Complete caps in projective space which are disjoint from a subspace of codimension two

David L. Wehlau

Working over the field of order 2 we consider those complete caps which are disjoint from some codimension 2 subspace of projective space. We derive restrictive conditions which such a cap must satisfy in order to be complete. Using these conditions we obtain explicit descriptions of complete caps which do not meet every hyperplane in at least 5 points. In particular, we determine the set of cardinalities of all such complete caps in all dimensions.

1 Introduction

A subset of the projective space $\Sigma = \mathbb{PG}(n,2)$ is a *cap* if no three points of S are collinear. A cap in Σ is said to be *complete* if it is not properly contained in any other cap of Σ. A cap S in Σ is called *large* if $|S| \geq 2^{n-1}+1$; otherwise S is said to be *small*. Much is known about the structure of large complete caps. In particular, it was shown in [5] that if $S \subset \Sigma$ is a large complete cap then $|S| = 2^{n-1}+2^i$ for some $i \in \{0,1,2,\ldots,n-3,n-1\}$. In [5] it is also shown that if S is a cap in Σ satisfying $|S| \geq 2^{n-1}+2$ then there exists a codimension 2 subspace $H_\infty \cong \mathbb{PG}(n-2,2)$ of Σ such that $S \cap H_\infty = \emptyset$. In [3] the same result was shown to hold for all large caps. Conversely in [1] it was shown that given a fixed $r \geq 1$ then for all sufficiently large n there exists a cap S in $\mathbb{PG}(n,2)$ such that S meets every codimension r subspace $H \cong \mathbb{PG}(n-r,2)$ of $\mathbb{PG}(n,2)$.

Here we will consider complete caps, $S \subset \Sigma = \mathbb{PG}(n,2)$ with the property that some codimension 2 subspace $H_\infty \cong \mathbb{PG}(n-2,2)$ of Σ satisfies the condition $S \cap H_\infty = \emptyset$. Fix such an H_∞ and denote the three hyperplanes of Σ which contain H_∞ by K_A, K_B and K_C. Furthermore we write $H_A := K_A \setminus H_\infty$, $H_B := K_B \setminus H_\infty$ and $H_C := K_C \setminus H_\infty$. Write $A := H_A \cap S = K_A \cap S$, $B := H_B \cap S = K_B \cap S$ and $C := H_C \cap S = K_C \cap S$. Finally define $A' = H_A \setminus A$, $B' = H_B \setminus B$, and $C' = H_C \setminus C$.

If $C = \emptyset$ then S is complete if and only if $S = \Sigma \setminus K_C$. Henceforth we will assume that none of the three sets A, B, and C is empty.

[0]Corrected version, September 1, 2001

[1]I thank A. Davydov for pointing out a error at the bottom of page 7, Sept. 15, 2003

Consider the subspace $\widetilde{F} \cong \mathbb{PG}(r,2)$ of K_C generated by C. Put $F := \widetilde{F} \cap H_\infty$ and $\widehat{C} := \widetilde{F} \setminus F$. Then $F \cong \mathbb{PG}(r-1,2)$ and $\widehat{C} \cong \mathbb{AG}(r,2)$. Consider the decomposition of H_A and of H_B into cosets of $\widetilde{C}$. Each such coset F' consists of 2^r points such that $F' \sqcup F \cong \mathbb{PG}(r,2)$. There are 2^{n-r-1} such cosets in each of H_A and H_B. Each of these cosets is isomorphic to $\widetilde{F} \setminus F \cong \mathbb{AG}(r,2)$. We will denote the cosets in H_A by $H_A(1), H_A(2), \ldots, H_A(2^{n-r-1})$. Similarly we denote the cosets in H_B by $H_B(1), H_B(2), \ldots, H_B(2^{n-r-1})$ ordered so that $H_B(i) = \widehat{C} + H_A(i)$ for all i. Write $A(i) := H_A(i) \cap A$, $A'(i) := H_A(i) \cap A'$, $B(i) := H_B(i) \cap B$ and $B'(i) := H_B(i) \cap B'$.

The cap S is complete if and only if

$$A + B = C', \quad A + C = B', \quad B + C = A'; \tag{1.1}$$

$$(A \oplus A) \sqcup (B \oplus B) \sqcup (C \oplus C) = H_\infty. \tag{1.2}$$

In particular, if S is complete we must have

$$A(i) + C = B'(i) \text{ and } B(i) + C = A'(i) \tag{1.3}$$

for all $i = 1, 2, \ldots, 2^{n-r-1}$.

One way to satisfy Equation 1.3 is to take $A(i) = H_A(i)$ and $B(i) = \emptyset$. Similarly we may take $A(i) = \emptyset$ and $B(i) = H_B(i)$. We will call these two solutions the *trivial* solutions. A second way to satisfy this equation is to take $A(i) = \{a_1\}$ and $B(i) = H_B(i) \setminus (a_1 + C)$. Symmetrically we may take $B(i) = \{b_1\}$ and $A(i) = H_A(i) \setminus (b_1 + C)$. If either $|A| = 1$ or $|B| = 1$ then we call the solution a singleton solution.

Once we have solutions to Equation 1.3 we still will need the extra conditions of $A + B = C'$ together with Equation 1.2. In Section 3 we explicitly describe all complete caps with $|C| = 1$. Using these results we are able in Section 4 to give detailed sufficient conditions on a cap S which satisfies Equation 1.3 for all i, for S to be complete.

In Sections 6, 7 and 8 we consider complete caps which meet at least one hyperplane in less than 5 points. For such caps we are able to find all solutions to Equation 1.3 and thus to explicitly describe such complete caps.

In particular, we are able to construct many small complete caps. Also note that the Plotkin doubling construction (described in Section 2), as well as its generalization the Black/White lift (described in [4]) can be used to build a great many more small complete caps from the small complete caps constructed here.

2 Preliminaries

Let $\mathbb{F}_2^{n+1}$ denote the $(n+1)$-dimensional vector space over $\mathbb{F}_2$, the field of order 2. The elements of $\mathbb{PG}(n,2)$ are the one dimensional subspaces of $\mathbb{F}_2^{n+1}$. Each such subspace may be uniquely represented by the non-zero vector it contains. Fix a basis $\{e_0, e_1, \ldots, e_n\}$ of $\mathbb{F}_2^{n+1}$ and consider a point $x \in \mathbb{PG}(n,2)$. For ease of notation, we write $x = s_1 s_2 \cdots s_r$ if $e_{s_1} + e_{s_2} + \ldots + e_{s_t}$ is the non-zero vector in x where $s_1, s_2, \ldots, s_r$ are distinct elements of $\{0, 1, \ldots, n\}$. For example, 013 denotes the element of $\mathbb{PG}(n,2)$ comprised by the one-dimensional subspace containing $e_0 + e_1 + e_3$. From this point of view, the three points of $\mathbb{PG}(n,2)$ represented by x, y and z are collinear if and only if x, y and z lie in a plane in $\mathbb{F}_2^{n+1}$ which occurs if and only if $x+y=z$ in $\mathbb{F}_2^{n+1}$. The usual inner product on $\mathbb{F}_2^{n+1}$ induces an inner product on $\mathbb{PG}(n,2)$. We will write $(x)^\perp$ to denote the hyperplane of points in $\mathbb{PG}(n,2)$ which are orthogonal to x with respect to this inner product. On occasion we will identify $\mathbb{F}_2^n$ with the affine space $\mathbb{AG}(n,2)$. In that setting we will use $\emptyset$ to denote the zero vector in $\mathbb{AG}(n,2)=\mathbb{F}_2^n$.

Let X and Y be two subsets of $\Sigma = \mathbb{PG}(n,2)$ and let $z \in \mathbb{PG}(n,2)$. We write $X \oplus Y$ to denote the set $\{x+y \mid x \in X, y \in Y, x \neq y\} \subseteq \Sigma$. If $X \cap Y = \emptyset$ we also write $X+Y = X \oplus Y$. If $z \notin X$ we write $z + X := \{z + x \mid x \in X\} = \{z\} + X$.

A subset X of $\mathbb{PG}(n,2)$ is said to be *periodic* if there exists a point $v \in \mathbb{PG}(n,2)$ such that $v + X = X$. If such a point v exists it is called a *vertex* of X. The (possibly empty) set of all vertices of X is denoted $V(X)$. Note that if X is periodic then necessarily $|X|$ is even.

Let X be a subset of $\Sigma = \mathbb{PG}(n,2)$. Embed Σ in $\widetilde{\Sigma} \cong \mathbb{PG}(n+1,2)$ and let $v \in \widetilde{\Sigma} \setminus \Sigma$. We define the *Plotkin double* of X (from the vertex v) by

$$\phi(X) := X \sqcup \{v + x \mid x \in X\}.$$

Then $\phi(X)$ is a periodic subset of $\widetilde{\Sigma}$ with $V(\phi(X)) = \phi(V(X)) \sqcup \{v\}$. It is straightforward to verify that $\phi(X)$ is a cap in $\widetilde{\Sigma}$ if and only if X is a cap in Σ; also, $\phi(X)$ is complete if and only if X is.

A slightly more general version of the following lemma may be found in [3, Lemma 3.9]. We include a proof here for the reader's convenience.

Lemma 2.1 *Let $S \subset \Sigma$ be complete. Suppose there exists a hyperplane L of Σ such that $S \cap L$ is periodic. Then S is periodic. Furthermore, if v is a vertex for $S \cap L$ then v is also a vertex for S.*

Proof Let v be a vertex for $S \cap L$ and assume, by way of contradiction, that v is not a vertex for S. Then there exists a point x_1 of S such that $x_2 := x_1 + v \notin S$. Clearly $x_1 \notin L$ since $v \in L$ and thus $x_2 \notin L$. Since S is complete, x_2 lies on a secant of S: say $x_2 = y_1 + z_1$ where $y_1, z_1 \in S$. Without loss of generality $y_1 \in L$ since every line of Σ meets L. Therefore $y_2 := v + y_1 \in L \cap S$. But $y_2 = v + y_1 = v + x_2 + z_1 = x_1 + z_1$ and thus the line $\{y_2, x_1, z_1\}$ is fully contained in S, a contradiction. □

Lemma 2.2 *Let X be a subset of $\Sigma = \mathbb{PG}(n,2)$ and let $V(X)$ denote the (possibly empty) set of vertices for X. Then $V(X)$ is a projective subspace of Σ.*

Proof We must prove that $V(X)$ is closed under addition, i.e., if u, v are two distinct points of $V(X)$ then $u + v \in V(X)$. Accordingly let u, v be two distinct points of $V(X)$. We must show that $w = v + u$ is also a vertex of X. Let $p \in X$. Since u is a vertex, $u + p \in X$. Similarly, since $v \in V(X)$ we have $w + p = v + (u + p) \in X$. Hence $w + X \subseteq X$ and thus w is a vertex of X. □

3 Caps having a tangent hyperplane

In this section we characterize those complete caps S which have a tangent hyperplane. Many of the results of this section are related to results found in [3].

We suppose that K_C is a hyperplane which is tangent to S, i.e., that C consists of a single point, $C = \{c_o\}$. Choose a codimension 2 subspace $H_\infty \cong \mathbb{PG}(n-2,2)$ contained in K_C and disjoint from S. Clearly if S is complete we must have $B' = c_0 + A$ and $A' = c_0 + B$. Thus S is determined by A (or by B). We wish to investigate conditions on the set A which guarantee that S is complete.

Define $E = \Sigma \setminus (S \sqcup (S \oplus S))$ and $T = S \sqcup E$.

Lemma 3.1 *T is a complete cap in $\Sigma = \mathbb{PG}(n,2)$.*

Proof Assume, by way of contradiction that T contains a line, ℓ Clearly ℓ must contain at least two points of E, say e, e_1. Clearly $E \subset K_C$ and thus $|\ell \cap K_C| \geq 2$ from which it follows that $\ell \subset K_C$. Thus ℓ contains at least one point of H_∞, e say. Therefore, $e + A = A'$ and $e + B = B'$. Now if $e_1 \in H_\infty$ then $e_1 + A = A'$ and $e_1 + B = B'$. Conversely if $e_1 \in H_C$ then $e_1 + A = B'$ and $e_1 + B = A'$. In both of these cases we have $e + e_1 + (A \sqcup B) = (A \sqcup B)$. This shows that $e + e_1 \in (A \sqcup B) \oplus (A \sqcup B) \subset S \oplus S$. Therefore $e + e_1 \notin T$. This contradiction proves the lemma. □

Proposition 3.2 *Let A and B be as defined above. Then $V(A) = V(B)$. In particular, A is periodic if and only if B is periodic.*

Proof Suppose A is periodic, i.e., suppose there exists $v \in H_\infty$ such that $v + A = A$. Since $v + H_A = H_A$ we have $v + A' = A'$. Now $c_0 + B = A'$ and $c_0 + A = B'$. Therefore $v + B = v + c_o + A' = c_0 + v + A' = c_0 + A' = B$. □

Next we give a sufficient condition on A to guarantee that the set $S = \{c_0\} \sqcup A \sqcup B$ is a *complete* cap. The following theorem is similar to [3, Theorem 4.1].

Theorem 3.3 *Let S be a cap in $\Sigma = \mathbb{PG}(n,2)$ with $|C| = 1$ and $B' = c_0 + A$. Suppose that $|A| \neq 2^{n-2}$. If A (or B) is not periodic then S is a complete cap.*

Proof Clearly every point of $\Sigma \setminus L_C$ lies on a secant to S through c_0. Let $\alpha = |A|$ and $\beta = |B|$. Then $\alpha + \beta = 2^{n-1}$. Since $\alpha \neq 2^{n-1}$ one of α or β, say without loss of generality α, exceeds 2^{n-2}. Therefore $A \oplus A = H_\infty$, i.e., every point of H_∞ lies on a secant to A.

Finally we prove that if S is not complete (and $|A| \neq 2^{n-2}$) then A is periodic. Thus we suppose that S is not complete. By the foregoing this implies that there exists $x \in H_C$ with $x \neq c_0$ such that $x \notin A + B$. Then $x + A = B'$ and $c_0 + A = B'$. Therefore $v + A = x + c_0 + A = x + B' = A$. Therefore A is periodic with vertex v. □

Remark 3.4 *Note that the hypothesis that $|A| \neq 2^{n-2}$ is required. See Example 3.12.*

Suppose that S is a cap with $|C| = 1$ and $B' = c_0 + A$. Then $|S| = 2^{n-1} + 1$. Since $|S|$ is odd, S cannot be periodic. Thus by Lemma 2.1, we have that both A and B are not periodic. Hence we have proved the following result.

Theorem 3.5 *In $\Sigma = \mathbb{PG}(n,2)$, let S be a cap which meets a hyperplane K_C of Σ in a single point, c_0. Choose a codimension 2 subspace $H_\infty \cong \mathbb{PG}(n-2,2)$ contained in K_C and disjoint from S. Suppose $B' = c_0 + A$ and that $|A| \neq 2^{n-2}$. Then S is complete if and only if A is not periodic if and only if B is not periodic.*

Next we consider in detail the case where $|A| = |B| = 2^{n-2}$.

Suppose first that $A \oplus A \neq H_\infty$, i.e., suppose there exists $p \in H_\infty \setminus (A \oplus A)$. Then $p + A = A'$. Therefore $A \oplus A = (p + A') \oplus (p + A') = A' \oplus A' = (c_0 + A') \oplus (c_0 + A') = B \oplus B$. In particular, $p \notin B \oplus B$. Note that we have shown that whenever $|A| = |B| = 2^{n-2}$, we have $A \oplus A = B \oplus B = A' \oplus A' = B' \oplus B'$.

Lemma 3.6 *If $|E \cap H_\infty| \geq 2$ then A, B, A' and B' are each periodic.*

Proof Let $e_1, e_2 \in E \cap H_\infty$. Then $e_1 + e_2 + A = e_1 + A' = A$ and thus A is periodic. Similarly $e_1 + e_2$ is a vertex for B, A' and B'. □

Suppose A is not periodic. By the preceding lemma, $|E| \leq 1$. Hence either S is complete or $T = S \sqcup \{e\}$. Thus we have proved the following theorem.

Theorem 3.7 *In $\Sigma = \mathbb{PG}(n,2)$, let S be a cap which meets a hyperplane K_C of Σ in a single point, c_0. Choose a codimension 2 subspace $H_\infty \cong \mathbb{PG}(n-2,2)$ contained in K_C and disjoint from S. Suppose $B' = c_0 + A$. If S is complete then A and B are not periodic. If A (or B) is not periodic then either*

(i) *S is complete or*

(ii) *$|A| = |B| = 2^{n-2}$ and $\Sigma = S \sqcup (S \oplus S) \sqcup \{e\}$ where $e \in H_\infty$ and the complete cap $T = S \sqcup \{e\}$ is periodic with $V(T) = \{e + c_0\}$.*

Remark 3.8 *If $n \leq 4$ and $|A| = 2^{n-2}$ then $A \oplus A \neq H_\infty$ since then $|A \oplus A| \leq \binom{|A|}{2} < |H_\infty|$.*

The following four examples show that $A \oplus A$ may or may not equal H_∞ and A may or may not be periodic in all possible combinations when $|A| = 2^{n-2}$.

Example 3.9 *In $\mathbb{PG}(9,2)$ take $A_0 = \{8\}$, $A_1 = \{08, 18, 28, 38, 48, 58, 68, 78\}$, $A_5 = \{ijklm8 \mid 0 \leq i < j < k < l < m \leq 7\}$, $A_8 = \{012345678\}$ and $A^\sharp = A_0 \sqcup A_1 \sqcup A_5 \sqcup A_8$. With $K_A = (9)^\perp$ and $K_B = (8)^\perp$ we have $A^\sharp \oplus A^\sharp = H_\infty$ and $|A^\sharp| = 66$. Now the point $v := 01 \in H_\infty$ lies on 16 secants to $A^\sharp$ and thus $|A^\sharp \sqcup (v + A^\sharp)| = 116$. Hence we may easily extend $A^\sharp$ to a set $A \subset (K_A \setminus H_\infty)$ with $|A| = 128$ and $v + A = A$ by adding the 12 points of 6 new secant lines of v to $A^\sharp$. Then $A \oplus A = H_\infty$ and A is periodic.*

Example 3.10 *In* $\mathbb{PG}(5,2)$ *take* $A = \{4, 04, 14, 24, 34, 01234, 0134, 0234\}$ *where* $K_A = (5)^\perp$ *and* $K_B = (4)^\perp$. *Then* A *is not periodic and* $A \oplus A = H_\infty$. *Thus* $S = A \sqcup (45 + A) \sqcup \{45\}$ *is complete.*

Example 3.11 *In* $\mathbb{PG}(5,2)$ *take* $A = \{4, 04, 14, 014, 24, 024, 34, 034\}$ *where* $K_A = (5)^\perp$ *and* $K_B = (4)^\perp$. *Then* A *is periodic (with vertex 0) and* $A \oplus A \neq H_\infty$. *Here if* $c_0 = 45$ *then* $E = \{045, 123, 0123\}$.

Example 3.12 *In* $\mathbb{PG}(4,2)$ *take* $A = \{3, 03, 13, 23\}$ *where* $K_A = (4)^\perp$ *and* $K_B = (3)^\perp$. *Then* A *is not periodic and* $A \oplus A \neq H_\infty$. *Thus* $H_\infty \setminus (A \oplus A) = \{e = 012\}$ *and* $S = A \sqcup (c_0 + A) \sqcup \{c_0, e\}$ *is a complete cap.*

4 General Case

Next we suppose that the sets A and B are such that Equation 1.3 is satisfied for all i. Under this hypothesis we want to investigate conditions on A and B which are sufficient to guarantee that the S is a *complete* cap.

We suppose that $\{1, 2, \ldots, t\} = \{i \mid H_A(i) \subset S\}$ for some t with $0 \leq t \leq 2^{n-r-1}$. Thus $\{1, 2, \ldots, t\} = \{i \mid H_B(i) \cap S = \emptyset\}$. Further suppose that $\{t+1, t+2, \ldots, t+u\} = \{i \mid H_A(i) \cap S = \emptyset\}$ for some u with $0 \leq u \leq 2^{n-r-1} - t$. Then $\{t+1, t+2, \ldots, t+u\} = \{i \mid H_B(i) \subset S\}$. Thus the pair of cosets corresponding to $i = 1, 2, \ldots, t+u$ are precisely those having a trivial solution to Equation 1.3.

Let $\overline{\Sigma} \cong \mathbb{PG}(n-r, 2)$ denote the quotient geometry of Σ by F. This is just the projective geometry associated to the quotient of the vector space associated to $\Sigma = \mathbb{PG}(n, 2)$ by the vector space associated to F.

Let $\overline{a_i}$ and $\overline{b_i}$ denote the points of $\overline{\Sigma}$ corresponding to the cosets $H_A(i)$ and $H_B(i)$ respectively. Write $\overline{C}$ for the point of $\overline{\Sigma}$ corresponding to $\widetilde{F}$. Similarly we define $\overline{H_A}$, $\overline{H_B}$, etc. for the subsets of $\overline{\Sigma}$ corresponding to H_A, H_B, etc. respectively.

Corresponding to the full, empty and non-empty cosets of A and B we define the following subsets of $\overline{\Sigma}$. Put $\overline{A_f} = \{\overline{a_1}, \overline{a_2}, \ldots, \overline{a_t}\}$, $\overline{B_f} = \{\overline{b_{t+1}}, \overline{b_{t+2}}, \ldots, \overline{b_{t+u}}\}$, $\overline{A_e} = \{\overline{a_{t+1}}, \overline{a_{t+2}}, \ldots, \overline{a_{t+u}}\}$, $\overline{B_e} = \{\overline{b_1}, \overline{b_2}, \ldots, \overline{b_t}\}$, $\overline{A_{ne}} = \overline{H_A} \setminus \overline{A_e}$ and $\overline{B_{ne}} = \overline{H_B} \setminus \overline{B_e}$. Recall that $\widehat{C} = \widetilde{F} \setminus F$.

Theorem 4.1 *Suppose that* A *and* B *are such that Equation 1.3 is satisfied for all* $i = 1, 2 \ldots, 2^{n-r-1}$. *Suppose* $0 < t+u < 2^{n-r-1}$, *that* $t \neq u$, *and that either* $\overline{A_f}$ *or* $\overline{B_f}$ *is non-periodic. Further suppose that* $\widehat{C} \setminus C \subseteq \cup_{i=1}^{2^{n-r-1}} (A(i) + B(i))$. *Then* S *is a complete cap in* Σ.

Proof By construction, S is a cap in Σ. Also Equation 1.3 guarantees that every point of A' and every point of B' lies on at least one secant of S through a point of C. Thus we need to show that every point of H_∞ and every point of $H_C \setminus C$ lies on a secant of S.

Without loss of generality suppose that $\overline{A_f}$ is non-periodic. Consider the cap $\overline{S_1}$ in $\overline{\Sigma}$ given by $\overline{S_1} = \overline{A_f} \sqcup \overline{B_{ne}} \sqcup \{\overline{C}\}$. Applying Theorem 3.7 we see that for every point $\overline{c_k}$ of $\overline{H_C}$ different from $\overline{C}$ there exists $i \leq t$ and $j \geq t+1$

such that $\overline{a_i} + \overline{b_j} = \overline{c_k}$. Since the coset $H_A(i)$ is entirely contained in S, and $H_B(j) \cap S \neq \emptyset$, this means that every point of the coset $H_C(k) = H_A(i) + H_B(j)$ lies on a secant to S.

By assumption, every point of $\widehat{C} \setminus C$ is contained in $A + B$.

Hence it only remains to prove that every point of H_∞ lies on at least one secant to S. Note that every point of F lies in $H_A(1) \oplus H_A(1)$ (or in $H_B(1) \oplus H_B(1)$ if $t = 0$). Thus we consider a point $z \in H_\infty \setminus F$ and write $\overline{z}$ for the point of $\overline{\Sigma}$ corresponding to the coset of F generated by z. As above, if $\overline{z} \in \overline{A_f} + \overline{A_{ne}}$ or $\overline{z} \in \overline{B_f} + \overline{B_{ne}}$ then z lies on a secant line to S. Thus we assume, by way of contradiction, that $\overline{z} + \overline{A_f} \subset \overline{A_e}$ and $\overline{z} + \overline{B_f} \subset \overline{B_e}$. The first inclusion implies that $t \leq u$ while the second implies that $u \leq t$. Thus $t = u$ contradicting our hypothesis. □

Remark 4.2 *Note that if there exists at least one value of i for which a singleton solution occurs then $\widehat{C} \setminus C \subseteq \cup_{i=1}^{2^{n-r-1}} (A(i) + B(i))$. Conversely if $t + u = 2^{n-r-1}$ then $A(i) + B(i) = \emptyset$ for all i and thus $\widehat{C} \setminus C \not\subseteq \cup_{i=1}^{2^{n-r-1}} (A(i) + B(i))$. Hence if $t + u = 2^{n-r-1}$ then S is not complete. Furthermore, in order that $1 \leq t + u \leq 2^{n-r-1} - 1$ we must have $r \leq n - 2$.*

Remark 4.3 *Note that, if t or u is odd, then $\overline{A_f}$ or $\overline{B_f}$ respectively is non-periodic. Thus for a fixed value of $t + u$ with $1 \leq t + u \leq 2^{n-r-1} - 1$ we may take $t = 1$, for example, to arrange that $\overline{A_f}$ is non-periodic. Furthermore, if $t + u \neq 2$ we may simultaneously arrange that $t \neq u$.*

Remark 4.4 *There do exist complete caps for which $t + u = 0$ (and thus $\overline{A_f} = \overline{B_f} = \emptyset$). See Example 5.4. The condition that $t \neq u$ is also sufficient but not necessary.*

5 A family of examples

In this section we consider the case where $|C| = 2^r - 1$, i.e., where $\widehat{C} \setminus C$ is a single point $\{c_0\}$. If $r = 0$ then C is empty contrary to our earlier assumption. If $r = 1$ then $|C| = 1$ and we are in the case considered in Section 3. Thus we will assume that $r \geq 2$. Accordingly, we must have $n \geq 3$.

Using the cap property of S we see that if $|S \cap H_A(i)| \geq 2$ then $S \cap H_B(i) = \emptyset$. Furthermore if $S \cap H_B(i)$ is a single point, $\{\alpha_i\}$, then using the completeness of S we must have $S \cap H_B(i) = \{\beta_i := \alpha_i + c_0\}$. Finally if $S \cap H_B(i) = \emptyset$ then the completeness of S implies that $H_B(i) \subset S$.

Thus the only solutions to Equation 1.3 are the trivial solutions and the singleton solutions. Let s denote the number of these pairs of cosets which have the singleton solution.

Then $|S| = (2^r - 1) + 2s + 2^r(2^{n-r-1} - s) = 2^{n-1} + 2^r - 1 - (2^r - 2)s$. By Remark 4.2, s cannot equal 0. Furthermore by Theorem 4.1 (and Remark 4.3) there do exist complete caps of this form for all $s = 1, 2, \ldots, 2^{n-r-1} - 3$ and $s = 2^{n-r-1} - 1$. Thus we find complete caps of this form of all cardinalities: $2^{n-r} + k(2^r - 2) + 1$ for $k = 2, 4, 5, \ldots, 2^{n-r-1}$ where $2 \leq r \leq n - 2$.

Next we consider the case $s = 2^{n-r-1} - 2$. Then $u + v = 2$ and either $u = v$ or else both $\overline{A_f}$ and $\overline{B_f}$ are periodic. Reordering the cosets we may suppose $|H_A(i)| = |H_B(i)| = 1$ for $i = 3, 4, \ldots, 2^{n-r-1}$. Write $\{\alpha_i\} = H_A(i)$ and $\{\beta_i\} = H_B(i)$ for $i = 3, 4, \ldots, 2^{n-r-1}$. We consider first the case that $t = u = 1$. Examining the proof of Theorem 4.1 we see that every point of $\mathbb{PG}(n, 2)$ lies on a secant to S except possibly the 2^r points in the coset $H_A(1) + H_A(2) = H_B(1) + H_B(2) = \{z_1, z_2, \ldots, z_{2^r}\} \subset H_\infty$. These points z_k can only lie on secants of the form $\alpha_i + \alpha_j = z_k$ or $\beta_i + \beta_j = z_k$. Since $\beta_i + \beta_j = \alpha_i + \alpha_j$ we may restrict our attention to the α_i. Thus we see that S is complete if and only if for every $k = 1, 2, \ldots, 2^r$ there exists $3 \le i(k) < j(k) \le 2^{n-r-1}$ such that $\alpha_{i(k)} + \alpha_{j(k)} = z_k$. Since the α_i all lie in different cosets, it is clear that $\alpha_{i(1)}, \alpha_{j(1)}, \ldots, \alpha_{i(2^r)}, \alpha_{j(2^r)}$ must be distinct. Thus we must have $2(2^r) \le 2^{n-r-1} - 2$ or equivalently $2r + 3 \le n$.

The case $(t, u) = (2, 0)$ (or $(0, 2)$) is entirely similar to the above case. The only difference being that the 2^r points which may not lie on secants to S form a coset in H_C instead of in H_∞. In summary we see that there exists a complete cap S with $|\widehat{C} \setminus C| = 1$ and with $|S| = 2^{n-r} + 2^{r+1} + 2^r - 5$ (and $s = 2^{n-r-1} - 2$) if and only if $n \ge 2r + 3$.

The only remaining possibility is that $s = 2^{n-r-1}$, i.e., $t = u = 0$. Thus we now consider caps with $\widehat{C} \setminus C = \{c_0\}$ and with $|H_A(i)| = |H_B(i)| = 1$ for all $i = 1, 2, \ldots, 2^{n-r-1}$.

Lemma 5.1 *Let $2 \le r \le n - 1$ and suppose that $\widehat{C} \setminus C$ is a single point $\{c_0\}$ and that $|A(i)| = |B(i)| = 1$ for all $i = 1, 2, \ldots, 2^{n-r-1}$. Then S is a complete cap if and only if $A \oplus A = H_\infty \setminus F$.*

Proof Note that $A \oplus A = B \oplus B$ since $\alpha_i + \alpha_j = (\alpha_i + c_0) + (\alpha_j + c_0) = \beta_i + \beta_j$. Furthermore $A \oplus A \subseteq H_\infty \setminus F$ and $C \oplus C = F$. Thus $H_\infty \subset S \oplus S$ if and only if $A \oplus A = H_\infty \setminus F$.

Suppose now that $A \oplus A = H_\infty \setminus F$. By construction $A + C = B'$ and $B + C = A'$ thus S is complete if and only if $A + B = C'$. Now $A + B = A + (c_0 + A) = \{c_0\} \sqcup (c_0 + (A \oplus A)) = \{c_0\} \sqcup (c_0 + (H_\infty \setminus F)) = C'$. Hence we conclude that S is complete if and only if $A \oplus A = H_\infty \setminus F$. □

We will now attempt to determine those sets $A = \{\alpha_1, \alpha_2, \ldots, \alpha_{2^{n-r-1}}\}$ for which $A \oplus A = H_\infty \setminus F$. The remainder of this section will be devoted to answering this question. We will reduce this question to an equivalent geometric question in $\overline{K_A}$ where, as above, $\overline{K_A}$ denotes a hyperplane in $\overline{\Sigma}$, the quotient geometry of Σ with respect to F.

To do this we will fix some subspace $L \cong \mathbb{PG}(n - r - 1, 2) \subset K_A$ such that $L \cap F = \emptyset$. Choose an identification of one of the cosets $H_A(i)$, say $H_A(i)$, with $\mathbb{AG}(r, 2)$. Choose this identification so that the unique point $a_0 :=\in H_A(1) \cap L$ is the point identified with the zero vector in $\mathbb{AG}(r, 2)$. Note that with this identification if $u, v \in \mathbb{AG}(r, 2)$ correspond to $x, y \in H_A(1)$ respectively then $u + v \in \mathbb{AG}(r, 2)$ corresponds to $x + y + a_0 \in H_A(1)$.

Now for each point a of $K_A \setminus F$ there exists a unique point x of $\mathbb{AG}(r,2)$ such that $x + a \in L$. For each $x \in \mathbb{AG}(r,2)$ define $(K_A)_x := \{a \in (K_A \setminus F) \mid x + a \in L\}$. This partitioning of $K_A \setminus F$ induces a partitioning of A: $A = \sqcup_{x\in\mathbb{AG}(r,2)} A_x$ where $A_x := A \cap (K_A)_x$. Similarly we have the partitioning of H_∞: $H_\infty = \sqcup_{x\in\mathbb{AG}(r,2)}(H_\infty)_x$ where $(H_\infty)_x := H_\infty \cap (K_A)_x \cong \mathbb{PG}(n-r-2)$ for all $x \in \mathbb{AG}(r,2)$.

Example 5.2 *Take $n = 7$ and $r = 2$. Put $K_A = (7)^\perp$, $K_B = (6)^\perp$ and $K_C = (67)^\perp$. Then $H_\infty \cong \mathbb{PG}(5,2)$ is the projective subspace generated by $\{0,1,2,3,4,5\}$. Take $C = \{067, 167, 0167\}$ so that $c_0 = 67$. Then $\widehat{C} = \{067, 167, 0167, 67\}$, $\widetilde{F} \cong \mathbb{PG}(2,2)$ is generated by $\{0,1,67\}$ and F is the line $\{0,1,01\}$. Choose $L \cong \mathbb{PG}(4,2)$ to be the subspace of K_A generated by $\{2,3,4,5,6\}$. Then $L\cap F = \emptyset$ as required. We take $H_A(1)$ to be the coset containing 6 say. Thus $H_A(1) = \{6, 06, 16, 016\}$ and $a_0 = 6 \in H_A(1)\cap L$. Then for example if we choose our identification so that $u = 1 \in \mathbb{AG}(2,2)$ corresponds to $16 \in H_A(1)$ we have $(H_\infty)_u = (H_\infty)_{16} = \{1, 12, 13, 14, 15, 123, 124, 125, 134, 135, 145, 1234, 1235, 1245, 1345, 12345\}$.*

Now the useful property of this partition is that for $u, w \in \mathbb{AG}(r,2)$ we have $A_u \oplus A_v \subseteq (H_\infty)_{u+v}$. Thus $A \oplus A = H_\infty \setminus F$ if and only if for every $w \in \mathbb{AG}(r,2)$ we have $\cup_{u\in\mathbb{AG}(r,2)} (A_u \oplus A_{u+w}) = (H_\infty)_w \setminus F$. Note that the partition of A yields a partition of $\overline{A} = \overline{H_A}$:

$$\overline{H_A} = \sqcup_{x\in\mathbb{AG}(r,2)} \overline{A_x}.$$

Furthermore $A \oplus A = H_\infty \setminus F$ if and only if for every $w \in \mathbb{AG}(r,2)$ we have

$$\cup_{u\in\mathbb{AG}(r,2)} \left(\overline{A_u} \oplus \overline{A_{u+w}}\right) = \overline{(H_\infty)}. \tag{5.1}$$

Thus we have proved the following proposition:

Proposition 5.3 *Let $C \cong \mathbb{AG}(r,2) \setminus \{c_0\}$ and suppose $|A(i)| = |B(i)| = 1$ for all $i = 1, 2, \ldots, 2^{n-r-1}$. Then S is a complete cap if and only if the partition of $\mathbb{AG}(n-r,2) = \overline{H_A}$ induced by A satisfies Equation 5.1 for all $w \in \mathbb{AG}(r,2)$.*

Example 5.4 *We continue with the notation of the previous example. Thus $\overline{H_A} \cong \mathbb{AG}(4,2) \subset \overline{K_A} \cong \mathbb{PG}(4,2)$. Hence to obtain a complete cap S we require a partition of $\mathbb{AG}(4,2)$ into 4 subsets, $\overline{A_0}, \overline{A_1}, \overline{A_{01}}, \overline{A_\emptyset}$ satisfying Equation 5.1 for all $w \in \mathbb{AG}(2,2) = \{0, 1, 01, \emptyset\}$. It is not too difficult to find such partitions. For example, $\overline{A_\emptyset} := \emptyset$, $\overline{A_0} := \{6, 246, 346, 256, 2346, 2456\}$ $\overline{A_1} := \{236, 356, 456, 2356, 3456\}$, and $\overline{A_{01}} := \{26, 36, 46, 56, 23456\}$. This partition corresponds to the complete cap $S = A \sqcup B \sqcup C$ where which $A = \{1236, 1356, 1456, 12356, 13456, 06, 0246, 0346, 0256, 02346, 02456, 0126, 0136, 0146, 0156, 0123456\}$ and $B = c_0 + A = 01 + A$.*

Lemma 5.5 *Let $\mathbb{AG}(k,2) \subset \Lambda = \mathbb{PG}(k,2)$ denote the complement of a hyperplane H of Λ. Suppose there exists a partition of $\mathbb{AG}(k,2)$ indexed by $\mathbb{AG}(r,2)$: $\mathbb{AG}(k,2) = \sqcup_{w\in\mathbb{AG}(r,2)} X_w$ such that*

$$\cup_{u\in\mathbb{AG}(r,2)} \left(\overline{X_u} \oplus \overline{X_{u+w}}\right) = H \tag{5.1}$$

for all $w \in \mathbb{AG}(r, 2)$*. Then* $r \le k-2$*.*

Proof The partition of $\mathbb{AG}(k, 2)$ induces a partition of the secant lines of $\mathbb{AG}(k, 2)$ as follows. If $a_1 \in X_u$ and $a_2 \in X_v$ then we say the secant line through a_1 and a_2 is of type $u+v \in \mathbb{AG}(r, 2)$. Equation 5.1 is the condition that every point of H lies on at least one secant line of type w for all $w \in \mathbb{AG}(r, 2)$. Since $|H| = 2^k - 1$ and since there are $\binom{2^k}{2} = 2^{k-1}(2^k-1)$ secant lines we must have $2^{k-1}(2^k-1) \ge 2^r(2^k-1)$. Thus $k-1 \ge r$. Furthermore, if $r = k-1$ then every point of H must lie on exactly one secant line of each type $w \in \mathbb{AG}(r, 2)$.

Suppose then that $r = k-1$ and assume by way of contradiction that a partition of $\mathbb{AG}(k, 2)$ satisfying Equation 5.1 exists. Define $x_w := |X_w|$ for $w \in \mathbb{AG}(k, 2)$. Then we have $\sum_{u\in\mathbb{AG}(r,2)} x_w x_u = 2^k - 1$ for all w different from zero in $\mathbb{AG}(r, 2)$. Also $\sum_{u\in\mathbb{AG}(r,2)} \binom{x_u}{2} = 2^k - 1$. Since $\sum_{u\in\mathbb{AG}(r,2)} x_u = 2^r - 1$, this gives $\sum_{u\in\mathbb{AG}(r,2)} x_u^2 = 2^k - 2$. Thus

$$\begin{aligned} 2^{2k} &= (\sum_{u\in\mathbb{AG}(r,2)} x_u)^2 = \sum_{u\neq v} x_u x_v + \sum_{u\in\mathbb{AG}(r,2)} x_u^2 \\ &= 2^r(2^k-1) + 2^k - 2 = 2^{k-1}(2^k-1) + (2^k-1) - 1 = (2^k-1)(2^k). \end{aligned}$$

This contradiction shows that $r \le k-2$. □

Remark 5.6 *Clearly, if a partition of* $\mathbb{AG}(k, 2)$ *exists which satisfies Equation 5.1 for some value* r_0 *of* r *then such a partition exists for all values of* r *less than* r_0*.*

6 $|C| = 2$

Lemma 6.1 *Let* S *be a complete cap in* $\Sigma = \mathbb{PG}(n, 2)$ *satisfying* $|C| = 2$*. Then* $S = \phi(S')$ *where* $S' = S\cap\Sigma'$ *is a complete cap in a hyperplane* $\Sigma' \cong \mathbb{PG}(n-1, 2)$*. In particular,* $|S| = 2^{n-1} + 2$*.*

Proof Write $C = S \cap K_C = \{c_1, c_2\}$. Then C is periodic with vertex $v = c_1 + c_2 \in H_\infty$. By Lemma 2.1, S is also periodic with vertex v. Choose any hyperplane Σ' of Σ not containing v and put $S' = S \cap \Sigma'$. Then $S = \phi(S')$ (with respect to the vertex v). Since S is complete in Σ, it follows that S' is complete in Σ'. Also the hyperplane $K_C \cap \Sigma'$ of Σ' is a tangent hyperplane for S' and thus $|S'| = 2^{n-2} + 1$. Hence $|S| = 2|S'| = 2(2^{n-2} + 1) = 2^{n-1} + 2$. □

7 $|C| = 3$

Now suppose that S is a complete cap meeting a hyperplane K_C in 3 points. Choose a codimension 2 subspace $H_\infty \cong \mathbb{PG}(n-2, 2)$ contained in K_C and disjoint from S. Write $C = \{c_1, c_2, c_3\}$. Then $F = \{c_1 + c_2, c_1 + c_3, c_2 + c_3\}$ is a line in H_∞. Put $c_0 = c_1 + c_2 + c_3$. Then $\widetilde{F} = F \cup C \cup \{c_0\} \cong \mathbb{PG}(2, 2)$ and $\widehat{C} := C \cup \{c_0\} \cong \mathbb{AG}(2, 2)$. Thus $|C| = 3$ implies that we are in the case described in Section 5 with $r = 2$.

Thus by Theorem 4.1 (and Remark 4.3) we have maximal caps with $|C| = 3$ of all cardinalities $3+2m+4(2^{n-3}-m) = 2^{n-1}-2m+3$ where $1 \leq m \leq 2^{n-3}-3$ or $m = 2^{n-3} - 1$, i.e., of cardinalities $2^{n-2} + 5, 2^{n-2} + 9, 2^{n-2} + 11, 2^{n-2} + 13, \ldots, 2^{n-1} + 1$ for $n \geq 4$.

Now we show that there for all $n \geq 3$ there exist complete caps with $|C| = 3$ and with $|A(i)| = |B(i)| = 1$ for all $i = 1, 2, \ldots, 2^{n-3}$. By Proposition 5.3 we must find a partition of $\mathbb{AG}(n-2, 2)$ into 4 subsets indexed by $\mathbb{AG}(2, 2)$ satisfying Equation 5.1. There are very many ways to do this. We give one construction here. We proceed by induction on $k = n - 2$. Since we have assumed that $r \geq 2$ the first case is $k = 4$. We exhibited such a partition for $k = 4$ in Example 5.2. Now we will inductively construct from this example a partition for all values of $k \geq 5$. Suppose we have a partition of $\mathbb{AG}(k, 2)$ into four sets $\overline{A_\emptyset}$, $\overline{A_0}$, $\overline{A_1}$ and $\overline{A_{01}}$ where we further suppose that $\overline{A_\emptyset} = \emptyset$. Write $\mathbb{PG}(k + 1, 2) = \phi(\mathbb{PG}(k, 2))$ with respect to some vertex $z \in PG(k + 1, 2) \setminus \mathbb{PG}(k, 2)$. Then $\mathbb{AG}(k + 1, 2) = \phi(\mathbb{AG}(k, 2))$. Let $\widetilde{H} = \mathbb{PG}(k + 1, 2) \setminus \mathbb{AG}(k + 1, 2) = \phi(H)$. We partition $\mathbb{AG}(k + 1, 2)$ into 4 sets: $\mathbb{AG}(k + 1, 2) = \widetilde{A_\emptyset} \sqcup \widetilde{A_0} \sqcup \widetilde{A_1} \sqcup \widetilde{A_{01}}$ by defining $\widetilde{A_w} := \phi(\overline{A_w}) = \overline{A_w} \sqcup (z + \overline{A_w})$. This partition almost satisfies Equation 5.1. However the point $z \in \widetilde{H}$ is not contained in any of the three sets $\widetilde{A_0} + \widetilde{A_1}$, $\widetilde{A_0} + \widetilde{A_{01}}$ and $\widetilde{A_1} + \widetilde{A_{01}}$. To overcome this defect we modify the partition slightly. Note that every point of $\widetilde{H}$ different from z lies on at least two secants of each type $u \in \mathbb{AG}(2, 2)$. Consider the three points of $\mathbb{AG}(k, 2)$: $a_0 := 6$, $a_1 := 236$ and $a_{01} := 26$ and the line in H: $z_{01} := a_0 + a_1$, $z_1 := a_0 + a_{01}$ and $z_0 := a_1 + a_{01}$. Each of the three points of this line lies on secants of all types which do not utilize a_0, a_1 nor a_{01}. Specifically we have $z_{01} = 23 = 256 + 356 = 2346 + 46 = 456 + 23456$, $z_1 = 2 = 2456 + 456 = 246 + 46 = 356 + 56$ and $z_0 = 3 = 256 + 2356 = 346 + 46 = 356 + 56$. We shift $z + a_0$ into $\widetilde{A_1}$, $z + a_1$ into $\widetilde{A_{01}}$ and $z + a_{01}$ into $\widetilde{A_0}$. This modified partition now satisfies Equation 5.1. Also note that we have not shifted any of the points a_0, a_1, a_{01} nor indeed any of the points of $\mathbb{AG}(k, 2)$ and so our induction proceeds using the same points a_0, a_1, a_{01} at each stage.

In conclusion we see that there are complete caps, S in $\mathbb{PG}(n, 2)$ with $|C| = 3$ of cardinality $2^{n-2} + 3$ for all $n \geq 7$. Thus we have the following.

Proposition 7.1 *Suppose $n \geq 4$ and that S is a complete cap in $\mathbb{PG}(n, 2)$. Further suppose that S meets a hyperplane K_C of $\mathbb{PG}(n, 2)$ in exactly 3 points. Then $|S|$ is one of the numbers: $2^{n-2}+5, 2^{n-2}+9, 2^{n-2}+11, 2^{n-2}+13, \ldots, 2^{n-1}+1$ or, if $n \geq 7$, $|S|$ may also be $2^{n-2} + 3$ or $2^{n-2} + 7$. Moreover complete caps of all these sizes exist for all corresponding values of n.*

8 $|C| = 4$

In this section we assume that S is a complete cap meeting a hyperplane K_C in 4 points. As usual we choose a codimension 2 subspace $H_\infty \cong \mathbb{PG}(n - 2, 2)$ contained in K_C and disjoint from S. Note that when $|K_C \cap S| = 4$ there are two possibilities; either these four points lie in a Fano plane or they do not. Now if they do lie in a Fano plane they form a periodic set and it follows as in Section 6 that if S complete then $|S| = 2^{n-1} + 4$ and S is large.

It remains to consider the case where $S \cap K_C$ consists of four points spanning a three dimensional projective subspace of K_C. In order to determine all solutions to Equation 1.3 we first assume that this three dimensional subspace is K_C and thus that $n = 4$. Without loss of generality, $C = \{04, 14, 24, 34\}$, $K_C = (01234)^\perp$, $K_B = (4)^\perp$ and $K_A = (0123)^\perp$. Thus $H_\infty = \{01, 02, 03, 12, 13, 23, 0123\}$ and $H_A = H_A(1) = \{4, 014, 024, 034, 124, 134, 234, 01234\}$.

Let G denote the group of projectivities of $\Sigma = \mathbb{PG}(4,2)$ which stabilize, C, K_A, K_B and K_C. Then G contains the subgroup L of projectivities which permute the points $0, 1, 2$ and 3 and which also fix the point 4. The group G also contains the projectivities f and h defined by: $f(0) = 0$, $f(1) = 0$, $f(2) = 0124$, $f(3) = 0134$ and $f(4) = 014$ and $h(0) = 123$, $h(1) = 023$, $h(2) = 013$, $h(3) = 012$, and $h(4) = 01234$.

Suppose that A and B satisfy Equation 1.3 and further that A and B do not correspond to either trivial or singleton solutions of that equation. Thus $|A| \geq 2$ and $|B| \geq 2$. Since G acts transitively on H_A we may assume without loss of generality that $01234 \in A$. Now the subgroup of G fixing 01234 contains the group L. The action of L decomposes the remaining points of H_A into two orbits: $\{4\}$ and $\{014, 024, 034, 124, 134, 234\}$. Thus without loss of generality we have two cases to consider: (i) $A \supset \{01234, 4\}$ and (ii) $A \supset \{01234, 014\}$. These two cases are distinguished by the fact that in the former case the point $z_0 := c_1 + c_2 + c_3 + c_3 = 04 + 14 + 24 + 34 = 0123$ lies on the secant through the two given points of A while in the latter case it does not.

In the first case, we see that $B' = A + C \supset \{01234, 4\} + C = H_B$ and thus we have the trivial solution $A = H_A$ and $B = \emptyset$. In the second case $B' = A + C \supset \{01234, 014\} + C = H_B \setminus \{2, 3\}$. Since $|B| \geq 2$, we must have $B = \{2, 3\}$ and $A = \{01234, 014\}$.

Thus we find only one new solution to Equation 1.3. This solution is determined by $A(i) = \{a_1, a_2\}$, $a_1 + a_2 \neq z_0$, and $B(i) = \{b_1, b_2\} = H_B(i) \setminus (A(i) + C)$ where $z_0 = c_1 + c_2 + c_3 + c_4$. Note that $b_1 + b_2 = a_1 + a_2$. Thus we have five solutions in total: the two trivial solutions, the two singleton solutions and this new solution. Notice that for the new solution and for both of the singleton solutions we have $z_0 \notin A(i) \oplus A(i)$ and $z_0 \notin B(i) \oplus B(i)$. Since $z_0 \notin C \oplus C$, if S is complete there must be at least one i for which the trivial solution occurs. In particular, we may apply Theorem 4.1 to obtain complete caps S with $|C| = 4$ and C not periodic.

Suppose that S is a complete cap with $|C| = 4$. If C is periodic then $|S| = 2^{n-1} + 4$. Suppose then that C is not periodic and that m of the pairs of cosets $H_A(i), H_B(i)$ utilize the new solution, s of them utilize singleton solutions and $t + u = 2^{n-4} - s - m$ of them utilize the trivial solution. Note that if $s = 0$ then we must have $m \geq 2$ in order that $\widehat{C} \setminus C \subseteq \cup_{i=1}^{2^{n-r-1}} (A(i) + B(i))$. The cardinality of S is then given by $|S| = 4 + 4m + 5s + 8(2^{n-4} - s - m) = 2^{n-1} - 3(s + m) - m + 4$ where $0 \leq m, s \leq 2^{n-4} - 1$ and $1 \leq m + s \leq 2^{n-4} - 1$ (and $(m, s) \neq (1, 0)$). Thus if $|C| = 4$ and C is not periodic then $n \geq 5$ and we find complete caps S of all cardinalities satisfying $2^{n-2} + 8 \leq |S| \leq 2^{n-1} - 2$. In summary we have the following.

Proposition 8.1 *Suppose $n \geq 5$ and that S is a complete cap in $\mathbb{PG}(n,2)$. Further suppose that S meets a hyperplane K_C of $\mathbb{PG}(n,2)$ in exactly 4 points. Then $2^{n-2}+8 \leq |S| \leq 2^{n-1}-2$ or else $|S| = 2^{n-1}+1$. Moreover except for $n=6$ for sizes 29 and 30, complete caps of all these sizes and this structure exist for all $n \geq 5$.*

Acknowledgements

I thank Aiden Bruen for many useful conversations. This research was supported by grants from ARP and NSERC.

References

[1] A.E. Brouwer, A. A. Bruen, and D.L. Wehlau, There exist caps which block all subspaces of fixed codimension in $\mathbb{PG}(n,2)$, *J.C.T. A* **73**, #1 (1996) 168–169.

[2] Aiden A. Bruen, Lucien Haddad and David L. Wehlau, Binary Codes and Caps, *J. Combin. Des.* **6**, #4 (1998) 275–284.

[3] Aiden A. Bruen and David L. Wehlau, Long Binary Linear Codes and Large Caps in Projective Space, *D.C.C.* **17**, (1999) 37–60.

[4] Aiden A. Bruen and David L. Wehlau, New Codes From Old; A New Geometric Construction (to appear in *J.C.T. A*).

[5] A.A. Davydov and L.M. Tombak, Quasiperfect Linear Binary Codes with Distance 4 and Complete Caps in Projective Geometry, *Problems of Information Transmission* **25** No. 4 (1990) 265–275.

David L. Wehlau
Department of Mathematics and Computer Science
Royal Military College
Kingston, Ontario
Canada K7K 7B4
`wehlau@rmc.ca`