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A family of small complete caps in $\mathbb{PG}(n, 2)$

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Abstract

The smallest known complete caps in $\mathbb{PG}(n, 2)$ have size $23(2^{(n-6)/2}) - 3$ if $n \geq 10$ is even and size $15(2^{(n-5)/2}) - 3$ if $n \geq 9$ is odd. Here we give a simple construction of complete caps in $\mathbb{PG}(n, 2)$ of size $24(2^{(n-6)/2}) - 3$ if n is even and size $16(2^{(n-5)/2}) - 3$ if n is odd. Thus these caps are only slightly larger than the smallest complete caps known in $\mathbb{PG}(n, 2)$.

1. Introduction

A *cap* is a set with no three points collinear. A cap in $\mathbb{PG}(n, 2)$ is called *complete* if it is not properly contained in any other cap lying in $\mathbb{PG}(n, 2)$. The smallest known complete caps in $\mathbb{PG}(n, 2)$ were described by Gabidulin et al. [2]. These smallest known caps have size $23(2^{(n-6)/2}) - 3$ if $n \geq 10$ is even and size $15(2^{(n-5)/2}) - 3$ if $n \geq 9$ is odd. Here we describe some complete caps which are almost as small. Specifically the caps, S_n, constructed here have size $3(2^{n/2} - 1) = 24(2^{(n-6)/2}) - 3$ for n even and size $2^{(n+3)/2} - 3 = 16(2^{(n-5)/2}) - 3$ for n odd.

The caps we describe here are constructed using the black/white lift, as described in [1]. We briefly recall this construction. Let S be a cap in $\Sigma = \mathbb{PG}(n, 2)$. Given a point x of Σ not lying in S we partition the set S into two subsets: the *Black points* and the *White points*. The black points, $\mathcal{B}(x, S)$, are the points of the cap, S, lying on the secant cone of x and the white points, $\mathcal{W}(x, S)$, are the points of S lying on the tangent cone of x. A point, w, of $\Sigma \backslash S$ is a *dependable point for* S if there does not exist any other point $x \in \Sigma \backslash S$ with $\mathcal{W}(w, S) \subseteq \mathcal{W}(x, S)$.

Let S be a complete cap in $\Sigma = \mathbb{PG}(n, 2)$ and $w \in \Sigma \backslash S$. Embed Σ in a projective space $\widetilde{\Sigma}$ of one dimension more. Fix $v \in \widetilde{\Sigma} \backslash \Sigma$. The black/white lift of S with respect to

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the apex, v, is the cap $\psi_w(S)$ in $\widetilde{\Sigma} = \mathbb{PG}(n+1, 2)$ defined by $\psi_w(S) := S \sqcup \{x + v \mid x \in \mathcal{W}(w, S)\} \sqcup \{v + w\}$.

The following is a combination of Theorems 2.2 and 2.8 of [1].

Theorem 1.1. *Let S be a complete cap in $\Sigma = \mathbb{PG}(n, 2)$ where $n \geq 2$ and let $w \in \Sigma \backslash S$. Then $\psi_w(S)$ is a cap in $\widetilde{\Sigma} = \mathbb{PG}(n+1, 2)$ of size $\#\psi_w(S) = \#S + \#\mathcal{W}(w, S) + 1 = 2\#S - \#\mathcal{B}(w, S) + 1$. Moreover, if w is a dependable point for S then $\psi_w(S)$ is complete.*

2. The family of small complete caps

Let S_3 be an ovoid in $\mathbb{PG}(3, 2)$. Each of the ten points of $\mathbb{PG}(3, 2)\backslash S_3$ lies on a unique secant line to S_3. Choose two points $u, w \in \mathbb{PG}(3, 2)\backslash S_3$ such that the two corresponding secant lines do not intersect. Thus we have $S_3 = \{s_1, s_2, s_3, s_4, s_5\}$ and $u = s_1 + s_2$, $w = s_3 + s_4$ and $s_5 = s_1 + s_2 + s_3 + s_4 = w + u$.

For higher dimensions we define S_n inductively by $S_n := \begin{cases} \psi_w(S_{n-1}), & \text{if } n \text{ is even;} \\ \psi_u(S_{n-1}), & \text{if } n \text{ is odd.} \end{cases}$

We will denote by v or v_n the apex used in constructing S_n from S_{n-1}.

Thus S_n is a cap in $\mathbb{PG}(n, 2)$. To prove that S_n is a complete cap we will show that w is a dependable point for S_{n-1} when n is even and that u is a dependable point for S_{n-1} when n is odd.

Remark 2.1. It can be seen that all possible choices of the ovoid S_3 together with the ordered pair (w, u) are equivalent up to collineations.

3. Properties of S_n

Lemma 3.1. *$\mathcal{W}(w, S_n) = \mathcal{B}(u, S_n) \sqcup \{w + u\}$ and $\mathcal{W}(u, S_n) = \mathcal{B}(w, S_n) \sqcup \{w + u\}$ for $n \geq 3$.*

Proof. We proceed by induction on n. The case $n = 3$ is easy and so we consider $n \geq 4$. By the symmetry between w and u, we may assume that $S_n = \psi_w(S_{n-1})$. By construction, $w + u \in S_{n-1}$ and thus $w + u \in \mathcal{W}(w, S_{n-1}) \cap \mathcal{W}(u, S_{n-1})$. We first show that $\mathcal{W}(w, S_n) \subseteq \mathcal{B}(u, S_n) \sqcup \{w + u\}$ and then that $\mathcal{B}(w, S_n) \sqcup \{w + u\} \subseteq \mathcal{W}(u, S_n)$. This suffices since we have $\mathcal{W}(w, S_n) \sqcup \mathcal{B}(w, S_n) = S_n = \mathcal{W}(u, S_n) \sqcup \mathcal{B}(u, S_n)$.

Using [1, Proposition 3.2(3)] and induction, we see $\mathcal{W}(w, S_n) = \mathcal{W}(w, S_{n-1}) \sqcup (S_n \backslash S_{n-1}) = \mathcal{B}(u, S_{n-1}) \sqcup \{w+u\} \sqcup (S_n \backslash S_{n-1})$. Clearly $\mathcal{B}(u, S_{n-1}) \sqcup \{w+u\} \subseteq \mathcal{B}(u, S_n) \sqcup \{w + u\}$, and thus to prove the first inclusion it remains to prove that $S_n \backslash S_{n-1} \subseteq \mathcal{B}(u, S_n)$. To prove this, let $x' = x + v \in S_n \backslash S_{n-1}$. We consider three cases. The first case is $x' = v + w$. We have $x' + u = v + (w + u) \in S_n$ since $w + u \in \mathcal{W}(w, S_{n-1})$ and thus $x' \in \mathcal{B}(u, S_n)$. The second case is $x' = v + w + u$. Since $v + w$ and $w + u \in S_n$, we see that $v + w + u \in \mathcal{B}(u, S_n)$. The third case is $x = v + x' \in \mathcal{W}(w, S_{n-1}) \backslash \{w + u\}$. Then $x \in \mathcal{B}(u, S_{n-1})$ by induction. Thus $x + u \in \mathcal{B}(u, S_{n-1})$ and therefore $x + u \in \mathcal{W}(w, S_{n-1})$ by induction. Hence $v + x + u = x' + u \in S_n$. Since $x' \in S_n$, this gives $x' \in \mathcal{B}(u, S_n)$ which proves the first inclusion.

For the other inclusion, we again apply [1, Proposition 3.2(3)] and induction to get $\mathcal{B}(w, S_n) \sqcup \{w + u\} = \mathcal{B}(w, S_{n-1}) \sqcup \{w + u\} = \mathcal{W}(u, S_{n-1}) \subseteq \mathcal{W}(u, S_n)$. $\square$

Proposition 3.2. *For all $n \geq 3$, $\mathcal{W}(w, S_n) \oplus \mathcal{W}(u, S_n) = \mathbb{PG}(n, 2) \backslash (S_n \sqcup \{w, u\})$.*

Proof. Clearly, $\mathcal{W}(w, S_n) \oplus \mathcal{W}(u, S_n) \subseteq \mathbb{PG}(n, 2) \backslash (S_n \sqcup \{w, u\})$. We will prove the opposite inclusion by induction on n. The case $n = 3$ is left to the reader to verify.

Let $n \geq 4$. By the symmetry between u and w we may assume that $S_n = \psi_w(S_{n-1})$. Let $z' \in \mathbb{PG}(n, 2) \backslash (S_n \sqcup \{w, u\})$. If $z' \in \mathbb{PG}(n-1, 2)$, then by the induction hypothesis, $z' \in \mathcal{W}(w, S_{n-1}) \oplus \mathcal{W}(u, S_{n-1}) \subseteq \mathcal{W}(w, S_n) \oplus \mathcal{W}(u, S_n)$. Thus we may assume that $z' \notin \mathbb{PG}(n-1, 2)$. Hence $z := v + z' \in \mathbb{PG}(n-1, 2)$.

We distinguish three cases. If $z \in \{w, u\}$ then since $z' \notin S_n$, we know $z' \neq w + v$ and thus $z = u$ and $z' = v + u$. Thus $z' = (v + w) + (u + w) \in \mathcal{W}(w, S_n) \oplus \mathcal{W}(u, S_n)$. Secondly, if $z \in S_{n-1}$ then since $z' \notin S_n$, we must have $z \in \mathcal{B}(w, S_{n-1})$. Thus $z + w \in \mathcal{B}(w, S_{n-1}) \subseteq \mathcal{W}(u, S_{n-1}) \subseteq \mathcal{W}(u, S_n)$. Since $w + v \in \mathcal{W}(w, S_{n-1})$, we have $z' = (w + v) + (z + w) \in \mathcal{W}(w, S_n) \oplus \mathcal{W}(u, S_n)$. Finally, if $z \notin (S_{n-1} \sqcup \{w, u\})$ then by induction, $z = x + y$ where $x \in \mathcal{W}(w, S_{n-1})$ and $y \in \mathcal{W}(u, S_{n-1})$. Therefore $z' = (x + v) + y$ where $x + v \in \mathcal{W}(w, S_n)$ and $y \in \mathcal{W}(u, S_n)$. □

Corollary 3.3. *For all $n \geq 3$, S_n is complete and w and u are dependable points for S_n.*

Proof. To see that both w and u are dependable, consider any point $z' \notin S_n$ with $z' \neq w$ and $z' \neq u$. Then by Proposition 3.2, there exist $x \in \mathcal{W}(w, S_n)$ and $y \in \mathcal{W}(u, S_n)$ such that $x + y = z'$. Thus $x, y \in \mathcal{B}(z', S_n)$. Therefore $x \in \mathcal{B}(z', S_n) \backslash \mathcal{B}(w, S_n)$ and $y \in \mathcal{B}(z', S_n) \backslash \mathcal{B}(u, S_n)$. It is clear from Lemma 3.1 that $\mathcal{B}(w, S_n) \nsubseteq \mathcal{B}(u, S_n)$ and $\mathcal{B}(u, S_n) \nsubseteq \mathcal{B}(w, S_n)$. Thus both u and w are dependable and by Theorem 1.1, S_n is complete. □

It is easy to verify by induction that $\#(S_n) = 2^{\lceil n/2 \rceil}(n) + 2^{\lfloor (n+2)/2 \rfloor} - 3$. Thus S has size $24(2^{(n-6)/2}) - 3$ if n is even and size $16(2^{(n-5)/2}) - 3$ if n is odd.

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