lemmas is $(V, G) = (\sigma(7) \otimes \mathbf{C}^4 \oplus \mathbf{C}^7, \mathbf{Spin}(7) \times \mathbf{SL}(4))$. To apply the method of ladders we need to compute the $\mathbf{SL}(4)$-variety $V//\mathbf{Spin}(7)$. We use the previous two lemmas to show that this variety is a hypersurface, Y in $V' = (S^2\mathbf{C}^4 \oplus \bigwedge^2 \mathbf{C}^4 \oplus \theta_3, \mathbf{SL}(4))$. We do this by taking V_1 to be one copy of $\sigma(7)$ and f_1 the corresponding invariant inner product. Then $H \cong \mathbf{G}_2$ and $N \cong \mathbf{G}_2 \rtimes \mathbf{Z}/2\mathbf{Z}$. Then calculating the restriction of $\mathbf{C}[Y]$ to $V_1^H \oplus W$ we prove the isomorphism $\mathbf{C}[V_1^H \oplus W]_{f_1}^N \cong \mathbf{C}[Y]_{f_1}$. Thus by Lemma 4.6.9 we have $\mathbf{C}[V]_{f_1}^{\mathbf{Spin}(7)} \cong \mathbf{C}[Y]_{f_1}$. Choosing f_2 to be the inner product in any one of the other three copies of $\sigma(7)$ we see that the identical argument shows that $\mathbf{C}[V]_{f_2}^{\mathbf{Spin}(7)} \cong \mathbf{C}[Y]_{f_2}$. Therefore applying Lemma 4.6.10 we see that $\mathbf{C}[V]^{\mathbf{Spin}(7)} \cong \mathbf{C}[Y]$.

4.7. *Strata*

Here we use the method of [Li1, Sect. 2.7] to refine the ideas of the previous section. This gives us a method for computing $\dim Z_G(V)$ exactly, when $X := V//G_1$ is an equidimensional G_2-variety. We consider the stratification of $V//G_1$ indexed by the closed isotropy classes of (V, G_1). Restricting this stratification we obtain a partitioning

$$Z_{G_2}(X) = \bigsqcup_{(K)} (Z_{G_2}(X) \cap X_{(K)})$$

which, via $\pi_{G_1,V}$ induces a partitioning of the null cone of V. Adding the dimension of the fibres above $X_{(K)}$ to $\dim(Z_{G_2}(X) \cap X_{(K)})$ we obtain the dimension of the subset in our partition which corresponds to (K).

If K is a closed isotropy group of (V, G_1) and (S, K) is the associated slice representation, then the fibres of $\pi_{G_1,V}$ above $(V//G_1)_{(K)}$ are all isomorphic to $G_1 *_K Z_K(S)$. Therefore, if we define $D_K := \dim Z_{G_2}(X) \cap (X)_{(K)}$ and $F_K := \dim G_1 *_K Z_K(S) = \dim G_1 - \dim K + \dim Z_K(S)$, then

$$\dim Z_G(V) = \max_{(K)} (D_K + F_K), \tag{11}$$

where the maximum is taken over all closed isotropy classes of (V, G_1). (We define the dimension of the empty set to be $-\infty$).

Next we show that we need only consider those closed isotropy classes whose associated slice representations are not equidimensional. If (S, K) is equidimensional, then $\dim Z_K(S) = \dim S - \dim S//K = \dim S - \dim V//G_1 = \dim V + \dim K - \dim G_1 - \dim V//G_1$. Thus $F_K = \dim V - \dim V//G_1$. Since we are assuming that $(V//G_1, G_2)$ is equidimensional, $D_K \leqslant \dim V//G_1 - \dim V//G$. Hence $D_K + F_K \leqslant \dim V - \dim V//G$. Therefore, in using Eq. (11) to determine whether or not (V, G) is equidimensional, we need only consider those closed isotropy classes whose associated slice representation is not equidimensional.

To apply this method we need to determine all the closed isotropy classes of (V, G_1). This can usually be done by using [Sch1, Lemma 3.8].

Computing the values of F_K is usually the simpler half of this method since only the (usually simple) group K is involved. In [Sch2] many techniques for calculating $\dim Z_K(V)$ when K is simple are given. The most important of these is the method of one parameter subgroups described above.

To compute D_K we make use of the G_2-equivariant (finite) normalization morphism

$$\eta\colon (V^K)//(N_{G_1} K) \to \overline{(V//G_1)_{(K)}}.$$

From this morphism we see that $D_K \leqslant \bar{D}_K := \dim Z_{G_2}((V^K)//(N_{G_1} K))$.

Since $\overline{(V//G_1)_{(K)}} = \bigsqcup_{(L) \geqslant (K)} (V//G_1)_{(L)}$, we have $D_L \leqslant D_K$ for all $(L) \geqslant (K)$. Therefore, if $\bar{D}_L < \bar{D}_K$ for all $(L) > (K)$, then $D_K = \bar{D}_K$.

Note however, that it not necessary to compute the values D_K; knowing the numbers $\bar{D}_K$ is sufficient. If $\bar{D}_K + F_K \leqslant \dim Z_G(V)$ for all closed isotropy classes (K) then $D_K + F_K \leqslant \dim Z_G(V)$ for all (K) and hence (V, G) is equidimensional. Conversely suppose there exists a closed isotopy class (K) with $\bar{D}_K + F_K > \dim Z_G(V)$. Since $\bar{D}_K = \dim Z_{G_2}(\overline{(V//G_1)_{(K)}})$, we see from the decomposition of $\overline{(V//G_1)_{(K)}}$ into strata that there is some closed isotopy class $(L) \geqslant (K)$ with $D_L = \bar{D}_K$. But $(L) \geqslant (K)$ implies $F_L \geqslant F_K$. Therefore, $D_L + F_L \geqslant \bar{D}_K + F_K > \dim Z_G(V)$ and thus (V, G) is not equidimensional.

EXAMPLE 4.7.1. Let $(V, G) = (2\mathbf{C}^4 \otimes \mathbf{C}^2 \oplus \mathbf{S}^2\mathbf{C}^2, \mathbf{SP}(4) \times \mathbf{SL}(2))$. Then $X := V//\mathbf{SP}(4) \cong \bigwedge^2 (2\mathbf{C}^2) \oplus \mathbf{S}^2\mathbf{C}^2 \cong 2\mathbf{S}^2\mathbf{C}^2 \oplus \theta_3$. The representation $(V, \mathbf{SP}(4))$ has three closed isotropy classes: $(\mathbf{SP}(4))$, $(\mathbf{SL}(2))$, and $\{e\}$. The slice representation, $(2\mathbf{C}^2, \mathbf{SL}(2))$, associated to $\mathbf{SL}(2)$ is cofree as is the slice representation associated to $\{e\}$. Thus we need only consider the stratum associated to $(\mathbf{SP}(4))$. By Eq. (9), $F_{\mathbf{SP}(4)} = 11$. Now $X_{(\mathbf{SP}(4))} = \{0\} \times \mathbf{S}^2\mathbf{C}^2$ and thus $D_{\mathbf{SP}(4)} = \dim Z_{\mathbf{SL}(2)}(\mathbf{S}^2\mathbf{C}^2)$. Since $(\mathbf{S}^2\mathbf{C}^2, \mathbf{SL}(2))$ is equidimensional we obtain $D_{\mathbf{SP}(4)} = \dim \mathbf{S}^2\mathbf{C}^2 - \dim \mathbf{S}^2\mathbf{C}^2//\mathbf{SL}(2) = 3 - 1 = 2$. Thus $D_{\mathbf{SP}(4)} + F_{\mathbf{SP}(4)} = 13$. Since $\dim V - \dim V//G = 19 - 5 = 14$ we see that (V, G) is equidimensional.

EXAMPLE 4.7.2. Take $(V, G) = (\mathbf{C}^{n+3} \otimes \mathbf{C}^n \oplus ((n-1)/2)\mathbf{C}^{n*} \oplus 2\mathbf{C}^{n+3}, \mathbf{SP}(n+3) \times \mathbf{SL}(n))$. Note that n is odd here. For this representation we will need to consider the stratifications arising from the closed isotropy classes of both $(V, \mathbf{SL}(n))$ and $(V, \mathbf{SP}(n+3))$. Computing the values of D_K here is difficult for both stratifications. To compute these numbers for the former stratification we study the subsets of the partition more closely and develop an important method to bound the D_K. However, this method fails for $K = \{e\} \subset \mathbf{SL}(n)$. To bound $D_{\{e\}} + F_{\{e\}}$ we will need to consider the second stratification.

First we consider the stratification of $Z_{\mathbf{SP}(n+3)}(V//\mathbf{SL}(n))$ by the closed isotropy classes of $(V, \mathbf{SL}(n))$. These closed isotropy classes are $\{e\}$ and $\mathbf{SL}(n-t)$ for $0 \leqslant t \leqslant (n-1)/2$, where $(\mathbf{C}^n, \mathbf{SL}(n-t)) = \mathbf{C}^{n-t} \oplus \theta$. We set aside $\{e\}$ for a while and consider the strata associated to be other closed isotropy classes. For ease of notation we set $F_t := F_{\mathbf{SL}(n-t)}$, $D_t := D_{\mathbf{SL}(n-t)}$, and $\bar{D}_t := \bar{D}_{\mathbf{SL}(n-t)}$. Using Example (4.1.1) we compute that $\dim Z_{\mathbf{SL}(n-t)}((n-t+3)\mathbf{C}^n \oplus ((n-1)/2 - t)\mathbf{C}^{n*}) = (n-t)(n-t+3) + \lceil (n/2 - 1/2 - t - 3)^2/2 \rceil + \operatorname{err}(t)$, where

$$\operatorname{err}(t) = \begin{cases} 6, & \text{if } t = (n-1)/2; \\ 4, & \text{if } t = (n-3)/2; \\ 1, & \text{if } t = (n-5)/2; \\ 0, & \text{if } t \leqslant (n-7)/2. \end{cases}$$

Hence, $F_t \leqslant n^2 + 3(n-t) + \frac{1}{4}((n-1)/2 - t - 3)^2 + \operatorname{err}(t)$.

To compute D_t we take $N_t := N_{\mathbf{SL}(n)}\mathbf{SL}(n-t)/\mathbf{SL}(n-t) \cong \mathbf{GL}(t)$ and $W_t := V^{\mathbf{SL}(n-t)} = \mathbf{C}^{n+3} \otimes \mathbf{C}^t \oplus ((n-1)/2)\mathbf{C}^{t*} \oplus 2\mathbf{C}^{n+3}$. Set $Y_t := W_t//N_t$. Then we have an $\mathbf{SP}(n+3)$-equivariant finite map

$$\eta\colon Y_t \to \overline{(V//\mathbf{SL}(n))_{(\mathbf{SL}(n-t))}}.$$

Since η is $\mathbf{SP}(n+3)$-equivariant, η maps $(Y_t)_{\{e\}}$ to $(V//\mathbf{SL}(n))_{(\mathbf{SL}(n-t))}$. Thus we need to determine $\dim Z_{\mathbf{SP}(n+3)}(Y_t) \cap (Y_t)_{\{e\}}$.

Suppose that $\xi \in Z_{\mathbf{SP}(n+3)}(Y_t) \cap (Y_t)_{\{e\}}$, and take $x = (A, y, v) \in \pi^{-1}_{N_t, W_t}(\xi)$, where $A \in \mathbf{C}^{n+3} \otimes \mathbf{C}^t$, $y \in ((n-1)/2)\mathbf{C}^{t*}$ and $v \in 2\mathbf{C}^{n+3}$. Since $\xi \in (Y_t)_{\{e\}}$, we have $\mathbf{GL}(t)_x = \{e\}$. This implies that the rank of A must be t and that if we write $y = (y_1, \ldots, y_{(n-1)/2})$, then $y_1, \ldots, y_{(n-1)/2}$ span a t-dimensional space. Consider $\bar{x} := \pi_{\mathbf{SP}(n+3), W_t}(x)$. We take advantage of the fact that $\pi_{\mathbf{SL}(n), V} \circ \pi_{\mathbf{SP}(n+3), V//\mathbf{SL}(n)} = \pi_{\mathbf{SP}(n+3), V} \circ \pi_{\mathbf{SL}(n), V//\mathbf{SP}(n+3)}$. Since $\xi \in Z_{\mathbf{SP}(n+3)}(Y_t)$, we have $x \in Z_{\mathbf{GL}(t) \times \mathbf{SP}(n+3)}(W_t)$, and thus

$\bar{x} \in Z_{GL(t)}(V//\mathbf{SP}(n+3))$. Since $\pi_{\mathbf{SP}(n+3), W_t}$ restricted to $((n-1/2)\mathbf{C}^{t*}$ is the identity and y has rank t, this implies that $\bar{x} = y$. Therefore, $x \in Z_{\mathbf{SP}(n+3)}(\mathbf{C}^{n+3} \otimes \mathbf{C}^t \oplus 2\mathbf{C}^{n+3}) \oplus ((n-1)/2)\mathbf{C}^{t*}$. Hence

$$\dim \pi_{N_t, W_t}^{-1}(Z_{\mathbf{SP}(n+3)}(Y_t) \cap (Y_t)_{\{e\}}) \leqslant \dim(Z_{\mathbf{SP}(n+3)}(\mathbf{C}^{n+3} \otimes \mathbf{C}^t \oplus 2\mathbf{C}^{n+3})) + \frac{(n-1)t}{2}.$$

Using Eq. (9) we obtain the bound,

$$\bar{D}_t = \dim Z_{\mathbf{SP}(n+3)}(Y_t) \cap (Y_t)_{\{e\}} \leqslant \dim(Z_{\mathbf{SP}(n+3)}(\mathbf{C}^{n+3} \otimes \mathbf{C}^t \oplus 2\mathbf{C}^{n+3})) + \frac{(n-1)t}{2} - t^2.$$

Thus $\dim V - D_t - F_t \geqslant f(t)$, where

$$f(t) := \frac{n-1}{2} n - tn + \frac{(t+2)(t+1)}{2} - \frac{n-1}{2} t + t^2 - \frac{1}{4}\left(\frac{n-1}{2} - t - 3\right)^2 + \operatorname{err}(t).$$

Applying the calculus we find that $f(t) \geqslant \dim V//G$ for all $0 \leqslant t \leqslant (n-1)/2$.

All that remains now is to consider the stratum associated to $\{e\}$. Since the slice representation associated to this class is equidimensional we would be done if we knew that $(V//\mathbf{SL}(n), \mathbf{SP}(n+3))$ were equidimensional. However, we see no easy method to show this. Thus the possibility remains that the set $U := \{x = (A, y, v) \in V \mid \pi_{\mathbf{SL}(n), V}(x) \in (V//\mathbf{SL}(n))_{\{e\}}\} \cap Z_G(V)$ has dimension greater than $\dim V - \dim V//G$. To prove that this is not the case we consider the stratification indexed by the closed isotropy classes of $(V, \mathbf{SP}(n+3))$. Let $x = (A, y, v) \in U$. Then the rank of A is n and this implies that $\pi_{\mathbf{SP}(n+3), V}(x)$ lies in one of the strata associated to $\mathbf{SP}(n+3-t)$, where $t = n+3$, $n+1$, $n-1$, or $n-3$. Now the slice representations, $((n+2-t)\mathbf{C}^{n+3-t}, \mathbf{SP}(n+3-t))$, associated to the first three of these strata are all cofree [Sch2], and $(V//\mathbf{SP}(n+3), \mathbf{SL}(n))$ is also cofree. Thus we need only consider the $\mathbf{SP}(6)$-stratum. Using Eq. (9) we find that $F_{\mathbf{SP}(6)} = \dim V - \dim V//\mathbf{SP}(n+3) + 1$ and hence it remains to show that $U \cap (V//\mathbf{SP}(n+3))_{(\mathbf{SP}(6))}$ has dimension less than $\dim Z_{\mathbf{SL}(n)}(V//\mathbf{SP}(n+3))$. But $(V//\mathbf{SP}(n+3))_{(\mathbf{SP}(6))}$ has normalization $\{(v, y) \mid v \in \bigwedge^2 (\mathbf{C}^n \oplus \theta_2)$, $y \in ((n-1)/2)\mathbf{C}^{n*}$ and $\operatorname{rank}(v) \leqslant n-3\}$. Examining $Z_{\mathbf{SL}(n)}(\bigwedge^2 \mathbf{C}^n \oplus 2\mathbf{C}^n \oplus ((n-1)/2)\mathbf{C}^{n*})$ (by considering 1-PSG's, for example) we easily obtain the desired bound on $\dim U$. Hence, (V, G) is equidimensional.

Remark 4.7.3. Note that we could have proven this representation was cofree by considering only the second stratification discussed above. However, we see no (easy) way to bound the values $D_{\mathbf{SP}(n+3-2t)}$.

For most representations to which we apply this method, the G_2-variety $V//G_1$ is coregular and thus it was sufficient to consider only the one stratification.

4.8. *Castling*

In [KiS] *Castling transformations* are defined for representations of the form $(W \otimes \mathbf{C}^n, H \times \mathbf{SL}(n))$. The relationship between the equidimensionality of such a representation and its castling transform was determined by Littelmann in [Li1, Lemma 2]. We begin by extending these ideas in a straightforward way. Let $(V, G) = (\mathbf{C}^n \otimes W \oplus \rho(H), \mathbf{SL}(n) \times H)$, where H is semi-simple and set $w := \dim W$.

If $w > n$, then the *castling transform* of (V, G) is $(V', G') = (\mathbf{C}^{n'} \otimes W^* \oplus \rho(H), \mathbf{SL}(n') \times H)$, where $n' = w - n$.

PROPOSITION 4.8.1. *Let* $(V, G) = (\mathbf{C}^n \otimes W \oplus \rho(H), \mathbf{SL}(n) \times H)$, *where* H *is semi-simple. Set* $w := \dim W$.

(1) *If* $w \leqslant n$ *then* $(V, \mathbf{SL}(n))$ *is coregular* (*resp. equidimensional*) *if and only if* $(\rho(H), H)$ *is coregular* (*resp. equidimensional*).

Suppose that $w > n$ *and set* $(V', G') = (\mathbf{C}^{n'} \otimes W^* \oplus \rho(H), \mathbf{SL}(n') \times H)$, *where* $n' = w - n$. *Further suppose* (*for* (4) *below*) *that* $n' \geqslant n$. *Then*

(2) $V//G \cong V'//G'$. *In particular,* $\dim V//G = \dim V'//G'$.

(3) *The generic isotropy groups of* (V, G) *are isomorphic to the generic isotropy groups of* (V', G').

(4) (V', G') *is equidimensional if and only if both* (V, G) *is equidimensional and* $\dim V//G \leqslant \dim \rho(H)//H + n + 1$.

The proof of this proposition is left to the reader.

Remark 4.8.2. There is an (in general outer) automorphism of H which carries any representation (W, H) into (W^*, H). Clearly this automorphism preserves each of the properties of representations in which we are interested. We will often take advantage of this automorphism and replace (V', G') by $(\mathbf{C}^{n'} \otimes W \oplus \rho(H)^*, \mathbf{SL}(n') \times H)$ without further comment.

EXAMPLE 4.8.3. Consider $(V_1, G) = (\sigma(7) \otimes \mathbf{C}^6 \oplus \mathbf{C}^7, \mathbf{Spin}(7) \times \mathbf{SL}(6))$ and $(V_2, G) = (V_1 \oplus \mathbf{C}^7, G)$. The castling transform of (V_1, G) is $(V_1', G') = (\sigma(7) \otimes \mathbf{C}^2 \oplus \mathbf{C}^7, \mathbf{Spin}(7) \times \mathbf{SL}(2))$ and that of (V_2, G) is

$(V'_2, G') = (\sigma(7)\otimes \mathbf{C}^2 \oplus 2\mathbf{C}^7, \mathbf{Spin}(7)\times \mathbf{SL}(2))$. (Note that $\sigma(7)$ is a self-dual representation.) Using the strata method we find that both (V'_1, G') and (V'_2, G') are cofree. Furthermore, $\dim V'_1//G' = 3$ and $\dim V'_2//G' = 7$. Therefore, (V_1, G) is cofree while (V_2, G) is not equidimensional.

Now we extend the notion of castling transforms further. Suppose that $(V, G) = (W\otimes \mathbf{C}^n \oplus r\mathbf{C}^{n*}, \mathbf{SL}(n)\times H)$, where $r < n < w := \dim W$. By classical invariant theory $\mathbf{C}[V]^{\mathbf{SL}(n)}$ is generated by a set of $\binom{w}{n}$ decomposable n-forms together with rw contractions. The first set of these generators transform under the H-action as $\bigwedge^n_{\text{pure}}(W)$, the variety of decomposable n-forms in $\bigwedge^n(W)$. The second set of generators transform as r copies of W.

Setting $n' := w - n$ we define the castling transform of (V, G) to be the $\mathbf{SL}(n')\times H$ subvariety, Y, of $(\tilde{Y}, G') := (\mathbf{C}^{n'}\otimes W^* \oplus rW, \mathbf{SL}(n')\times H)$ cut out by the rn' contractions of W with W^*. Then $\mathbf{C}[Y]^{\mathbf{SL}(n')}$ has $\binom{w}{n'}$ generators which transform as $\bigwedge^{n'}_{\text{pure}}(W^*)$ and another rw generators which transform as r copies of W.

THEOREM 4.8.4. *Suppose* $(V, G) = (\mathbf{C}^n\otimes W \oplus r\mathbf{C}^{n*}, \mathbf{SL}(n)\times H)$, *where* $r < n < w := \dim W$. *Set* $(\tilde{Y}, G') := (\mathbf{C}^{n'}\otimes W^* \oplus rW, \mathbf{SL}(n')\times H)$, *where* $n' = w - n$. *Let* Y *be the* G'*-subvariety of* $\tilde{Y}$ *cut out by the* rn' H*-invariant contractions on* $\tilde{Y}$ *of degree* $(1, 1)$ *which together transform under* $\mathbf{SL}(n')$ *as* $r\mathbf{C}^{n'}$. *Then* $V//\mathbf{SL}(n) \cong Y//\mathbf{SL}(n')$ *as* H*-varieties.*

Proof. First we prove the theorem for the case when $r = 1$. In this case we perform a chain of "ordinary" castling transformations to obtain the result. First, V has $(\bar{V}, \bar{G}) := (\mathbf{C}^n\otimes W \oplus \mathbf{C}^n\otimes \mathbf{C}^{n-1}, \mathbf{SL}(n)\times H\times \mathbf{SL}(n-1))$ as castling transform. Now if we castle $(\bar{V}, \bar{G})$ with respect to the $\mathbf{SL}(n)$-action we obtain $(\hat{V}, \hat{G}) := (\mathbf{C}^{w-1}\otimes W^* \oplus \mathbf{C}^{w-1}\otimes \mathbf{C}^{n-1*}, \mathbf{SL}(w-1)\times H\times \mathbf{SL}(n-1))$. Next castling $(\hat{V}, \hat{G})$ with respect to the $\mathbf{SL}(n-1)$-action gives $(\tilde{V}, \tilde{G}) := (\mathbf{C}^{w-1}\otimes W^* \oplus \mathbf{C}^{n'}\otimes \mathbf{C}^{w-1*}, \mathbf{SL}(w-1)\times H\times \mathbf{SL}(n'))$. Applying classical invariant theory we see that $\tilde{V}//\mathbf{SL}(w-1)\cong Y$. Tracing through this sequence we obtain the H-equivariant isomorphisms, $Y//\mathbf{SL}(n')\cong \tilde{V}//(\mathbf{SL}(w-1)\times \mathbf{SL}(n')) \cong \hat{V}//(\mathbf{SL}(w-1)\times \mathbf{SL}(n-1)) \cong \bar{V}//(\mathbf{SL}(n)\times \mathbf{SL}(n-1))\cong V//\mathbf{SL}(n)$.

Now we consider general values for r. We use the theorem for $r = 1$ to prove the general case. Writing $V = \mathbf{C}^n\otimes W\oplus \mathbf{C}^{n*}_{(1)}\oplus \cdots \oplus \mathbf{C}^{n*}_{(r)}$ and $\tilde{Y} = \mathbf{C}^{n'}\otimes W^* \oplus W_{(1)}\oplus \cdots \oplus W_{(r)}$ we define $V_{(i)} := \mathbf{C}^n\otimes W\oplus \mathbf{C}^{n*}_{(i)}$ and $\tilde{Y}_{(i)} := \mathbf{C}^{n'}\otimes W^* \oplus W_{(i)}$. We define $Y_{(i)}$ to be the subvariety of $\tilde{Y}_{(i)}$ cut out by the n' contractions of degree $(1, 1)$. Also set $R := \mathbf{C}[V]^{\mathbf{SL}(n)}$, $R_{(i)} := \mathbf{C}[V_{(i)}]^{\mathbf{SL}(n)}$, $S := \mathbf{C}[Y]^{\mathbf{SL}(n')}$ and $S_{(i)} := \mathbf{C}[Y_{(i)}]^{\mathbf{SL}(n')}$. Now by the theorem for the case $r = 1$ there exist H-equivariant isomorphisms $\phi_{(i)}\colon R_{(i)} \to S_{(i)}$ and $\psi_{(i)} = \phi^{-1}_{(i)}\colon S_{(i)} \to R_{(i)}$.

By classical invariant theory $R_{(i)}$ is generated by $\binom{w}{n}$ determinants together with w contractions. Let A_i be the set of these generators. Applying classical invariant theory again we see that R is generated by $A := \bigcup_{i=1}^{r} A_i$. Similarly if we let B_i be a generating set for the relations among the elements of A_i then $B := \bigcup_{i=1}^{r} B_i$ is a generating set for the relations among the elements of A. From this it follows that the $\phi_{(i)}$ taken together define an H-equivariant ring homomorphism $\phi: R \to S$.

Similarly, we may use classical invariant theory to see that the $\psi_{(i)}$ together give an H-equivariant ring homomorphism $\psi: S \to R$. Clearly $\psi = \phi^{-1}$ and thus $R \cong S$. ▌

As with the simpler form of castling we may extend this (continuing with the same notation) to obtain $(V \oplus \rho(H), \mathbf{SL}(n) \times H)$ and $(Y \oplus \rho(H), \mathbf{SL}(n') \times H)$ as castling transforms.

With these castling transforms it is clear that coregularity is preserved. To determine how equidimensionality behaves under these castling transformations we proceed as with the simpler castling transforms and so we consider the $\mathbf{SL}(n)$ and $\mathbf{SL}(n')$ strata. However, now instead of having just two of each of these strata we have $r+1$ of each. Nevertheless the method of considering strata does work and allows us to determine the equidimensionality of $V \oplus \rho(H)$ or $Y \oplus \rho(H)$.

EXAMPLE 4.8.5. Take $(V, G) = (\mathbf{C}^8 \otimes \mathbf{C}^{10} \oplus 3\mathbf{C}^{8*} \oplus \sigma^+(10), \mathbf{SL}(8) \times \mathbf{Spin}(10))$. We will show that (V, G) is cofree. In order to bound the dimension of $Z_G(V)$ we consider the stratification induced by the closed isotropy classes of $(V, \mathbf{SL}(8))$. There are five such classes: $(\mathbf{SL}(8))$, $(\mathbf{SL}(7))$, $(\mathbf{SL}(6))$, $(\mathbf{SL}(5))$ and $(\{e\})$. We may compute that $\dim V - F_{\mathbf{SL}(t)} - \bar{D}_{\mathbf{SL}(t)} \geqslant 12$ for $5 \leqslant t \leqslant 8$. Since $\dim V//G = 12$, as we shall see, it follows that (V, G) is equidimensional if and only if $X := V//\mathbf{SL}(8)$ is an equidimensional $\mathbf{Spin}(10)$-variety. In order to show this we consider the castling transform of V. This castling transform, Y, is the subvariety of $(Y, G') := (\mathbf{C}^2 \otimes \mathbf{C}^{10} \oplus 3\mathbf{C}^{10} \oplus \sigma^+(10), \mathbf{SL}(2) \times \mathbf{Spin}(10))$ cut out by the six $\mathbf{Spin}(10)$-invariants of degree $(1, 1, 0)$.

Note that these six invariants are part of a minimal generating set for $\mathbf{C}[5\mathbf{C}^{10}]^{\mathbf{Spin}(10)}$. Since $(5\mathbf{C}^{10}, \mathbf{Spin}(10))$ is cofree these six functions form a (partial) regular sequence in $\mathbf{C}[\tilde{Y}]$ and thus $\dim Y = \dim \tilde{Y} - 6 = 60$. Furthermore, $(\tilde{Y}//\mathbf{Spin}(10), \mathbf{SL}(2)) = S^2\mathbf{C}^2 \oplus 4\mathbf{C}^2 \oplus \theta_{10}$ and hence $(Y//\mathbf{Spin}(10), \mathbf{SL}(2)) = S^2\mathbf{C}^2 \oplus \mathbf{C}^2 \oplus \theta_{10}$. Therefore, $V//G \cong X//\mathbf{Spin}(10) \cong (Y//\mathbf{SL}(2))//\mathbf{Spin}(10) \cong (Y//\mathbf{Spin}(10))//\mathbf{SL}(2)$ and thus (V, G) is coregular with $\dim V//G = 12$.

Considering the strata of $\tilde{Y}$ induced by the $\mathbf{Spin}(10)$-action we may show that Y and hence X and also V are equidimensional. Note that for these computations we may bound the values F_K by observing that $\pi_{G', Y}^{-1}(\eta) \subset \pi_{G', \tilde{Y}}^{-1}(\eta)$.

4.9. *Finite Principal Isotropy Groups* (FPIG)

LEMMA 4.9.1. *Suppose that G is reductive.*

(1) *If (V, G) has finite principal isotropy groups and $(W, G) \not\cong \theta$, then $(V \oplus W, G)$ is not equidimensional.*

(2) *Suppose that $G = G_1 \times G_2$. Let W be an equidimensional representation of G_1 (where G_2 acts trivially). If (W, G_1) has finite principal isotropy groups then $(V \oplus W, G)$ is equidimensional if and only if (V, G_2) is equidimensional and $(V//G_2, G_1) \cong \theta$.*

The proof of this lemma is left to the reader.

Remark 4.9.2. If G is semi-simple then a representation (V, G) has finite principal isotropy groups $\Leftrightarrow (V, G)$ has finite generic isotropy groups $\Leftrightarrow \dim V//G = \dim V - \dim G$ [LV].

EXAMPLE 4.9.3. Here we use castling to show that a representation has finite principal isotropy groups and then apply FPIG to see that some representations are not equidimensional.

Let $(V, G) = (\varphi_1(\mathbf{G}_2) \otimes \mathbf{C}^5 \oplus \varphi_1(\mathbf{G}_2) \oplus \rho(5), \mathbf{G}_2 \times \mathbf{SL}(5))$. Set $(W, G) = (\varphi_1(\mathbf{G}_2) \otimes \mathbf{C}^5 \oplus \varphi_1(\mathbf{G}_2), \mathbf{G}_2 \times \mathbf{SL}(5)) \subset (V, G)$. (W, G) has castling transform $(W', G') = (\varphi_1(\mathbf{G}_2) \otimes \mathbf{C}^2 \oplus \varphi_1(\mathbf{G}_2), \mathbf{G}_2 \times \mathbf{SL}(2))$ which has trivial principal isotropy group. Since castling preserves generic isotopy groups it follows that (W, G) also has finite principal isotropy groups. Thus if $\rho(5) \not\cong \theta$ then (V, G) is not equidimensional.

4.10. *Restriction*

It is often possible to find a complex reductive algebraic group N and a subspace $W \subset V$ such that the restriction map res: $\mathbf{C}[V]^G \to \mathbf{C}[W]^N$ is an isomorphism. If (W, N) is coregular then (V, G) is. Furthermore, if (W, N) is equidimensional, then so is (V, G) by [Sch2, Lemma 2.5].

Classical invariant theory often suffices to show that the morphism res is an isomorphism.

EXAMPLE 4.10.1. Consider $(V, G) = (\overline{\mathbf{C}^m} \otimes \mathbf{C}^n \oplus ((n-1)/2)\mathbf{C}^{n*} \oplus 2\mathbf{C}^m, \mathbf{SP}(m) \times \mathbf{SL}(n))$, where $m \geqslant n+3$ and n is odd. By classical invariant theory the invariants of (V, G) restrict to give the invariants of $(W, N) = (\overline{\mathbf{C}^{n+3}} \otimes \mathbf{C}^n \oplus ((n-1)/2)\mathbf{C}^{n*} \oplus 2\mathbf{C}^{n+3}, \mathbf{SP}(n+3) \times \mathbf{SL}(n))$. Now (W, N) is the representation which we proved equidimensional in Example 4.7.2 thus (V, G) is also equidimensional.

A general class of examples where res is an isomorphism is provided by the Luna–Richardson theorem [LRi]: If H is a principal isotropy group of (V, G), $N := N_G H$ and $W := V^H$, then res: $\mathbf{C}[V]^G \to \mathbf{C}[W]^N$ is an isomorphism. While we can apply this method to our representation (V, G) it is often equally useful when applied only to (V, G_1). If H is a principal isotropy group of (V, G_1), and $N := N_{G_1} K$, then res, $\mathbf{C}[V]^{G_1} \to \mathbf{C}[V^H]^N$ is an isomorphism. Hence res: $\mathbf{C}[V]^G \cong (\mathbf{C}[V]^{G_1})^{G_2} \to (\mathbf{C}[V^H]^N)^{G_2} \cong \mathbf{C}[V^H]^{N \times G_2}$ is also an isomorphism. Note that this result is of no use if $H = \{e\}$.

EXAMPLE 4.10.2. Let $(V, G) = (\mathbf{C}^m \otimes \mathbf{C}^n \oplus \mathbf{C}^n, \mathbf{SO}(m) \times \mathbf{SP}(n))$, where $m \geqslant n$ and consider $(V', G') = (\mathbf{C}^m \otimes \mathbf{C}^n \oplus \mathbf{C}^n, \mathbf{O}(m) \times \mathbf{SP}(n))$. Since $\mathbf{O}(m-n)$ is a principal isotropy group of $(V', \mathbf{O}(m))$, we may pass to $(W, H) = (\overline{\mathbf{C}^n} \otimes \mathbf{C}^n \oplus \mathbf{C}^n, \overline{\mathbf{O}(n)} \times \mathbf{SP}(n))$ which can be shown to be equidimensional by applying the inequality (6). The representations $(\mathbf{C}^m \otimes \mathbf{C}^n \oplus \mathbf{C}^n, \mathbf{SO}(m) \times \mathbf{SP}(n))$ for $m < n$ $(\mathbf{C}^m \otimes \mathbf{C}^n \oplus \mathbf{C}^m, \mathbf{SO}(m) \times \mathbf{SP}(n))$, $(\mathbf{C}^m \otimes \mathbf{C}^n, \mathbf{SO}(m) \times \mathbf{SO}(n))$, and $(\mathbf{C}^m \otimes \mathbf{C}^n \oplus a\mathbf{C}^m \oplus b\mathbf{C}^n, \mathbf{SP}(m) \times \mathbf{SP}(n))$, where $a + b = 2$ are all handled similarly.

4.11. *L-Method*

In this section we describe a method for showing that a representation is coregular.

LEMMA 4.11.1 [Schl, Lemma 3.5]. *Let G be a connected semi-simple group and let (V, G) be a G-representation having a closed isotropy class (L) such that* $\dim V^L//N_G(L) = d - 1$, *where* $d = \dim V//G$. *Let* $\mathrm{res}_L : \mathbf{C}[V]^G \to \mathbf{C}[V^L]^{N_G(L)}$ *be the restriction map. Suppose that $p_1, p_2, ..., p_d$ are elements of $\mathbf{C}[V]^G$ such that the elements $p_i' := \mathrm{res}_L(p_i)$ form a minimal generating set for $\mathrm{res}_L(\mathbf{C}[V]^G)$. Since G is connected and semi-simple, the ideal $I(V, L)$ in $\mathbf{C}[V]^G$ vanishing on V^L is principal and prime. Let f_L be a homogeneous generator of $I(V, L)$. Suppose that the relations among the p_i' are generated by $f(p_1', ..., p_d')$. Then $\mathbf{C}[V]^G$ is generated (perhaps non-minimally) by $\{f_L, p_1, ..., p_d\}$ with a relation of the form $f_L \cdot h(f_L, p_1, ..., p_d) = f(p_1, ..., p_d)$ (where h may be zero).*

EXAMPLE 4.11.2. We only required this method to handle a few representations, including $(V_1, G) := (\mathbf{C}^4 \otimes \mathbf{C}^7 \oplus \wedge^3 \mathbf{C}^7, \mathbf{SP}(4) \times \mathbf{SL}(7))$ and $(V_2, G) := (\mathbf{C}^4 \otimes \mathbf{C}^7 \oplus \wedge^4 \mathbf{C}^7, \mathbf{SP}(4) \times \mathbf{SL}(7))$. These two were the most difficult of all the representations we considered. Both had a closed isotropy group L of the desired form with $L \cong \mathbf{C}^*$. Moreover they both also had a closed isotropy group $M \cong \mathbf{C}^* \times \mathbf{C}^*$ with $L \subset M$. For V_2 we were able to show that $\mathrm{res}_L(\mathbf{C}[V_2]^G)$ is the subalgebra of $\mathbf{C}[V_2^L]^{N_G(L)}$ consisting of those elements whose restriction to V_2^M is $N_G(M)$-invariant. For V_1 this was not the case and we had to consider Hilbert series to determine $\mathrm{res}_L(\mathbf{C}[V_1]^G)$.

Having found, for each of the two cases, a set of five invariants whose restrictions $p'_1, \ldots, p'_5$ generate the ring, $\text{res}_L(\mathbf{C}[V_k]^G)$, we needed to show that h is a non-zero constant in order to prove coregularity. To do this we considered a slice representation. Both V_1 and V_2 have $(\mathbf{C}^4 \otimes \varphi_1(\mathbf{G}_2) \oplus \theta_1, \mathbf{SP}(4) \times \mathbf{G}_2)$ as a slice representation. Considering the image of the p_i in this slice representation we were able to show that for both representations f has degree $(24, a)$ for some a. Studying the algebras $\text{res}_L(\mathbf{C}[V_k]^G)$ we found that both f_L have degree $(24, b)$ for some b. Using this we were able to prove (after showing that h was non-zero) that in both cases $h \in \mathbf{C}^*$ and hence that $\mathbf{C}[V_1]^G$ and $\mathbf{C}[V_2]^G$ are both regular rings.

The question of the equidimensionality of V_2 is discussed in Example 4.12.1.

4.12. *Subdivision by Orbits*

For some representations we used the method described in [Sch2, Example 2.16]. A representation is *visible* if its null cone contains only finitely many orbits. Suppose our representation $(V = V_1 \oplus V_2, G)$ contains a visible subrepresentation V_1. Write $Z_G(V_1) = \bigcup_{i=1}^r G \cdot w_i$. Then decompose $Z_G(V)$ into $Z_G(V) = \bigcup_{i=1}^r Z_i$, where $Z_i := Z_G(V) \cap (G \cdot w_i \oplus V_2)$. Now let H_i denote the reductive part of the isotropy group G_{w_i}. Restricting the elements of $\mathbf{C}[V]^G$ to $(\{w_i\} \oplus V_2) \cong V_2$ gives elements of $\mathbf{C}[V_2]^{H_i}$. Suppose that (V_2, H_i) is equidimensional and that we can find a sequence of a_i elements of $\mathbf{C}[V]^G$ which restrict to give a regular sequence in $\mathbf{C}[V_2]^{H_i}$. Then writing $b_i := \text{codim}_{V_1}(G \cdot w_i)$ we obtain $\text{codim}_V(Z_G(V)) = \min\{\text{codim}_V(Z_i) \mid 1 \leqslant i \leqslant r\} \geqslant \min\{a_i + b_i \mid 1 \leqslant i \leqslant r\}$.

EXAMPLE 4.12.1. Let $(V, G) = (\mathbf{C}^4 \otimes \mathbf{C}^7 \oplus \bigwedge^3 \mathbf{C}^7, \mathbf{SP}(4) \times \mathbf{SL}(7))$, one of the representations considered in Example 4.11.2. In that example we showed that V is coregular by proving that $\mathbf{C}[V]^G = \mathbf{C}[p_1, \ldots, p_5]$ for five invariants p_i. One of these invariants, p_1, has degree $(0, 7)$ on V. In [G, 35.1–35.3], $Z_{\mathbf{SL}(7)}(\bigwedge^3 \mathbf{C}^7) = \{x \in \bigwedge^3 \mathbf{C}^7 \mid p_1(x) = 0\}$ is shown to consist of nine $\mathbf{SL}(7)$-orbits: $\mathbf{SL}(7) \cdot w_1, \ldots, \mathbf{SL}(7) \cdot w_9$. Studying these nine orbits we find that (keeping the ordering of [G]) $b_i \geqslant 5$ for all i except $i = 7$ and $i = 9$. We obtain $b_7 = 4$. However, the invariant p_3 of degree $(4, 1)$ on V is non-zero on $\overline{\mathbf{C}^4 \otimes \mathbf{C}^7 \oplus \mathbf{SL}(7) \cdot w_7}$ and thus $a_7 \geqslant 1$.

The orbit of w_9 is dense in $Z_{\mathbf{SL}(7)}(\bigwedge^3 \mathbf{C}^7)$ and we must work harder to bound this portion of the null cone. The reductive part of G_{v_9} has identity component $H = \mathbf{SP}(4) \times \mathbf{SL}(2) \times \overline{\mathbf{SL}(2)}$. Restricting to the action of H on $\mathbf{C}^4 \otimes \mathbf{C}^7 \times \{w_9\}$ we obtain the representation $W = \mathbf{C}^4 \otimes \mathbf{S}^2\mathbf{C}^2 \oplus \mathbf{C}^4 \otimes \mathbf{C}^2 \otimes \overline{\mathbf{C}^2}$. We need to show that $p_2, \ldots, p_5$ restricts to a regular sequence in $\mathbf{C}[W]$. We see no direct way to show this. However, $(\mathbf{C}^4 \otimes \mathbf{S}^2\mathbf{C}^2, H)$ is a visible representation and we may apply the method of subdivision to W itself.

The method used in Example 4.11.2 gives a description of the five invariants as they are described by classical invariant theory, that is, as sums and products of various seven by seven determinants. This description expresses these invariants as long sums, some of which involve more than 10^{24} terms. However, the restrictions of the invariants to $\mathbf{C}^4 \otimes \mathbf{C}^7 \times \{w_9\}$ are significantly less complicated and we were able to use a computer to express these restrictions explicitly as polynomials.

The restriction of p_2 to W is the invariant of degree $(4, 0)$ in $\mathbf{C}[W]^H$ which cuts out $Z_H(\mathbf{C}^4 \otimes \mathbf{S}^2\mathbf{C}^2)$. Hence we further restrict p_3, p_4, and p_5 to sets of the form $W_q := \{v_q\} \times \mathbf{C}^4 \otimes \mathbf{C}^2 \otimes \overline{\mathbf{C}^2}$. Doing this we find that there are only two orbits $H \cdot v_q$ which do not have $b_q \geqslant 4$. For the first of these $H \cdot v_1$ we again find that $b_1 = 3$ and p_3 restricts to a non-zero function on $\{v_1\} \times \mathbf{C}^4 \otimes \mathbf{C}^2 \otimes \overline{\mathbf{C}^2}$. Thus we have only one orbit to consider. This orbit $H \cdot v_2$ is dense in $Z_H(\mathbf{C}^4 \otimes \mathbf{S}^2\mathbf{C}^2)$ and thus $b_2 = 1$. Now we need to show that the restrictions of p_3, p_4, and p_5 to $\{w_2\} \times \mathbf{C}^4 \otimes \mathbf{C}^2 \otimes \overline{\mathbf{C}^2}$ form a regular sequence. By considering the restriction of p_3, p_4, and p_5 to $\{w_2\} \times T$ for a carefully chosen subspace, T, of $\mathbf{C}^4 \otimes \mathbf{C}^2 \otimes \overline{\mathbf{C}^2}$ we were able to show this. Hence $\operatorname{codim} Z_G(V) = \operatorname{codim} Z_9 = 5 = \dim V//G$ and thus (V, G) is equidimensional.

Remark 4.12.2. It is not necessary that V_1 be visible to use this method. It is clear that the method may be applied if, for example, all but finitely many of the orbits $G \cdot w_i$ in $Z_G(V_1)$ satisfy $b_i \geqslant \dim V//G$.

4.13. *Expanding*

Here we describe a method which is especially useful for dealing with representations that have $S^2\mathbf{C}^{n^{(*)}}$ or $\bigwedge^2 \mathbf{C}^{n^{(*)}}$ as subrepresentations. We illustrate this method with two examples.

The first example illustrates many of the techniques of expanding without being unduly complicated or long and so we include it, even though it deals with a representation of $\mathbf{SL}(n) \times \mathbf{SL}(m)$.

EXAMPLE 4.13.1. Take $(V, G) = (\tilde{\mathbf{C}}^m \otimes \overline{\mathbf{C}}^n \oplus S^2\tilde{\mathbf{C}}^m \oplus S^2\overline{\mathbf{C}}^n, \tilde{\mathbf{SL}}(m) \times \overline{\mathbf{SL}}(n))$, where $m > n$. Consider $(V_1, G_1) = (\tilde{\mathbf{C}}^m \otimes \overline{\mathbf{C}}^n \oplus \tilde{\mathbf{C}}^m \otimes \hat{\mathbf{C}}^m \oplus \overline{\mathbf{C}}^n \otimes \mathbf{C}^n, \tilde{\mathbf{SL}}(m) \times \overline{\mathbf{SL}}(n) \times \hat{\mathbf{O}}(m) \times \mathbf{O}(n))$. Note that the quotient of V_1 by $\hat{\mathbf{O}}(m) \times \mathbf{O}(n)$ is isomorphic to (V, G) and thus the coregularity of (V, G) would follow from the coregularity of (V_1, G_1). By introducing the two new groups we allow the possibility of castling to a representation which may be easier to handle. Before we castle however we introduce an extra wrinkle. We kill one of the generators of the invariants by using a slightly different representation. This greatly simplifies the rest of the calculation.

Hence we define $(V_2, G_2) := (\tilde{\mathbf{C}}^m \otimes \bar{\mathbf{C}}^n \oplus \tilde{\mathbf{C}}^m \otimes \hat{\mathbf{C}}^{m-1} \oplus \bar{\mathbf{C}}^n \otimes \mathbf{C}^n, \widetilde{\mathbf{SL}}(m) \times \overline{\mathbf{SL}}(n) \times \hat{\mathbf{O}}(m-1) \times \mathbf{O}(n))$. If we quotient V_2 by $\hat{\mathbf{O}}(m-1) \times \mathbf{O}(n)$ we obtain the hypersurface in V defined by the discriminant, $f_0 \colon S^2\tilde{\mathbf{C}}^m \to \mathbf{C}$. Thus $\dim V_2//G_2 = \dim V//G - 1$ and $\mathbf{C}[\tilde{\mathbf{C}}^m \otimes \hat{\mathbf{C}}^m]^{\mathbf{SL}(m) \times \hat{\mathbf{O}}(m)} = \mathbf{C}[f_0]$. Write $\mathbf{C}[V_2]^{G_2} = \mathbf{C}[f_1, ..., f_r]$. Then clearly $\mathbf{C}[f_0, f_1, ..., f_r] = \mathbf{C}[V_1]^{G_1}$. Now $\dim V_1//G_1 = \dim V//G = n+1$. Hence (V_1, G_1) is coregular if and only if $r = n$ if and only if (V_2, G_2) is coregular.

Now castle to obtain $(V_2', G_2') = (\tilde{\mathbf{C}}^{n-1} \otimes \bar{\mathbf{C}}^{n*} \oplus \tilde{\mathbf{C}}^{n-1} \otimes \hat{\mathbf{C}}^{m-1} \oplus \bar{\mathbf{C}}^n \otimes \mathbf{C}^n, \widetilde{\mathbf{SL}}(n-1) \times \overline{\mathbf{SL}}(n) \times \hat{\mathbf{O}}(m-1) \times \mathbf{O}(n))$. The representation $(V_2', \overline{\mathbf{SL}}(n))$ is coregular and quotienting V_2' by $\overline{\mathbf{SL}}(n)$ we obtain the representation $(W_1, K_1) = (\tilde{\mathbf{C}}^{n-1} \otimes \hat{\mathbf{C}}^{m-1} \oplus \tilde{\mathbf{C}}^{n-1} \otimes \mathbf{C}^n \oplus \omega, \widetilde{\mathbf{SL}}(n-1) \times \hat{\mathbf{O}}(m-1) \times \mathbf{O}(n))$, where ω is a one dimensional representation on which only $\mathbf{O}(n)/\mathbf{SO}(n)$ acts non-trivially. Now $(W_1, \hat{\mathbf{O}}(m-1) \times \mathbf{O}(n))$ is again coregular and thus we quotient W_1 by $\hat{\mathbf{O}}(m-1) \times \mathbf{O}(n)$ to obtain $(W_2, K_2) = (S^2\tilde{\mathbf{C}}^{n-1} \oplus S^2\tilde{\mathbf{C}}^{n-1} \oplus \omega^2, \widetilde{\mathbf{SL}}(n-1))$, where $\widetilde{\mathbf{SL}}(n-1)$ acts trivially on the one dimensional representation ω^2. Thus (W_2, K_2), and thus also both (V_2', G_2') and (V_2, G_2), are coregular with quotient dimension $n+1$.

EXAMPLE 4.13.2. Let $(V, G) = (\sigma(7) \otimes \mathbf{C}^n \oplus \bigwedge^2 \mathbf{C}^n, \mathbf{Spin}(7) \times \mathbf{SL}(n))$, where n is odd and $n \geqslant 7$. Both $V//\mathbf{Spin}(7)$ and $V//\mathbf{SL}(n)$ are extremely complicated. To get around this difficulty we "expand" (V, G) to $(\tilde{V}, \tilde{G}) := (\sigma(7) \otimes \mathbf{C}^n \oplus \hat{\mathbf{C}}^{n-1} \otimes \mathbf{C}^n, \mathbf{Spin}(7) \times \mathbf{SL}(n) \times \widehat{\mathbf{SP}}(n-1))$. Then $\tilde{V}//\widehat{\mathbf{SP}}(n-1) \cong V$. Now we may castle $(\tilde{V}, \tilde{G})$ to obtain $(\tilde{V}', \tilde{G}') = (\sigma(7) \otimes \mathbf{C}^7 \oplus \hat{\mathbf{C}}^n \otimes \mathbf{C}^7, \mathbf{Spin}(7) \times \mathbf{SL}(7) \times \widehat{\mathbf{SP}}(n-1))$. Then $(\tilde{V}'//\hat{\mathbf{C}}^n, \mathbf{Spin}(7) \times \mathbf{SL}(7)) \cong (\sigma(7) \otimes \mathbf{C}^7 \oplus \bigwedge^2 \mathbf{C}^7, \mathbf{Spin}(7) \times \mathbf{SL}(7)) =: (V', G')$. Now the principal isotropy group of $(\sigma(7) \otimes \mathbf{C}^7, \mathbf{Spin}(7) \times \mathbf{SL}(7))$ is a copy of $\mathbf{G}_2$ and the corresponding slice representation of V' is $(\varphi_1(\mathbf{G}_2) \oplus \varphi_2(\mathbf{G}_2) \oplus \theta, \mathbf{G}_2)$ which is neither equidimensional nor coregular. Therefore (V, G) is not coregular. Also $(\tilde{V}', \tilde{G}')$ and $(\tilde{V}, \tilde{G})$ are not equidimensional. To prove that (V, G) is not equidimensional we consider the $\widehat{\mathbf{SP}}(n-1)$ stratification of $\tilde{V}$. This stratification is indexed by the $(n+1)/2$ closed isotropy classes $(\widehat{\mathbf{SP}}(n-1)), ..., (\widehat{\mathbf{SP}}(2)), (\{e\})$. Computing we find that $\dim \tilde{V} - D_{\widehat{\mathbf{SP}}(t)} - F_{\widehat{\mathbf{SP}}(t)} \geqslant 8 = \dim V//G$ for $t \geqslant 2$. From this it follows that (V, G) is equidimensional if and only if $(\tilde{V}, \tilde{G})$ is and thus we are done.

4.14. *Substrata*

Sometimes it was necessary to further subdivide the Luna strata by using the action of 1-PSG's. The following example illustrates this method.

EXAMPLE 4.14.1. Consider $(W, H) := (\mathbf{S}^2\mathbf{C}^n, \mathbf{SL}(n))$, where $n \geqslant 3$. Let λ_p be a 1-PSG of $\mathbf{SL}(n)$ having weights $a_1, ..., a_n$ on $\mathbf{C}^n$ where $|a_1| > \cdots > |a_n| > 0$ and $a_i < 0$ if and only if $p+1 \leqslant i \leqslant 2p+1$. Fix such a

λ_p for each $1 \leqslant p \leqslant \lceil (n-1)/2 \rceil$. Now let λ be a generic 1-PSG of $\mathbf{SL}(n)$ whose weights on $\mathbf{C}^n$ are $b_1, \ldots, b_n$. Using the action of the Weyl group of $\mathbf{SL}(n)$ we may permute these weights into any order. Thus we need only consider those λ whose weights satisfy $|b_1| > \cdots > |b_n| > 0$. But if we examine the action of such a 1-PSG, λ, we find there exists a p such that $Z_\lambda(W) \subseteq Z_{\lambda_p}(W)$ (cf. [Sch2, Eq. 2.12]). Therefore, $Z_H(W) = \bigcup_p H \cdot Z_{\lambda_p}(W)$.

Now consider $(V_1, G) = (\mathbf{C}^m \otimes \mathbf{C}^n, \mathbf{SO}(m) \times \mathbf{SL}(n))$, where $m > n \geqslant 3$, $V_2 = \mathbf{S}^2\mathbf{C}^n$ and $V = V_1 \oplus V_2$. Observe that $(V_1//\mathbf{SO}(m), \mathbf{SL}(n)) \cong (W, H)$ and define $X_p := \{v = (v_1, \ldots, v_n) \in V_1 \mid \pi_{\mathbf{SO}(m), V_1}(v) \in Z_{\lambda_p}(\mathbf{S}^2\mathbf{C}^n)\}$. Then $Z_G(V_1) = \bigcup_p (\mathbf{SL}(n) \cdot X_p) = \bigcup_p (U_{\lambda_p} \cdot X_p)$, where U_{λ_p} is the unipotent subgroup of $\mathbf{SL}(n)$ not preserving $Z_{\lambda_p}(\mathbf{S}^2\mathbf{C}^n)$. Hence we have $Z_G(V) = \bigcup_p (U_{\lambda_p} \cdot (X_p \oplus Z_{\lambda_p}(V_2)))$. We partition X_p as the union $X_p = \bigcup_{r=0}^{p+1} X_{p,r}$, where $X_{p,r} := \{(v_1, \ldots, v_n) \in X_p \mid \dim \operatorname{span}\{v_{p+1}, \ldots, v_{2p+1}\} = r\}$. Then considering the action of $\mathbf{SO}(m)$ we compute that $\dim X_{p,r} = mn - \binom{r+1}{2} - (p+1-r)(m-r) - (n-2p-1)r$. Applying the calculus we find that $\dim X_p = \dim X_{p,p+1} = mn + \frac{3}{2}p^2 + \frac{3}{2}p - n - np$. Now $\dim U_{\lambda_p} + \dim X_p + \dim Z_{\lambda_p}(V_2) = \dim V - (n+1) = \dim V - \dim V//G$ for all p. Hence (V, G) is equidimensional.

Remark 4.14.2. If $n = 2$ then the condition $|a_1| > |a_2|$ is impossible. However taking this into account, a slight modification of the proof in the above example shows that $(\mathbf{C}^m \otimes \mathbf{C}^2 \oplus \mathbf{S}^2\mathbf{C}^2, \mathbf{SO}(m) \times \mathbf{SL}(2))$ is equidimensional.

5. The Classification

In this section we outline the steps we used to obtain the classification.

Let G be a connected (and simply connected) complex algebraic semisimple group having two simple factors. Write $G = G_1 \times G_2$, where the G_i are the simple factors. It is possible to form representations of G using representations of G_1 and G_2 which do not interact: $(V, G) = (V_1 \oplus V_2, G) = (V_1, G_1) \oplus (V_2, G_2)$. For such representations, (V, G) is equidimensional (resp. coregular) if and only if both (V_1, G_1) and (V_2, G_2) are equidimensional (resp. coregular). Thus in this case we are reduced to considering representations of simple groups.

If (V, G) does not decompose into representations of simple groups then (V, G) must contain an irreducible subrepresentation $\rho(G_1) \otimes \rho'(G_2)$, where both $(\rho(G_1), G_1)$ and $(\rho'(G_2), G_2)$ are non-trivial irreducible representations. We call such a representation, $\rho(G_1) \otimes \rho'(G_2)$ *bi-simple*. We will classify the remaining equidimensional representations (V, G) by considering their bi-simple subrepresentations. A list of all the bi-simple equidimensional representations (up to castling transformation) may be found in [Li1].

The methods of Section 4 are sufficient to classify the equidimensional representations of $G_1 \times G_2$ when both factors are not isomorphic to $\mathbf{SL}(n)$ for any n. These representations are listed in Tables I–IV. However, if we suppose that, say, $G_2 = \mathbf{SL}(n)$ then any representation $(W \otimes \mathbf{C}^n, G_1 \times \mathbf{SL}(n))$ with $\dim W \leqslant n$ is cofree. Since, a priori, we know nothing about the representation W other than this bound on its dimension we require some extra results to classify those representations which contain $W \otimes \mathbf{C}^n$ as a subrepresentation.

In [Li1] all the equidimensional irreducible representations of the form $W \otimes \mathbf{C}^n$ are given (up to castling transformation). To this list we need to add all the reducible representations of this form. We shall assume for the rest of this section that G_2 is a special linear group and that G_1 is either $\mathbf{Spin}(m)$ with $m \geqslant 5$, $\mathbf{SP}(m)$ with $m \geqslant 4$ or is one of the simple exceptional groups.

LEMMA 5.0.1. *Let G_1 be as above and let W a representation of G_1. Set $w := \dim W$.*

(1) *If $3w \leqslant \dim G_1$ then W is plain.*

(2) *If $2w < \dim G_1$ and W is non-plain then (W, G_1) is one of the representations*: $(\sigma(7) \oplus \theta, \mathbf{Spin}(7))$, $(\sigma(9) \oplus \theta, \mathbf{Spin}(9))$, $(\sigma^{\pm}(10) \oplus \theta, \mathbf{Spin}(10))$, *or* $(\sigma^{\pm}(12) \oplus \theta, \mathbf{Spin}(12))$.

(3) *If W is an irreducible non-plain representation with $w \leqslant 13$ (and $G_1 \not\cong \mathbf{SL}(4)$) then $W = (\sigma(7), \mathbf{Spin}(7))$ or $W = (\mathbf{S}^2\mathbf{C}^4, \mathbf{SP}(4))$.*

Proof. A list of all the (nontrivial) irreducible representations (W, H) with $w := \dim W \leqslant \dim H + 2$ may be found in [Li1]. Examining this list we arrive at the proof of the lemma. Note that $(\sigma^{\pm}(8), \mathbf{Spin}(8))$ is plain since $(\sigma^{\pm}(8), \mathbf{Spin}(8)) \cong (\mathbf{C}^8, \mathbf{Spin}(8))$ by an outer automorphism of $\mathbf{Spin}(8)$.

The techniques of Section 4 suffice to classify all the equidimensional plain representations (for $G_2 = \mathbf{SL}(n)$). These representations are listed in Tables VI–X.

Representations which contain spinor subrepresentations are especially difficult to classify, principally because classical invariant theory does not apply. Plain representations containing spinor subrepresentations are listed separately from the other plain representations in Tables IX and X.

Most of the non-plain representations $(V, G_1 \times \mathbf{SL}(n))$ contain a subrepresentation W of the form $W \cong W_1 \otimes \mathbf{C}^n$ with $\dim W = n+1$. (Thus (W_1, G_1) is the castling transform of W.) Tables XIV–XVI list all the non-plain equidimensional representations having such a subrepresentation W.

We need the following lemmas to handle the non-plain representations.

LEMMA 5.0.2. *Suppose that* $(V, G) = (W \otimes \mathbf{C}^n \oplus \rho(G_1), G_1 \times \mathbf{SL}(n))$ *is equidimensional where* W *is reducible and* G_1 *is semi-simple. Set* $w := \dim W$.

(1) *If* $w \leqslant n$ *then* (V, G) *is equidimensional (resp. coregular) if and only if* $(\rho(G_1), G_1)$ *is equidimensional (resp. coregular).*

(2) *If* $w \geqslant 2n$ *and* $n \geqslant 7$ *then* W *is plain.*

(3) *If* $w \geqslant 2n$ *and* $3 \leqslant n \leqslant 6$ *then either* W *is plain or* (W, G_1) *is one of* $(\sigma^{\pm}(12) \oplus \theta, \mathbf{Spin}(12))$, $(\sigma^{\pm}(10) \oplus \theta, \mathbf{Spin}(10))$, $(\sigma(9) \oplus \theta, \mathbf{Spin}(9))$ *or* $(\sigma(7) \oplus \theta, \mathbf{Spin}(7))$.

(4) *If* $w \geqslant 2n$, $n = 2$ *and* W *is not plain then either* $(W, G_1) = (\mathbf{C}^8 \oplus \sigma^{\pm}(8), \mathbf{Spin}(8))$, $(W, G_1) = (\mathbf{C}^6 \oplus \sigma^{\pm}(6), \mathbf{Spin}(6))$ *or* $W = W_1 \oplus \theta$, *where* W_1 *is irreducible. Furthermore, in this case* W_1 *is one of the following:* $(\varphi_3(6), \mathbf{SP}(6))$, $(\sigma^{\pm}(12), \mathbf{Spin}(12))$, $(\sigma(11), \mathbf{Spin}(11))$, $(\sigma^{\pm}(10), \mathbf{Spin}(10))$, $(\sigma(9), \mathbf{Spin}(9))$, $(\sigma(7), \mathbf{Spin}(7))$ *or* $(S^2\sigma^{\pm}(6), \mathbf{Spin}(6)) \cong (S^2\mathbf{C}^{4(*)}, \mathbf{SL}(4))$.

(5) *If* $n + 1 < w < 2n$ *and* W *is not plain then* $w \leqslant n + 6$ *and* $W = W_1 \oplus \theta$, *where* W_1 *is one of the irreducible representations listed in* (4) *above.*

(6) *If* $w = n + 1$ *then* $\dim W//G_1 \leqslant 2$. *Moreover,* $\dim V//G = \dim(W \oplus \rho(G_1))//G_1 \leqslant \dim \rho(G_1)//G_1 + 2 \leqslant 4$.

Proof. (1) This is just part 1 of Proposition 4.8.1.

Recall that by Eq. (8), $\operatorname{codim} Z_{\mathbf{SL}(n)}((k+n)\mathbf{C}^n) = k + 1$. Hence $\operatorname{codim} Z_G(V) \leqslant \operatorname{codim} Z_{\mathbf{SL}(n)}(V) = w - n + 1$. Now $\dim V//G \geqslant \dim V - \dim G = wn - n^2 + 1 - \dim G_1$. Since V is equidimensional this implies that $(w-n)(n-1) \leqslant \dim G_1$.

(2) If $n \geqslant 7$ and $2n \leqslant w$, this gives $3w \leqslant \dim G_1$ and so W must be plain.

(3) Similarly if $3 \leqslant n \leqslant 6$ then using the above inequality and Lemma 5.0.1 (and the tables in [Li1] for the case $n = 3$) we find that either W is plain or (W, G_1) is one of the representations listed.

(4) From the tables of [Li1] we see that if $n = 2$ then each irreducible subrepresentation of W is one of the representations listed in the statement.

Recalling that W is reducible and examining all the possible pairs we find that if W is not plain then $W = W_1 \oplus \theta$, where W_1 is one of the irreducible representations listed above or $(W, H) = (\mathbf{C}^8 \oplus \sigma^{\pm}(8), \mathbf{Spin}(8))$ or $(W, H) = (\mathbf{C}^6 \oplus \sigma^{\pm}(6), \mathbf{Spin}(6))$.

(5) Suppose that $(W \otimes \mathbf{C}^n \oplus \rho(G_1), G_1 \times \mathbf{SL}(n))$ is equidimensional and $w \leqslant 2n$. Then the castling transform $(V', G') = (W^* \otimes \mathbf{C}^{n'} \oplus \rho(G_1), G_1 \times \mathbf{SL}(n'))$ is also equidimensional and since $n' := w - n < n$ we have

$2n' < n' + n = w$. Hence (V', G') satisfies the hypothesis for one of the cases (2), (3), or (4) above. From this (5) follows.

(6) Here the castling transform $(V', G') \cong (W^* \oplus \rho(G_1), G_1)$. Hence by Proposition 4.8.1, (V, G) is equidimensional if and only if both (V', G') is cofree and $\dim V//G = \dim V'//G_1 \leqslant \dim \rho(G_1)//G_1 + 2$. In particular, $\dim W^*//G_1 \leqslant 2$.

LEMMA 5.0.3. *Suppose that* $(V, G) = (W \otimes \mathbf{C}^n \oplus \bigwedge^2 \mathbf{C}^n, G_1 \times \mathbf{SL}(n))$ *is equidimensional where n is odd. Set* $w := \dim W$.

(1) *If* $9 \leqslant w \leqslant n+1$ *then W is plain.*

(2) *If* $w \leqslant n+1$ *and* $w \leqslant 8$ *then either W is plain or* $(W, G_1) = (\sigma(7), \mathbf{Spin}(7))$.

Proof. Consider $(V', G') = (W \otimes \mathbf{C}^n \oplus \hat{\mathbf{C}}^{n-1} \otimes \mathbf{C}^n, G_1 \times \mathbf{SL}(n) \times \widehat{\mathbf{SP}}(n-1))$. Since $V'//(\mathbf{SP}(n-1)) \cong V$ we have $\dim V//G = \dim V'//G'$. Castling (V', G') we obtain the representation $(V'', G'') = (W \otimes \mathbf{C}^{w-1} \oplus \hat{\mathbf{C}}^{n-1} \otimes \mathbf{C}^{w-1}, G_1 \times \mathbf{SL}(w-1) \times \widehat{\mathbf{SP}}(n-1))$. Set $W' := V''//(\widehat{\mathbf{SP}}(n-1))$ and $K = G_1 \times \mathbf{SL}(w-1)$. Then $W'//K \cong V//G$. But $\dim W'//K \geqslant \dim W' - \dim K = \frac{1}{2}(w-1)(w-2) + w - \dim G_1$. Combining this with the fact that $\operatorname{codim} Z_G(V) \leqslant \operatorname{codim} Z_{\mathbf{SL}(n)}(V) = w$ we obtain $\frac{1}{2}w(w-3) + 1 \leqslant \dim G_1$. If $w \geqslant 9$ this gives $3w < \dim G_1$ which implies that W is plain. The proof of the lemma then follows from this by Lemma 5.0.1.

LEMMA 5.0.4. (1) *Let* $(V, G) = (W \otimes \mathbf{C}^n \oplus \mathbf{S}^2\mathbf{C}^{n(*)}, G_1 \times \mathbf{SL}(n))$, *where* $n \geqslant 5$. *If (V, G) is equidimensional then W is plain.*

(2) *Let* $(V, G) = (W \otimes \mathbf{C}^n \oplus \bigwedge^2 \mathbf{C}^{n(*)})$, *where n is even and* $n \geqslant 4$. *If (V, G) is equidimensional then either W is plain or* $W = (\sigma(7), \mathbf{Spin}(7))$.

(3) *Let* $(V, G) = (W \otimes \mathbf{C}^n \oplus \bigwedge^2 \mathbf{C}^{n*})$, *where n is odd and* $\dim W \leqslant n$. *If (V, G) is equidimensional then either W is plain or* $W = (\sigma(7), \mathbf{Spin}(7))$.

Proof. (1) and (2) follow immediately by taking the slice representation of the principal isotropy group of $\mathbf{S}^2\mathbf{C}^{n(*)}$ or $\bigwedge^2 \mathbf{C}^{n(*)}$ and comparing with Tables I–IV.

For (3) we observe that $V//\mathbf{SL}(n) \cong \bigwedge^2 (W) \oplus \theta$. Now (V, G) is equidimensional implies that $(\bigwedge^2 (W), G_1)$ is equidimensional (Lemma 4.6.1). Hence we see by examining the tables in [Sch2] that either W is plain or $W = \sigma(7)$.

LEMMA 5.0.5. *Suppose that* $(V, G) = (W \otimes \mathbf{C}^n \oplus sl(n), G_1 \times \mathbf{SL}(n))$ *is equidimensional.*

(1) *If* $n \geqslant 8$ *then* W *is plain.*

(2) *If* $n \geqslant 5$ *then either* W *is plain or* $W = \sigma^{\pm}(10) \oplus \theta$ *or* $W = \sigma^{\pm}(9)$ *or* $W = \sigma(7) \oplus \theta$.

Proof. The slice representation at a principal isotropy group of $sl(n)$ is $(V', G') \oplus \theta_{n-1} = (W \otimes \mathbf{C}^n, G_1 \times (\mathbf{C}^*)^{n-1}) \oplus \theta_{n-1}$. Here we have $\dim V'//G' \geqslant wn - n + 1 - \dim G_1$. Also $\operatorname{codim} Z_{G'} V' \leqslant \frac{1}{2}(\dim V')$. Hence $n(w/2 - 1) + 1 \leqslant \dim G_1$. If $n \geqslant 8$ then we have $3w + (w - 7) \leqslant \dim G_1$ and thus either $3w \leqslant \dim G_1$ or $w < 7$. Hence W is plain.

(2) If $n \geqslant 5$ the above inequality gives $\frac{5}{2}w - 4 \leqslant \dim G_1$. Hence either $w \leqslant 8$ or $2w < \dim G_1$.

LEMMA 5.0.6. *Let* $(V, G) = (W_1 \otimes \mathbf{C}^n \oplus W_1 \otimes \mathbf{C}^{n^*}, G_1 \times \mathbf{SL}(n))$, *where* $w_1 := \dim W_1 \leqslant n$ *and* $w_2 := \dim W_2 < n$. *If* (V, G) *is equidimensional then* $W_1 \cong \theta$ *and* W_2 *is plain or* W_1 *is plain and* $W_2 \cong \theta$ *or* $W_1 = W_2 = (\mathbf{C}^4, \mathbf{SP}(4))$.

Proof. Since $(V, \mathbf{SL}(n))$ is coregular, $(V//\mathbf{SL}(n), G_1) \cong (W_1 \otimes W_2 \oplus \theta, G_2)$ is equidimensional. From this the lemma follows.

Remark 5.0.7. If $W_1, W_2 \cong (\sigma^{\pm}(6) \oplus \theta, \mathbf{Spin}(6))$ then V is considered as a representation of $\mathbf{SL}(4) \times \mathbf{SL}(n)$ and so not treated here.

Finally we have the case $V = W \otimes \mathbf{C}^n \oplus b\mathbf{C}^{n^*}$ for $b \geqslant 1$.

LEMMA 5.0.8. *Suppose that* $(V, G) = (W \otimes \mathbf{C}^n \oplus b\mathbf{C}^{n^*}, G_1 \times \mathbf{SL}(n))$ *is equidimensional where* $b \geqslant 2$ *and* $n \geqslant \dim W$.

(1) *Then either* W *is plain or* $b \leqslant 3$.

(2) *If* W *is not plain and* $b = 3$ *then* $(W, G_1) = (\sigma(7) \oplus \theta, \mathbf{Spin}(7))$.

(3) *If* W *is not plain and* $b = 2$ *then* $(W, G_1) = (\sigma(2r+1) \oplus \theta, \mathbf{Spin}(2r+1))$ *with* $r \leqslant 5$, $(W, G_1) = (\sigma^{\pm}(2r) \oplus \theta, \mathbf{Spin}(2r))$ *with* $r \leqslant 6$ *or* $(W, G_1) = (\varphi_2(m) \oplus \theta, \mathbf{SP}(m))$ *with* $m \leqslant 6$.

Proof. If we quotient V by $\mathbf{SL}(n)$ we find that (bW, G_1) must be cofree. Examining the tables of [Sch2] we see the only possibilities are the ones listed in the lemma.

Lemmas 5.0.2–5.0.8 give some restrictions on those equidimensional representations containing $(W \otimes \mathbf{C}^n, G_1 \times \mathbf{SL}(n))$. Given these restrictions we may use the methods of Section 4 to classify the equidimensional non-plain representations of this form. Representations of this form are listed in Tables XI–XVI with those having such a W of dimension $n-1$ appearing in Tables XIV–XVI.

Representations having a bi-simple subrepresentation of the form $(W_1 \otimes W_2, G_1 \times \mathbf{SL}(n))$ with W_1 plain and W_2 non-plain are listed in Table XVII. Finally those representations containing a bi-simple subrepresentation of the form $W_1 \otimes W_2$, where both W_1 and W_2 are non-plain are listed in Table XVIII. Note that this only happens for $n = 2$ and that both of the equidimensional representations containing such a $W_1 \otimes W_2$ are irreducible.

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