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Equidimensional Varieties and Associated Cones*

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INTRODUCTION

Let G be a complex reductive algebraic group and V a finite dimensional complex representation of G. We denote by $\mathcal{O}(V)$ the ring of regular (=polynomial) functions on V and by $\mathcal{O}(V)^G$ the subring of G-invariant functions. Hilbert's famous theorem states that $\mathcal{O}(V)^G$ is a finitely generated $\mathbf{C}$-algebra (see [Kr, II.3.2] or [MF]). The affine variety $V/\!\!/G$ with coordinate ring $\mathcal{O}(V)^G$ is called the *quotient* of V by G, and the morphism $\pi_V: V \to V/\!\!/G$ dual to the inclusion $\mathcal{O}(V)^G \subsetneq \mathcal{O}(V)$ the *quotient map*. Similar notations are used for an arbitrary affine G-variety X and its quotient $\pi_X: X \to X/\!\!/G$. The quotient map has a number of important properties: It is surjective, separates disjoint G-stable closed subvarieties, and is universal with respect to morphisms which are constant on G-orbits (loc. cit.).

An important class of representations are the *cofree* representations V of G. This means that π_V is flat, or equivalently, that $\mathcal{O}(V)$ is a free $\mathcal{O}(V)^G$-module. These representations share many nice geometric properties with the adjoint representation of G (cf. [Ko]). In particular, they are *coregular* (i.e., $\mathcal{O}(V)^G$ is a polynomial ring or equivalently, $V/\!\!/G$ is smooth), *equidimensional* (i.e., all fibers of π_V have the same dimension, namely, $\dim V - \dim V/\!\!/G$), and the quotient map $\pi_V: V \to V/\!\!/G$ is an *open morphism*. Conversely, one can show that a coregular and equidimensional representation V is cofree (see [Sch1, Sch2]).

It has been conjectured by V. L. Popov that *every equidimensional representation of a connected reductive group G is cofree* [Po]. This is true for simple groups G and for tori due to results of Schwarz and Wehlau (cf. [Sch2, We]), and is also proved for irreducible representations of

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semisimple groups by Littelmann [Li]. Our results in this paper give further indication that the Popov conjecture might be true in general.

For any representation V of a reductive group G define the *zero fiber* by $V^0 := \pi_V^{-1}(\pi(0))$. It is clear already from the work of Hilbert that the geometry of V^0 is strongly related to the structure of the quotient $V/\!/G$ and the quotient map π_V (see [Kr]). For example, it is easy to see that V is equidimensional if and only if $\dim V^0 = \dim V - \dim V/\!/G$. In order to compare a general fiber $F = \pi_V^{-1}(z)$ with the zero fiber, we define the *associated cone* of F by

$$\mathscr{C}(F) := \overline{\mathbf{C}^*F} \setminus \mathbf{C}^*F,$$

where $\bar{X}$ denotes the *Zariski-closure* of a subset $X \subset V$ and $\mathbf{C}^*$ acts via scalar multiplication on V. (This construction is introduced and discussed in [BK], cf. [Kr, II.4.2].)

If V is cofree then for all $z \in V/\!/G$, $z \neq \pi_V(0)$ the schemes V^0 and $\mathscr{C}(\pi_V^{-1}(z))$ are equal. In fact, the existence of a principal z with $V^0 = \mathscr{C}(\pi_V^{-1}(z))$ as schemes implies that V is cofree. For an equidimensional representation we have the following theorem.

THEOREM. *Let V be an equidimensional representation. Then for all $z \in V/\!/G$, $z \neq \pi(0)$ the sets $\mathscr{C}(\pi^{-1}(z))$ and V^0 coincide.*

Let $V = V_1 \oplus V_2$ be a direct sum of representations. If V is coregular or cofree, then so are V_1 and V_2 [Sch1, Sch2]. A similar result holds for equidimensional representations.

PROPOSITION (Vinberg, Wehlau). *If the direct sum $V_1 \oplus V_2$ is equidimensional then V_1 and V_2 are both equidimensional, too.*

Recently, I learned that Vinberg has also proven the above theorem. In fact, his proof works in a slightly more general situation.

1. ASSOCIATED CONES

Let X be an irreducible affine variety.

1.1. DEFINITION. X is called *conical* if the coordinate ring of X is equipped with a positive graduation $\mathcal{O}(X) = \bigoplus_{i=0}^{\infty} \mathcal{O}(X)_i$ such that $\mathcal{O}(X)_0 = \mathbf{C}$. Equivalently, there is a $\mathbf{C}^*$-action on X with a unique fixed point x_0 which is attractive, i.e., $\lim_{t \to 0} t \cdot x = x_0$ for all $x \in X$. The elements of $\mathcal{O}(X)_i$ are called *homogeneous of degree i.*

Let X be conical. If $f \in \mathcal{O}(X)$ we can write $f = f_0 + f_1 + \cdots + f_q$, where f_i is homogeneous of degree i and $f_q \neq 0$. We define $\mathrm{gr}(f)$ to be the homogeneous function f_q. If $\mathscr{I}$ is any ideal of $\mathcal{O}(X)$ then $\mathrm{gr}(\mathscr{I})$ denotes the ideal generated by the set $\{\mathrm{gr}(f) \mid f \in \mathscr{I}\}$. Hence

$$\mathrm{gr}(\mathscr{I}) = \bigoplus_{q=0}^{\infty} \{\mathrm{gr}(f) \mid f \in \mathscr{I} \quad \text{and} \quad \deg(f) = q\}.$$

(By convention, the zero element 0 has every possible degree and $\mathrm{gr}\, 0 = 0$.)

1.2. Definition (see [BK]). Let $S \subset X$ be any subset and $\mathscr{I}(S) \subseteq \mathcal{O}(X)$ the ideal of functions vanishing on S. We define the *associated cone* of S to be the zero set of the ideal $\mathrm{gr}\, \mathscr{I}(S)$:

$$\mathscr{C}(S) := \{x \in X \mid (\mathrm{gr}\, f)(x) = 0 \text{ for all } f \in \mathscr{I}(S)\}.$$

We will also be interested in the structure of $\mathscr{C}(S)$ as a scheme. If we begin with a (possibly radical) ideal $\mathscr{I}$ the homogeneous ideal $\mathrm{gr}\, \mathscr{I}$ will not in general be radical. Hence in general the scheme $\mathscr{C}(S)$ is non-reduced. If we have a scheme S, the notation $\mathcal{O}(S)$ denotes the schematic ring of regular functions. In particular, if $\mathscr{I}$ is the ideal defining the scheme $S \subset X$ then $\mathcal{O}(\mathscr{C}(S)) = \mathcal{O}(X)/\mathrm{gr}\, \mathscr{I}$.

This construction has a number of important properties. We refer to [BK, Sect. 3] for details (cf. [Kr, I.6 and II.4.2]).

1.3. Proposition. *Let S, $T \subset X$ be closed subvarieties.*

(a) $\dim \mathscr{C}(S) = \dim S$.

(b) *If S is irreducible then $\mathscr{C}(S)$ has pure dimension.*

(c) $\mathscr{C}(S \cup T) = \mathscr{C}(S) \cup \mathscr{C}(T)$.

(d) $m_\lambda(\mathcal{O}(S)) = m_\lambda(\mathcal{O}(\mathscr{C}(S)))$ *where $m_\lambda(Y)$ is the multiplicity of the irreducible representations of type λ in the G-module Y.*

(A variety has *pure dimension* if all irreducible components have the same dimension.)

The following lemma shows how the operation gr behaves with respect to radical ideals. (The proof is easy and left to the reader.)

1.4. Lemma. *If $\sqrt{\mathscr{I}} = \sqrt{\mathscr{T}}$ then $\sqrt{\mathrm{gr}(\mathscr{I})} = \sqrt{\mathrm{gr}(\mathscr{T})}$.*

Recall that a variety X is called *Cohen–Macaulay* if the coordinate ring $\mathcal{O}(X)$ is Cohen–Macaulay. We refer to [Ma, Chap. 5 and 6] for the notion of a *system of parameters*, a *regular sequence*, and a *Cohen–Macaulay ring* and its properties (see also [St] and [HR]).

1.5. PROPOSITION. *Assume that X is Cohen–Macaulay, and let $f_1, ..., f_d$ be a (partial) regular sequence of homogeneous elements of $\mathcal{O}(X)$. Define $\rho = (f_1, ..., f_d): X \to \mathbf{C}^d$, and let $a = (a_1, ..., a_d)$ be any point in $\mathbf{C}^d$. Then the schemes $\mathscr{C}(\rho^{-1}(a))$ and $\rho^{-1}(0)$ coincide.*

Proof. Set $F := \rho^{-1}(a)$. By definition, $\mathscr{I}(F) = (f_1 - a_1, ..., f_d - a_d)$. Since $f_i = \operatorname{gr}(f_i - a_i)$, we have $\mathscr{I}(\rho^{-1}(0)) = (f_1, ..., f_d) \subseteq \operatorname{gr} \mathscr{I}(F)$.

For the opposite inclusion observe that $\mathcal{O}(X)$ is a free $\mathbf{C}[f_1, ..., f_d]$-module (since the f_i form a regular sequence),

$$O(X) = \bigoplus_{\beta \in B} \mathbf{C}[f_1, ..., f_d]\, s_\beta,$$

where the s_β are homogeneous [St, Prop. 3.1]. Consider $h = \sum_{i=1}^d g_i(f_i - a_i) \in \mathscr{I}(F)$. Then $h = \sum_{i=1}^d \sum_{\beta \in B} k_{i,\beta} s_\beta (f_i - a_i)$, where $k_{i,\beta} \in \mathbf{C}[f_1, ..., f_d]$. Setting $h_\beta := \sum_{i=1}^d k_{i,\beta}(f_i - a_i)$ we have $h = \sum_{\beta \in B} h_\beta s_\beta$, where the h_β are in the ideal of $\mathbf{C}[f_1, ..., f_d]$ generated by $f_1 - a_1, ..., f_d - a_d$. Now $\operatorname{gr}(h) = \sum_{\beta \in B'} \operatorname{gr}(h_\beta) s_\beta$ for some subset $B' \subseteq B$. But $\operatorname{gr}(h_\beta)$ is a non-constant homogeneous element of $\mathbf{C}[f_1, ..., f_d]$, and hence lies in the unique homogeneous maximal ideal $(f_1, ..., f_d) \cdot \mathbf{C}[f_1, ..., f_d]$. Therefore $\operatorname{gr}(h) \in (f_1, ..., f_d) \cdot \mathcal{O}(X) \subseteq \mathscr{I}(\rho^{-1}(0))$. ∎

2. CONICAL QUOTIENTS

Let G be a connected reductive group. We assume that G acts (algebraically) on the irreducible conical variety X, compatibly with the $\mathbf{C}^*$-action. Then the quotient $X/\!\!/G$ is also conical, and the quotient map $\pi_X: X \to X/\!\!/G$ is $\mathbf{C}^*$-equivariant. In particular, the zero fiber $X^0 := \pi_X^{-1}(\pi_X(x_0))$ is conical, too. We first give a geometric description of the associated cone of a subset S of a general fiber of π_X (see [BK, 3.4 Satz]).

2.1. LEMMA. *Let $S \subset X$ be a closed subvariety contained in a fiber $\pi_X^{-1}(y)$ different from the zero fiber X^0. Then*

$$\mathscr{C}(S) = \overline{\mathbf{C}^* S} \setminus \mathbf{C}^* S = \overline{\mathbf{C}^* S} \cap X^0.$$

In particular, $\mathscr{C}(S) \subseteq X^0$, and $\mathscr{C}(S)$ is G-stable in case S is G-stable.

2.2. Recall that the quotient $\pi_X: X \to X/\!\!/G$ is *equidimensional* if all components of all fibers of π_X have the same dimension. It follows from the lemma above that this holds if and only if

$$\dim X^0 = \dim X - \dim X/\!\!/G.$$

2.3. The quotient $X/\!/G$ contains an open dense subset U whose elements are called *principal points.* A fibre $\pi_X^{-1}(z)$ for $z \in U$ is called a *principal fibre.* If F_1 and F_2 are any two principal fibres of X then the G-modules $\mathcal{O}(F_1)$ and $\mathcal{O}(F_2)$ are isomorphic (see [Lu, III.4]).

2.4. PROPOSITION. *The G-variety X is cofree if and only if there exists a principal point $z \in X/\!/G$, $z \neq \pi_X(x_0)$, such that the schemes $\mathscr{C}(\pi_X^{-1}(z))$ and X^0 are equal.*

Proof. If X is cofree then there exists a regular sequence $f_1, ..., f_d$ in $\mathcal{O}(X)$ such that $\mathcal{O}(X)^G$ is the polynomial ring $\mathbf{C}[f_1, ..., f_d]$. Then $\pi_X = (f_1, ..., f_d)$ and applying Proposition 1.5 we see that the schemes X^0 and $\mathscr{C}(\pi_X^{-1}(z))$ coincide for all z.

To prove the opposite implication let $K := \pi_X^{-1}(z)$ be a principal fibre different from X^0 such that $\mathrm{gr}(\mathscr{I}(K)) = \mathscr{I}(X^0)$. Let $S \subset \mathcal{O}(X)$ denote a G-stable graded complement to $\mathscr{I}(X^0)$. Consider the surjection $\mu\colon \mathcal{O}(X)^G \otimes S \to \mathcal{O}(X)$. We will prove that X is cofree by showing that μ is an isomorphism.

Let F be an arbitrary principal fibre and consider the restriction map $\psi_F\colon S \to \mathcal{O}(F)$. Since S generates $\mathcal{O}(X)$ as an $\mathcal{O}(X)^G$-module, we have that ψ_F is a surjection. Now let $W_\lambda \subseteq \ker \psi_F$ be an irreducible G-representation of type λ. The multiplicity of W_λ in S, $m_\lambda(S)$ is finite ([Kr, II 3.2 Zusatz]). But $m_\lambda(S) = m_\lambda(\mathcal{O}(X^0)) = m_\lambda(\mathcal{O}(\mathscr{C}(K))) = m_\lambda(\mathcal{O}(K)) = m_\lambda(\mathcal{O}(F))$, which contradicts the fact that $m_\lambda(\ker \psi_F) \geqslant 1$. Hence ψ_F is injective and so ψ_F is an isomorphism.

Now let $h = \sum_i f_i \otimes s_i \in \ker \mu$ where each $f_i \in \mathcal{O}(X)^G$ and where we assume the union of the s_i is a linearly independent set. Then $\sum_i f_i s_i \in \mathscr{I}(X)$. In particular, $\sum_i f_i(F)\, s_i \in \mathscr{I}(F)$. Hence $\sum_i f_i(F)\, s_i \in \ker \psi_F = \{0\}$. Since the s_i are linearly independent, this means that $f_i(F) = 0\ \forall i$. Since F was an arbitrary principal fibre and since the union of the principal fibres is dense in X, we have $f_i \in \mathscr{I}(X)^G\ \forall i$ and thus $h \in \mathscr{I}(X)^G \otimes S$. Hence μ is injective and so μ is an isomorphism. ∎

2.5. THEOREM. *Let X be a normal conical equidimensional G-variety. Then the sets $\mathscr{C}(\pi_X^{-1}(z))$ and X^0 are equal for all $z \in X/\!/G$, $z \neq \pi_X(x_0)$.*

Proof. The quotient variety, $X/\!/G$ is normal since X is normal [Kr, II 3.3 Satz 1]). Hence, π_X is an open morphism since it is equidimensional and dominant ([Bo, AG 18.4]). Therefore, the set $U := \pi_X(X \setminus \overline{\mathbf{C}^* \pi_X^{-1}(z)})$ is open in $X/\!/G$. Clearly $z \notin U$ and therefore U does not meet $\overline{\mathbf{C}^* z} = \overline{\pi_X(\mathbf{C}^* \pi_X^{-1}(z))}$. Hence, $\pi_X(0) \notin U$, because $\pi_X(0) \in \overline{\mathbf{C}^* z}$, and so $X^0 \subset \overline{\mathbf{C}^* \pi_X^{-1}(z)}$. ∎

After I proved the following result, I learned that Vinberg had also proven it. His proof works without the assumptions that X and Y be conical and normal.

2.6. Proposition. *Let X be an irreducible normal conical G-variety and $Y \subset X$ a closed irreducible normal conical G-stable subvariety. Assume that there exists a G-equivariant retraction $\rho: X \to Y$, $\rho|_Y = \mathrm{id}_Y$. If $\pi_X: X \to X/\!/G$ is equidimensional, then so is $\pi_Y: Y \to Y/\!/G$.*

Proof. Let $\bar{\rho}: X/\!/G \twoheadrightarrow Y/\!/G$ denote the map induced by ρ. For every $z \in Y/\!/G \subseteq X/\!/G$ we have $\pi_Y^{-1}(z) \subseteq \pi_X^{-1}(z) \overset{\rho}{\twoheadrightarrow} \pi_Y^{-1}(z)$, where the composition is the identity. Hence $\mathscr{C}(\pi_Y^{-1}(z)) \subseteq \mathscr{C}(\pi_X^{-1}(z)) \overset{\rho}{\twoheadrightarrow} \mathscr{C}(\pi_Y^{-1}(z))$. Thus $Y^0 = \rho(X^0) = \rho(\mathscr{C}(\pi_X(z)) \subseteq \mathscr{C}(\pi_Y^{-1}(\bar{\rho}(z)))$, which shows that $\dim Y^0 \leqslant \dim Y - \dim Y/\!/G$. Therefore by Proposition 2.2 π_Y is equidimensional. ∎

2.7. The proposition applies in particular when the equidimensional G-variety X is a product $Y_1 \times Y_2$ of two G-varieties. Thus we obtain a proof of the proposition of the Introduction.

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