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Partitioning Quadrics, Symmetric Group Divisible Designs and Caps *

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Dedicated to Hanfried Lenz on the occasion of his 80th birthday.

Abstract. Using partitionings of quadrics we give a geometric construction of certain symmetric group divisible designs. It is shown that some of them at least are self-dual. The designs that we construct here relate to interesting work — some of it very recent — by D. Jungnickel and by E. Moorhouse. In this paper we also give a short proof of an old result of G. Pellegrino concerning the maximum size of a cap in $AG(4, 3)$ and its structure. Semi-biplanes make their appearance as part of our construction in the three dimensional case.

Keywords: Quadrics, Symmetric group divisible designs, Caps, Semi-biplanes, Ovoids

For $n \geq 2$, we denote by $PG(n, q)$ the finite projective space of dimension n over $F := GF(q)$, the field of order q. Similarly $AG(n, q)$ denotes the affine space of dimension n over F. A subset of $PG(n, q)$ or $AG(n, q)$ is a *cap* if no three of its points are collinear. For $1 \leq k \leq n$, a subset $K \subseteq PG(n, q)$ is a *k-flat* if K is isomorphic to $PG(k, q)$. A *line* is a 1-flat, a *plane* is a 2-flat and a *solid* is a 3-flat. The complement in $PG(n, q)$ of an $(n-1)$-flat is isomorphic to $AG(n, q)$. We will denote by $\ell(x, y)$ the unique line containing both x and y.

We begin by recalling the structure of maximal caps in low dimensions. Proofs of these results may be found in [6] and [7]. A conic Ω in $PG(2, q)$ consists of $q + 1$ points no three of which are collinear. A line of $PG(2, q)$ meets a conic in 0, 1 or 2 points. The lines which meet Ω in 2 points are called *secant* lines; those that meet Ω in a single point are called *tangents* and the lines which miss Ω are called *exterior* lines. For each point $p \in \Omega$ there exists a unique line tangent to Ω at p. There are $(q^2 + q)/2$ secant lines to Ω and $(q^2 - q)/2$ exterior lines. If q is odd then every maximum cap in $PG(2, q)$ is a conic. If q is even then the $q + 1$ tangents to Ω are all concurrent. Their common point is called the *nucleus* of Ω. Every line through the nucleus, N, is a tangent line to Ω. For q even every maximum cap contains $q + 2$ points. As above, one way to realize such a maximum cap is to take a conic together with its nucleus.

A maximum cap in $PG(3, q)$ with $q \neq 2$ consists of $q^2 + 1$ points. (A maximum cap in $PG(3, 2)$ contains 8 points.) Such a set of $q^2 + 1$ points is called an *ovoid*. If q is odd or

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$q = 4$ then every ovoid is an elliptic quadric. Conversely an elliptic quadric is an ovoid for all q. If $\mathcal{E} \subset PG(3,q)$ is an ovoid then at each point of $\mathcal{E}$ there exists a unique tangent plane. Other than these q^2+1 tangent planes, each of the remaining $q(q^2+1)$ planes in $PG(3,q)$ meets $\mathcal{E}$ in $q+1$ points which form a cap. If $\mathcal{E}$ is an ovoid and $p \notin \mathcal{E}$ then there are exactly $q+1$ tangent planes to $\mathcal{E}$ which pass through p. If $\mathcal{E}$ is any ovoid in $PG(3,q)$ and ℓ is a line containing no points of $\mathcal{E}$ then ℓ is in exactly 2 tangent planes of $\mathcal{E}$; each of the remaining $q-1$ planes through ℓ meets $\mathcal{E}$ in $q+1$ points.

To construct a maximum cap in $AG(3,q)$ where $q > 2$ we begin with an ovoid $\mathcal{E} \subset PG(3,q)$. Choose any point $z \in \mathcal{E}$ and let Π_z be the unique tangent plane to $\mathcal{E}$ at z. Then $\mathcal{E} \setminus \{z\}$ is a maximum cap in $AG(3,q)$. We call such a cap a *punctured ovoid* with *puncture point* z. If $\mathcal{E}$ is an elliptic quadric we also use the term *punctured elliptic quadric* for $\mathcal{E} \setminus \{z\}$.

1. The Three Dimensional Case

1.1. The Construction

Let q be a prime power and put $\Pi_\infty := PG(2,q)$. Choose a conic $\Omega \subset \Pi_\infty$. Embed $\Pi_\infty \subset PG(3,q)$ and choose $V \in PG(3,q) \setminus \Pi_\infty$. Then we define the following two sets:

$$C = C(V,\Omega) := \bigcup_{z\in\Omega} \ell(V,z) \qquad \text{and} \qquad \hat{C} := C \setminus (\Omega \cup \{V\}) \ .$$

Since $|\Omega| = q+1$, we have $|C| = q(q+1)+1 = q^2+q+1$ and $|\hat{C}| = q^2-1$.

We define two structures $\mathcal{P}$ (resp. $\hat{\mathcal{P}}$) having points and blocks as follows. The points of $\mathcal{P}$ (resp. $\hat{\mathcal{P}}$) are just the points of C (resp. $\hat{C}$). For $z \in \Omega$ we denote by ℓ_z the unique tangent line at z to Ω in Π_∞. Now there are $q+1$ planes $\Pi_\infty, \Pi_{z,0}, \ldots, \Pi_{z,q-1}$ through ℓ_z. Reordering these planes we may arrange that $V \in \Pi_{z,0}$.

The blocks of $\mathcal{P}$ are defined to be the intersections $B_{z,i} := \Pi_{z,i} \cap C$ for $z \in \Omega$ and $i = 0\ldots,q-1$. In addition, $\Omega = \Pi_\infty \cap C$ is also declared to be a block of $\mathcal{P}$. Thus $\mathcal{P}$ has $q(q+1)+1 = q^2+q+1 = |\mathcal{P}|$ blocks. We note that, by definition, the lines $\ell(V,z) = \Pi_{z,0} \cap C$ for $z \in \Omega$ are each blocks of $\mathcal{P}$.

The blocks of $\hat{\mathcal{P}}$ are defined to be the intersections $\hat{B}_{z,i} := \Pi_{z,i} \cap \hat{C}$ for $z \in \Omega$ and $i = 1\ldots,q-1$. It follows that, unlike $\mathcal{P}$, none of the blocks of $\hat{\mathcal{P}}$ are the intersections of $\hat{\mathcal{P}}$ with lines of $PG(3,q)$. Also $\hat{\mathcal{P}}$ has q^2-1 blocks as well as q^2-1 points.

THEOREM 1 *If q is even then $\mathcal{P}$ is the Desarguesian projective plane.*

Proof: If $V \notin \Pi_{z,i}$ then $B_{z,i}$ and Ω are in perspective from V, as are their nuclei. Then the nucleus M of $B_{z,i}$ is the point of intersection of the plane $\Pi_{z,i}$ with the line $\ell(V,N)$ where N is the nucleus of Ω. Since ℓ_z is tangent to Ω, $N \in \ell_z$ which in turn is contained in $\Pi_{z,i}$. Thus the line $\ell(V,N)$ and the plane $\Pi_{z,i}$ meet exactly in N and therefore $M = N$. In particular, all of the blocks of $\mathcal{P}$, other than the $q+1$ lines $\ell(V,z)$, are conics which share the same nucleus, N. Note also that all the planes Π_∞ and $\Pi_{z,i}$ contain N. Moreover each one of the q^2+q+1 planes on N is some $\Pi_{z,i}$ or is Π_∞.

We now show that no two points x_1, x_2 of $\mathcal{P}$ are collinear with N. This is clear unless $x_1 \notin \Omega$ and $x_2 \notin \Omega$. In that case let y be the intersection of $\ell(x_1, x_2)$ with Π_∞. The points $x_1' := \ell(V, x_1) \cap \Pi_\infty$, $x_2' := \ell(V, x_2) \cap \Pi_\infty$ and y are three collinear points in Π_∞. Since x_1' and x_2' are in Ω, and Ω has no three of its points collinear, it follows that $y \notin \Omega$. Moreover $y \neq N$, since y is on a secant line to Ω. Hence no two points of $\mathcal{P}$ are collinear with N.

We have now that the blocks of $\mathcal{P}$ are the intersections with C of the planes in $PG(3, q)$ through N. The intersection of any two such planes is a line through N. Since $|C| = q^2 + q + 1$, and no two points of C are collinear with N, it follows that each of the $q^2 + q + 1$ lines through N meets C in exactly one point. Consequently, $\mathcal{P}$ can be thought of as the quotient geometry at N which is $PG(2, q)$. (For a definition of quotient geometry see [2, page 25]). ■

Recall (see [8]) that a semi-symmetric design is a connected incidence structure of points and blocks with constant block size and replication number and such that any pair of points (resp. blocks) lies on exactly 0 or λ blocks (resp. points). When $\lambda = 2$ we get a semi-biplane.

THEOREM 2 *Let q be odd. Then $\hat{\mathcal{P}}$ is not connected but rather has two connected components. Each of these components is a semi-biplane on $(q^2 - 1)/2$ points.*

Outline of proof: We use arguments similar to those in the proof of the previous theorem with the following important difference: for a conic with q odd each point not on the conic lies either on exactly 2 or 0 tangents to that conic. The connectedness properties follow from a coordinate calculation. ■

2. The Four Dimensional Case: A Symmetric Group Divisible Design

Note. In the rest of this paper for the sake of brevity alone we will assume that $q \neq 2$.

We proceed to construct certain symmetric group divisible designs. A symmetric (sometimes called square) group divisible design has an equal number of points and blocks. Recall (see [1]) that a group divisible design is an incidence structure with the following two properties:
(a) All blocks have the same size, k.
(b) The $v = nm$ points are partitioned into m point clases, each of n points (the groups). Any two points from distinct groups are joined by exactly $\mu > 0$ blocks whereas points in the same group are not joined by any block.

These group divisible designs have been of considerable importance in design theory (see [1]). See also the interesting survey paper [10]. Note that in the symmetric or square case there are various consequences. For example, the relation of being disjoint is an equivalence relation on the blocks and any two disjoint blocks meet precisely the same groups. For further details see [9].

Let $\mathcal{E}$ be an ovoid contained in $H_\infty \cong PG(3, q)$. For each $z \in \mathcal{E}$ let Π_z denote the unique plane in H_∞ which is tangent to $\mathcal{E}$ at z. Embed H_∞ into $PG(4, q)$ and choose

$V \in PG(4,q) \backslash H_\infty$. The *cone on* $\mathcal{E}$, is the set of points $C = C(V,\mathcal{E}) = \bigcup_{p \in \mathcal{E}} \ell(V,p)$. Define $S = S(V,\mathcal{E}) := C(V,\mathcal{E}) \setminus (\mathcal{E} \cup \{V\})$. Since $|\mathcal{E}| = q^2+1$, we have $|S| = (q-1)(q^2+1)$. We construct a symmetric group divisible design $\mathcal{D}$ on the points of S, as follows.

The groups of $\mathcal{D}$ are the sets $G_z := \ell(V,z) \setminus \{z, V\}$ for $z \in \mathcal{E}$. Thus each group contains $q-1$ points.

Now for each $z \in \mathcal{E}$, there are $q+1$ solids $H_\infty, H_{z,0}, \ldots, H_{z,q-1}$ in $PG(4,q)$ which contain Π_z. Since the union of these $q+1$ solids is all of $PG(4,q)$, by reordering the $H_{p,i}$ we may assume that $V \in H_{z,0}$. Then $H_{z,0} \cap S = \ell(V,z) \setminus \{V,z\}$ is the group G_z. The sets $B_{z,i} := H_{z,i} \cap S$ for $1 \leq i \leq q-1$ and $z \in \mathcal{E}$ are the blocks of $\mathcal{D}$. Thus there are $(q-1)(q^2+1) = |S|$ blocks of $\mathcal{D}$.

LEMMA 1 *If ℓ is a line in $PG(3,q)$ and ℓ contains more than 2 points of $S \cup \mathcal{E}$ then $\exists z \in \mathcal{E}$ such that $\ell = \ell(z,V)$.*

Proof: Suppose that x_1, x_2, x_3 are three distinct points in $(S \cup \mathcal{E}) \cap \ell$. For $i = 1,2,3$ denote by z_i the point of intersection of $\ell(x_i, V)$ with H_∞. Then z_1, z_2 and z_3 are collinear points of $\mathcal{E}$. Since every line of H_∞ meets $\mathcal{E}$ in at most 2 points, $|\{z_1,z_2,z_3\}| \leq 2$. Without loss of generality, $z_1 = z_2$. Therefore $\ell(V,x_1) = \ell(V,z_1) = \ell(V,z_2) = \ell(V,x_2) = \ell(x_1,x_2) = \ell$. ■

PROPOSITION 1 *1) Each block of $\mathcal{D}$ contains exactly q^2 points of $\mathcal{D}$.*
2) Each point of $\mathcal{D}$ is contained in exactly q^2 different blocks of $\mathcal{D}$.
3) If x_1, x_2 are not contained in any one group then there are exactly $q+1$ blocks of $\mathcal{D}$ which contain both x_1 and x_2.
4) If $z_1 \neq z_2$ then $|B_{z_1,i} \cap B_{z_2,j}| = q+1$ for all i,j.

Proof: 1) Note that $B_{z,i} \cup \{z\}$ is in perspective from V with $\mathcal{E}$. Therefore for each $z \in \mathcal{E}$ and all $1 \leq i \leq q-1$, $B_{z,i}$ is a punctured ovoid with puncture point z and $|B_{z,i}| = q^2$.

2) Fix a point $x \in \mathcal{D}$ and let $z' \in \mathcal{E}$ be the point of intersection of $\ell(V,x)$ with Π_∞. For each $z \in \mathcal{E}$ with $z \neq z'$, there exists a unique solid $H_{z,i(x)}$ which contains Π_z and x. Then $x \in B_{z,i(x)}$. Hence x lies in q^2 different blocks.

3) Suppose $x_1, x_2 \in \mathcal{D}$ do not lie in the same group (i.e., x_1, x_2 and V are not collinear). The line $\ell(x_1,x_2)$ meets H_∞ at some point z'. By Lemma 1, $z' \notin \mathcal{E}$. In H_∞ there are $q+1$ tangent planes to $\mathcal{E}$, $\Pi_{z_1}, \ldots, \Pi_{z_{q+1}}$ through z'. For each z_j, there exists a unique $i = i(j)$ such that $x_1 \in H_{z_j,i}$. Finally $x_2 \in \ell(x_1,z') \subset H_{z_j,i}$. Also by the definition of the blocks it is clear that no other block contains both x_1 and x_2.

4) Note that in H_∞ the two tangent planes Π_{z_1} and Π_{z_2} intersect in a line ℓ which does not intersect $\mathcal{E}$. All of the other planes through ℓ in H_∞ contain $q+1$ points of $\mathcal{E}$. Let Π be the plane of intersection of the two solids $H_{z_1,i}$ and $H_{z_2,j}$. We have that $\Pi \supset \ell$. Project Π to H_∞ from V to get the plane Π'. Since $H_{z_1,i}$ does not contain V, $\Pi' \neq \Pi_{z_1}$. Similarly, $\Pi' \neq \Pi_{z_2}$. Therefore Π' contains exactly $q+1$ points of $\mathcal{E}$. Joining these points to V and intersecting with Π gives that $B_{z_1,i}$ and $B_{z_2,j}$ have exactly $q+1$ points in common. ■

As in [1] a *correlation* of an incidence structure with itself is a bijection θ between the points and the blocks and satisfying $p \in B \iff \theta(B) \in \theta(p)$ for all points p and blocks

B. If θ has order 2 this correlation is called a *polarity*. Note that if θ is a polarity and θ_1 is any other correlation then their product is a collineation (automorphism of the incidence structure) and therefore θ_1 is expressible as a product of a collineation times the given polarity θ.

THEOREM 3 *If $\mathcal{E}$ is a quadric then $\mathcal{D}$ is self-dual.*

Proof: We will give a proof using homogeneous coordinates on $PG(4,q)$. We will write $(x_0 : x_1 : x_2 : x_3 : x_4)$ for the homogeneous coordinates of a point and we denote by $[a_0 : a_1 : a_2 : a_3 : a_4]$ the hyperplane of $PG(4,q)$, $[a_0 : a_1 : a_2 : a_3 : a_4] := \{(x_0 : x_1 : x_2 : x_3 : x_4) \in PG(4,q) \mid a_0x_0 + a_1x_1 + a_2x_2 + a_3x_3 + a_4x_4 = 0\}$. We may assume that $H_\infty = [0:0:0:0:1]$ and that $V = (0:0:0:0:1)$.

If q is odd then by [6, Theorem 5.1.6], we may assume that $\mathcal{E}$ is the set of points in H_∞ which satisfy the equation $x_0^2 + x_1^2 + x_2^2 + \nu x_3^2$ where ν is some non-zero element of F. Thus $\mathcal{E} = \{(x_0 : x_1 : x_2 : x_3 : 0) \mid x_0^2 + x_1^2 + x_2^2 + \nu x_3^2 = 0\}$. If q is even then by [6, Theorem 5.1.7] we may assume that $\mathcal{E}$ is the set of points in H_∞ which satisfy the equation $\nu x_0^2 + x_0x_1 + x_1^2 + x_2x_3$ for some $\nu \neq 0$. We will give the proof of Theorem 3 for q odd.

Working in H_∞, let $z = (z_0 : z_1 : z_2 : z_3) \in \mathcal{E}$. Then the tangent plane to $\mathcal{E}$ at z is the plane $\Pi_z = [z_0 : z_1 : z_2 : \nu z_3]$. Returning to $PG(4,q)$, let $p = (p_0 : p_1 : p_2 : p_3 : \lambda)$ be any point of $S \cup \mathcal{E}$. Recall that H_∞ has equation $x_4 = 0$. It follows that $z = (p_0 : p_1 : p_2 : p_3 : 0)$ is on $\mathcal{E}$. Moreover the solids, $[p_0 : p_1 : p_2 : \nu p_3 : \lambda]$ for $\lambda \in F_q$ are the q solids other than H_∞ which contain Π_z. We want to construct a map θ which gives a duality of $\mathcal{D}$. Define $\theta(p)$ to be the set $[p_0 : p_1 : p_2 : \nu p_3 : \lambda] \cap S$. (For q even take $\theta(p) := [p_1 : p_0 : p_3 : p_2 : \lambda] \cap S$.) Thus $\theta(p)$ is a block (resp. a group) of $\mathcal{D}$ if $p \in S$ (resp. $p \in \mathcal{E}$).

Let $B = [a_0 : a_1 : a_2 : a_3 : \mu] \cap S$. Then B is either a block or group of $\mathcal{D}$. Put $\theta(B) = (a_0 : a_1 : a_2 : a_3/\nu : \mu)$. (For q even take $\theta(B) = (a_1 : a_0 : a_3 : a_2 : \mu)$.) Then $\theta(B) \in S$ (resp. $\theta(B) \in \mathcal{E}$) if B is a block (resp. a group) of $\mathcal{D}$. Moreover $\theta(\theta(p)) = p$ for all $p \in \mathcal{D} \cup \mathcal{E}$ and $\theta(\theta(B)) = B$ for each block or group B of S. Finally, to see that θ is a polarity we must show that $p \in B$ if and only if $\theta(B) \in \theta(p)$. To see this, take $p = (p_0 : p_1 : p_2 : p_3 : \lambda) \in \mathcal{D}$, $z = (z_0 : z_1 : z_2 : z_3 : 0) \in \mathcal{E}$ and let $B = [z_0 : z_1 : z_2 : \nu z_3 : \mu] \cap S$ be a block. Then

$$\begin{aligned}
\theta(B) \in \theta(p) &\iff (z_0 : z_1 : z_2 : z_3 : \mu) \in [p_0 : p_1 : p_2 : \nu p_3 : \lambda] \cap S \\
&\iff z_0p_0 + z_1p_1 + z_2p_2 + \nu z_3p_3 + \mu\lambda = 0 \\
&\iff (p_0 : p_1 : p_2 : p_3 : \lambda) \in [z_0 : z_1 : z_2 : \nu z_3 : \mu] \\
&\iff p \in B
\end{aligned}$$ ∎

Remark. Note that statement 1) of Proposition 1 is dual to 2) and 3) is dual to 4) there. Thus by Theroem 5 it would have sufficed to prove 1) and 3) in Proposition 1 if $\mathcal{E}$ were not just an ovoid but in fact a quadric.

2.1. The Inherited Automorphism Group of $\mathcal{D}$

We begin with a lemma.

LEMMA 2 *Every automorphism ϕ of $\mathcal{D}$ inherited from $PGL(5, q)$ fixes V and stabilizes $\mathcal{E}$.*

Proof: Case 1: $q > 3$. Lemma 1 implies that any line in $PG(3, q)$ meets S in at most 2 points unless the line is one of the $\ell(z, V)$ for $z \in \mathcal{E}$. It follows that V is fixed by ϕ. Also since H_∞ is the unique hyperplane missing S, H_∞ is stabilized. Since $\mathcal{E} = \cup_{x \in S}(\ell(x, V) \cap H_\infty)$, this implies that $\mathcal{E}$ is stabilized by ϕ.
Case 2: $q = 3$. Again H_∞ is the unique solid missing S. We will see below in Corollary 11 that there are exactly 10 solids $H_1, \ldots, H_{10}$ in $PG(3, 3)$ which meet S in 2 points. Furthermore the planes $H_i \cap H_\infty$ are the 10 tangent planes to $\mathcal{E}$ and so they determine $\mathcal{E}$. Therefore, ϕ must stabilize $\mathcal{E}$. ∎

THEOREM 4 *The automorphism group G of $\mathcal{D}$ inherited from $PGL(5, q)$ consists precisely of all projective transformations obtained from matrices M in $GL(5, q)$ of the following form:*

$$M = \left(\begin{array}{c|c} & 0 \\ A & 0 \\ & 0 \\ & 0 \\ \hline 0\,0\,0\,0 & \lambda \end{array} \right)$$

where A is the matrix corresponding to an automorphism of the solid H_∞ which stabilizes the ovoid $\mathcal{E}$. In particular, if q is odd or $q = 4$, since $\mathcal{E}$ is then a quadric, A corresponds to an element of the (projective) orthogonal group in 4 variables.

Proof: We can choose homogeneous coordinates so that $V = (0 : 0 : 0 : 0 : 1)$ and H_∞ has equation $x_4 = 0$. The theorem now follows from the previous lemma. ∎

COROLLARY 1 *If $\mathcal{E}$ is a quadric then every correlation of $\mathcal{D}$ is the product of an element of G with the polarity θ.*

Remark. In such a case, using the existence of a cyclic collineation on the quadric we see that G contains an abelian subgroup K which acts regularly on both the points and blocks of S. The existence of a polarity on $\mathcal{D}$ then follows from an old result of M. Hall (see [1, page 37]).

3. Pellegrino's 20 Cap in AG(4,3)

We want to focus now on the case $q = 3$. In $AG(4, 3)$ the points of the design $\mathcal{D}$ give the points of a cap. A celebrated result of Pellegrino [13] (see also [5]) implies that this is in fact a maximum cap on $AG(4, 3)$. In fact, more generally in [13] Pellegrino showed that a maximum cap in $PG(4, 3)$ contains 20 points and he constructed such caps including one which lies in $AG(4, 3)$. However we must point out that the proof of Pellegrino is conceptually very difficult and the details are most intricate.

L. Haddad [4] shows directly that any set of 21 points in $AG(4,3)$ contains three points which are collinear. Here we give a transparent geometric proof that any cap of size 20 in $AG(4,3)$ is of the form $S = S(V,\mathcal{E})$. This structure theorem will also imply (see Theorem 5) that any cap of $AG(4,3)$ has at most 20 points.

Now let $S \subset AG(4,3)$ be any cap of cardinality 20. We will denote the set of all solids in $AG(4,3)$ by $\mathcal{H}$. Define h_i to be the cardinality of the set $\{H \in \mathcal{H} : |H \cap S| = i\}$ for $i = 0, \ldots, 20$. Note that $h_i \geq 0$. We now proceed to the standard incidence equations for points, pairs of points, and triples of points in $AG(4,3)$(cf. [3]). Recalling $|S| = 20$ we obtain

$$\sum_{i=0}^{20} h_i = |\mathcal{H}| = 120. \tag{0}$$

Counting incidences of points in S with solids in $AG(4,3)$ gives

$$\sum_{i=0}^{20} i h_i = 40|S| = 800. \tag{1}$$

Simlarly counting incidences of pairs and triples gives

$$\sum_{i=0}^{20} \binom{i}{2} h_i = 13\binom{|S|}{2} = 2470 \tag{2}$$

and

$$\sum_{i=0}^{20} \binom{i}{3} h_i = 4\binom{|S|}{3} = 4560. \tag{3}$$

Since $H \cap S$ is a cap in $AG(3,3)$ for every $H \in \mathcal{H}$ we know that $h_i = 0$ for $i \geq 10$. Furthermore, if some solid H met S in less than 2 points then at least one of the two solids parallel to H would have to contain at least 10 points of S. It follows that $h_0 = h_1 = 0$.

LEMMA 3 *There exists $H \in \mathcal{H}$ such that $|H \cap S| = 2$, i.e., $h_2 \geq 1$.*

Proof: Assume that $h_2 = 0$. The linear combination $6(\text{Eq. }3) - 28(\text{Eq. }2) + 78(\text{Eq. }1) - 168(\text{Eq. }0)$ yields the equality $-12h_3 + 2h_5 + 8h_8 + 30h_9 = 440$. Now if a solid H meets S in 3 points then one of the solids parallel to H meets S in 8 points and the other meets S in 9 points. Therefore $h_8 \geq h_3$ and $h_9 \geq h_3$. Substituting these inequalities into the previous equation yields $2h_5 + 26h_3 \leq 440$, and thus $h_3 \leq 16$.

The linear combination $6(\text{Eq. }3) - 32(\text{Eq. }2) + 98(\text{Eq. }1) - 224(\text{Eq. }0)$ yields the equality $-20h_3 + 6h_5 + 4h_6 + 10h_9 = -160$. Using $h_9 \geq h_3$ again, we deduce that $-10h_3 + 6h_5 + 4h_6 \leq -160$ and so $h_3 \geq 16 + 3h_5/5 + 2h_6/5$. Thus $h_3 = 16$ and $h_5 = h_6 = 0$. It follows that the only remaining unknowns are h_4, h_7, h_8 and h_9. We can then solve for them using Eq. 0–Eq. 3, giving $h_3 = h_9 = 16$, $h_8 = 19$, $h_4 = 9$ and $h_7 = 60$. From this we easily get a contradiction as follows.

(1) since $h_3 = h_9$ each solid meeting S in 9 points is parallel to one meeting S in 8 points (and to another meeting S in 3 points.)
(2) If H meets S in 4 points, then the solids parallel to H must both meet S in 8 points.
From (1) and (2) it follows that $h_8 \geq h_9 + 2h_4$. This is a contradiction. ■

THEOREM 5 *If $S \subset AG(4,3)$ is a cap with $|S| \geq 20$ then $|S| = 20$ and $S = S(V, \mathcal{C})$ for some quadric $\mathcal{C}$ and some vertex V. In particular, any cap S of $AG(4,3)$ has at most 20 points with equality if and only if S is of the form $S = S(V, \mathcal{C})$.*

Proof: We embed $AG(4,3)$ into $PG(4,3)$. We will use standard combinatorial results concerning the numbers of various subspaces in $AG(n,q)$ and $PG(n,q)$ when $n = 4$ and $q = 3$. See for example Appendix B on finite geometries in [12]. Let $H_\infty := PG(4,3) \setminus AG(4,3)$ be the "solid at infinity".

By the above lemma there exists an affine solid H meeting S in exactly two points. Denote its projective completion by $H_0 \subset PG(4,3)$. So $|H_0 \cap S| = 2$. Let H_1, H_2 be solids in $PG(4,3)$ such that H_0, H_1, H_2, H_∞ is the pencil of solids containing $H_0 \cap H_\infty$. Then $|H_1 \cap S| + |H_2 \cap S| = 18$. Therefore $H_i \cap S$ is a punctured elliptic quadric (see the introduction), and we write $\mathring{\mathcal{C}}_i := H_i \cap S = \mathcal{C}_i \setminus \{p_i\}$ for $i = 1, 2$ where C_i is an elliptic quadric. Hence $S = \{a, b\} \sqcup \mathring{\mathcal{C}}_1 \sqcup \mathring{\mathcal{C}}_2$, with $H_0 \cap S = \{a, b\}$.

Let ℓ be the line through a and b. There are thirteen planes in $PG(4,3)$ which contain ℓ. Of these, 4 are contained in H_0. Let $\Pi_1, \Pi_2, \ldots, \Pi_9$ be the 9 planes which contain ℓ and which are not themselves contained in H_0. Clearly each of the 18 points of S different from a and b lies in exactly one of the Π_i. Since no plane can contain more than 4 points of a cap and since ℓ has 2 points of the cap we see that $|\Pi_i \cap S| = 4$ for $i = 1, 2, \ldots, 9$.

Now there are 40 planes contained in H_1. Of these, 10 are tangent to $\mathcal{C}_1$ and the remaining 30 meet $\mathcal{C}_1$ in 4 points. For every $z \in \mathcal{C}_1$ there are exactly 12 secant planes to $\mathcal{C}_1$ which pass through z. In particular, for $z = p_1$ there are exactly 12 planes $\Gamma_1, \Gamma_2, \ldots, \Gamma_{12}$ in H_1 which meet $\mathring{\mathcal{C}}_1$ in three points and contain p_1.

Take any solid K in $PG(4,3)$ which contains ℓ and is different from H_0. Consider the pencil of 4 planes in K which contain ℓ. Apart from $K \cap H_0$, each of these planes is one of the Π_i. Thus $|K \cap S| = 2 + 3(2) = 8$ and therefore $|K \cap (\mathring{\mathcal{C}}_1 \cup \mathring{\mathcal{C}}_2)| = 6$. Now for $i = 1, 2$ we have $K \cap \mathcal{C}_i = K \cap H_i \cap \mathcal{C}_i$. But $K \cap H_i$ is plane in H_i and therefore $|K \cap \mathcal{C}_i| = 1$ or 4. Hence $|K \cap \mathring{\mathcal{C}}_i| \in \{0, 1, 3, 4\}$. Therefore $|K \cap \mathring{\mathcal{C}}_1| = |K \cap \mathring{\mathcal{C}}_2| = 3$ and $p_1, p_2 \in K$.

Since there are 13 solids containing ℓ there are 12 solids $K_1, K_2, \ldots, K_{12}$ different from H_0 which may be used in the role of K in the previous paragraph. By the result there, each of them meets H_1 in a plane containing p_1 and 3 points of $\mathring{\mathcal{C}}_1$. In particular, p_1 lies in K_j, as does p_2. Thus $p_1, p_2, a, b \in K_j$ for $j = 1, 2, \ldots, 12$. But no plane is contained in more than 4 solids. Therefore p_1, p_2, a, b are all collinear. Since $p_1, p_2 \in H_\infty$ this implies $\{p_1\} = \ell \cap H_\infty = \{p_2\}$. We will use p to denote the point $p_1 = p_2$.

Let V be the fourth point on the line ℓ different from a, b and p. We will now show that $\mathcal{C}_1$ and $\mathcal{C}_2$ are in perspective from V. Recall that $|\Pi_j \cap S| = 4$ and that $a, b \in \Pi_j \cap S$. By way of contradiction, assume that there exists $j \in \{1, 2, \ldots, 12\}$ such that $|\Pi_j \cap \mathring{\mathcal{C}}_1| = 2$. Then the line $\Pi_j \cap H_1$ meets $\mathcal{C}_1$ in 3 points since $p \in \Pi_j \cap H_1$, a contradiction. Hence, $|\Pi_j \cap \mathring{\mathcal{C}}_1| \leq 1$ for each j. Similarly, $|\Pi_j \cap \mathring{\mathcal{C}}_2| \leq 1$. But $|\Pi_j \cap (\mathring{\mathcal{C}}_1 \cup \mathring{\mathcal{C}}_2)| = 2$ and therefore

$|\Pi_j \cap \mathring{\mathcal{C}}_i| = 1$ for $i = 1, 2$ and $j = 1, \ldots, 12$. Define $u_j := \Pi_j \cap \mathring{\mathcal{C}}_1$ and $w_j := \Pi_j \cap \mathring{\mathcal{C}}_2$ for $j = 1, 2, \ldots, 12$. Since $u_j \in H_1 \setminus H_\infty$ and $w_j \in H_2 \setminus H_\infty$, the line $\ell(u_j, w_j)$ meets H_0 in a point z_j which does not lie in H_∞. Consider the 9 points of the affine plane $\overline{\Pi_j} = \Pi_j \setminus H_\infty$. Then $z_j \in \overline{\Pi_j} \cap H_0 = \{a, b, V\}$. Since S is a cap containing a, b and also u_j, w_j it follows that we must have $z_j = V$. Thus $\mathcal{C}_1$ and $\mathcal{C}_2$ are in perspective from V. Hence $S = S(V, \mathcal{C}_1)$.

It only remains to show that no cap in $AG(4, 3)$ contains more than 20 points. By way of contradiction, suppose T is a cap containing more than 20 points. Let S be a subset of T of size 20. Then by the above, $S = S(V, \mathcal{C})$ for some quadric $\mathcal{C}$ and vertex V. Take $t \in T \setminus S$. Project t from V to $y \in H_\infty$. Since $t \notin S$, we must have $y \notin \mathcal{C}$. Choose a secant line, ℓ to $\mathcal{C}$ through y. Suppose that ℓ intersects $\mathcal{C}$ in the two points z_1 and z_2. Let Π be the plane containg ℓ and V. Choose $s_1 \in \ell(V, z_1) \setminus \{V, z_1\}$ such that $s_1 \notin \ell(t, z_2)$. Now the two lines $\ell(s_1, t)$ and $\ell(V, z_2)$ of Π meet in a point s_2 of S. By construction, $s_2 \neq z_2$. Furthermore, since $t \notin S$, it is clear that $s_2 \neq V$. But then since $t \in \ell(V, z_2)$ we have $t \in S$, a contradiction. ■

COROLLARY 2 *There exist 10 solids which meet S in 2 points, i.e., $h_2 = 10$.*

Proof: Suppose K is a solid such that $|K \cap S| = 2$ and let $\Pi := H_0 \cap H_1 = H_0 \cap H_2$. Now Π misses both $\mathring{\mathcal{C}}_1$ and $\mathring{\mathcal{C}}_2$. Thus Π is both the tangent plane to $\mathcal{C}_1$ at p and the tangent plane to $\mathcal{C}_2$ at p. If K contains Π then $K \in \{H_0, H_1, H_2, H_\infty\}$ and thus $K = H_0$. If $\Pi \not\subset K$ then $K \cap \mathcal{C}_i \neq \{p\}$ and thus $K \cap H_i$ is a tangent plane to $\mathcal{C}_i$ at some point other than p for $i = 1, 2$. Write $K \cap \mathcal{C}_1 = \{u_j\}$ and $K \cap \mathcal{C}_2 = \{w_k\}$. Since $a, b, p \notin K$, we must have $K \cap \ell = \{V\}$. But then since u_j and V are in K, we must also have $w_j \in K$ since from above w_1 and w_2 are in perspective from V. Hence $j = k$ and there are at most 9 choices for K other than H_0: they correspond to the 9 lines through V other than ℓ coming from the perspective of $\mathcal{C}_1$ with $\mathcal{C}_2$.

Conversely, if K is the solid containing V and containing the tangent plane to $\mathcal{C}_1$ at u_j then it is clear that $K \cap H_2$ is the tangent plane to $\mathcal{C}_2$ at w_j. Furthermore, since $V \in K$ and $p \notin K$, $a, b \notin K$ and thus $K \cap S = \{u_j, w_j\}$. ■

4. Concluding Remarks

4.1. Polarity

We have shown in Theorem 3 that if $\mathcal{E}$ is a quadric then there exists a polarity of $\mathcal{D}$. We point out however that the structure of ovoids in $PG(3, q)$ is unknown for q even. This gives rise to the following question.

Question. Does there always exist a polarity of $\mathcal{D}$ whether or not $\mathcal{E}$ is a quadric?

4.2. Construction of Semi-Biplanes

In the three dimensional case $\hat{\mathcal{P}}$ consists of two isomorphic semi-biplanes each on $(q^2-1)/2$ points and having block size q. Semi-biplanes with these parameters are also described in [8]. The arguments in Section 1 in fact, remain valid over infinite fields. Thus the construction in Section 1 gives infinite semi-biplanes in that case.

4.3. Higher dimensions, other quadrics

Of course, our construction can be partially generalized to ruled quadrics and quadrics in higher dimensions. However since there are more than two kinds of pairs of points in those settings we will not get a group divisible design in those cases.

4.4. Quotient constructions and work of D. Jungnickel and E. Moorhouse

Our construction in Section 1, The Three Dimensional Case, yields by Theorem 2 two semi-biplanes on $(q^2-1)/2$ points which are isomorphic to each other. These semi-biplanes are examples of *homology semi-biplanes* in the terminology of Moorhouse (see [11]). In [11] Moorhouse poses the question as to whether all homology semi-biplanes are constructible as a quotient of a projective plane by an involutory homology. It is not clear, apart from small cases, whether or not the homology semi-biplanes constructed in Section 1 are constructible as such quotients.

As regards the designs constructed in Section 2, we should point out that in [10] D. Jungnickel has constructed symmetric group divisible designs with the same parameters. These designs in [10] are again obtained as quotients of projective planes. However, as is pointed out in [11], the isomorphism problem appears to be a difficult one. Our construction is valid for any ovoid in $PG(3,q)$. As pointed out in Remark 1 above the structure of such ovoids for q even is unknown.

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