On Games with Coupled Constraints

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***Abstract*— We study the problem of cost minimization in competitive resource allocation problems, motivated by our previous work on power minimization in MIMO interference systems. Our setup leads to a general cost minimization game in which each player wishes to minimize the cost of its resource consumption while achieving a target utility level. In general, the player strategies are coupled through both their cost functions and their utility functions. Equilibrium exists only for a certain set of target utility levels which in general is a proper set of all achievable utility levels. To characterize the set of equilibrium utility levels, we introduce the dual of a cost minimization game called a utility maximization game in which each player wishes to maximize its utility while keeping the cost of its resource consumption below a cost threshold. We associate the set of equilibrium utility levels with the set of equilibrium of the dual game corresponding to all cost thresholds, and show that the dual game always possesses an equilibrium. We also obtain an inner estimate of the set of equilibrium utility levels in the case of decoupled cost functions by a minimax approach. We then relax the hard constraint on achieving a target utility level, and introduce a weighted cost minimization game which always possesses an equilibrium. We recover the original equilibria through the equilibria of the weighted cost minimization game as the penalty on not achieving the target utility levels increases.**

I. NOTATION

- $:=$ stands for "defined as".
- $\equiv$ stands for "identically equal to".
- $\dagger$ denotes the conjugate transpose.
- I denotes an identity matrix of proper dimension.
- $\mathbb{R}$ denotes the set of real numbers.
- $\mathbb{R}^n$ denotes the $n-$dimensional Euclidian vector space.
- $\mathbb{R}^n_+ = \{x \in \mathbb{R}^n : x_i \geq 0, \text{ for all } i \in \{1, \dots, n\}\}$.
- $\mathbb{R}^n_{++} = \{x \in \mathbb{R}^n : x_i > 0, \text{ for all } i \in \{1, \dots, n\}\}$.
- $-k$ denotes the set of indices other than k; for example, for $x \in \mathbb{R}^n$, $x_{-k} = (x_1, \dots, x_{k-1}, x_{k+1}, \dots, x_n)$.
- $\mathcal{H}_+$ denotes the set of positive semi-definite matrices of proper dimension.
- $(a)^+ = \max\{a, 0\}$, for a real a.
- $\mathcal{F} : \mathcal{X} \rightrightarrows \mathcal{Y}$ indicates that $\mathcal{F}$ is a correspondence (a set-valued mapping) from $\mathcal{X}$ to the set of subsets of $\mathcal{Y}$.
- $\mathrm{cl}(\cdot)$ denotes the closure.
- $\mathrm{co}(\cdot)$ denotes the convex hull.

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II. INTRODUCTION

We introduce a problem of power optimization in MIMO interference systems, studied in our previous work [1], as a motivating application for this paper. A MIMO interference system consists of L communication links where each link has a transmitter and a receiver; see [2], [3] for further analysis and a literature review. There are n_t antennas in each transmitter site, and n_r antennas in each receiver site. The user of each link k sends an n_t dimensional complex signal vector x_k. As a result, an n_r dimensional complex signal vector y_k is received at the receiver of link k. The received signal vector y_k is given as

$$\mathrm{y}_k = \mathrm{H}_{k,k}\mathrm{x}_k + \sum_{\ell \neq k} \mathrm{H}_{k,\ell}\mathrm{x}_\ell + \mathrm{n}_k$$

where $\mathrm{H}_{k,\ell}$ is the complex channel matrix between the $\ell-$th transmitter and the $k-$th receiver, and n_k is the circularly symmetric complex Gaussian noise vector at the $k-$th receiver with $E(\mathrm{n}_k) = 0$ and $E(\mathrm{n}_k\mathrm{n}_k^\dagger) = \mathrm{I}$. The user of each link k decides on the distribution of x_k to satisfy a Quality of Service (QoS) requirement

$$I(\mathrm{x}_k; \mathrm{y}_k) \geq r_k \tag{1}$$

with minimum power consumption $E(\mathrm{x}_k^\dagger\mathrm{x}_k)$ where $I(\mathrm{x}_k; \mathrm{y}_k)$ is the mutual information for the link k and $r_k \in \mathbb{R}_+$ is a target rate. The user of each link k views its total interference $\sum_{\ell \neq k} \mathrm{H}_{k,\ell}\mathrm{x}_\ell$ as a zero-mean circularly symmetric complex Gaussian noise vector. In this case, the power consumption $E(\mathrm{x}_k^\dagger\mathrm{x}_k)$ is minimized by a zero-mean circularly symmetric complex Gaussian distribution satisfying (1); see [2]. Moreover, the mutual information for link k takes the form

$$I(\mathrm{x}_k; \mathrm{y}_k) = \log_2 \det\big(\mathrm{I} + \mathrm{R}_k^{-1/2}\mathrm{H}_{k,k}\mathrm{Q}_k\mathrm{H}_{k,k}^\dagger\mathrm{R}_k^{-1/2}\big)$$
$$\mathrm{R}_k := \mathrm{I} + \sum_{\ell \neq k} \mathrm{H}_{k,\ell}\mathrm{Q}_\ell\mathrm{H}_{k,\ell}^\dagger$$

where $\mathrm{Q}_k := E(\mathrm{x}_k\mathrm{x}_k^\dagger) \in \mathcal{H}_+$; see [2].

Therefore, the power minimization problem for the user of each link k reduces to choosing a covariance matrix Q_k from the set of feasible strategies

$$\mathcal{F}_k^p(\mathrm{Q}_{-k}) := \{\mathrm{Q}_k \in \mathcal{H}_+ : r_k - I(\mathrm{x}_k; \mathrm{y}_k) \leq 0\}$$

in order to minimize the cost $\mathrm{trace}(\mathrm{Q}_k)$. Note that, the set $\mathcal{F}_k^p(\mathrm{Q}_{-k})$ of feasible strategies for each link k depends on the decisions Q_{-k} of the other links, even though the cost $\mathrm{trace}(\mathrm{Q}_k)$ of each link k depends only on its own decision Q_k. This means that the users of L links are engaged in a noncooperative game with coupled constraints [4], [5], which motivates us to introduce a general cost minimization game in the next section.

III. A COST MINIMIZATION GAME

Motivated by the power optimization problem introduced in Section II, we abstract out a general cost minimization game. We start with a finite player set $\mathcal{P} := \{1, \ldots, P\}$. Each player $p \in \mathcal{P}$ has a strategy set $\mathcal{S}_p$ which is a nonempty closed convex cone (with the vertex at 0) in $\mathbb{R}^{n_p}$ where $n_p \geq 1$ is a finite integer. It is convenient to think of $\mathcal{S}_p$ as the set of resources available to player p. Hence, for any non-zero $s_p \in \mathcal{S}_p$, the semi-infinite ray $\{\alpha s_p : \alpha \in \mathbb{R}_+\}$ can be interpreted as the set of resources of the same type as that of s_p but with different intensities. Whereas, if $s_p^1, s_p^2 \in \mathcal{S}_p$ are linearly independent, then s_p^1 and s_p^2 can be interpreted as different types of resources. In our cost minimization game, all players choose their strategies simultaneously and their collective choices are represented by some profile of strategies $s \in \mathcal{S}$ where $\mathcal{S} := \bigtimes_{p \in \mathcal{P}} \mathcal{S}_p$. For any $s \in \mathcal{S}$ and $p \in \mathcal{P}$, $s_p \in \mathcal{S}_p$ denotes the strategy chosen by player p, i.e., $p-$th entry in s, whereas $s_{-p} \in \mathcal{S}_{-p}$ denotes the profile of strategies chosen by all players other than player p, where $\mathcal{S}_{-p} := \bigtimes_{q \in \mathcal{P} - \{p\}} \mathcal{S}_q$. We sometimes write $s \in \mathcal{S}$ as $s = (s_p, s_{-p})$, for some $p \in \mathcal{P}$.

If all players choose the strategy profile $s \in \mathcal{S}$, then each player $p \in \mathcal{P}$ incurs the cost $c_p(s)$ and receives the utility $u_p(s)$ where $c_p : \mathcal{S} \mapsto \mathbb{R}_+$ and $u_p : \mathcal{S} \mapsto \mathbb{R}_+$ denote player p's cost and utility functions, respectively. We make the following assumption throughout the paper without further mention.

Assumption 1: For all $p \in \mathcal{P}$, $(s_p, s_{-p}) \in \mathcal{S}$ with $s_p \neq 0$,

(i) c_p, u_p are continuous in $\mathcal{S}$
(ii) $c_p(\cdot, s_{-p})$ is convex in $\mathcal{S}_p$, $u_p(\cdot, s_{-p})$ is concave in $\mathcal{S}_p$
(iii) $c_p(0, s_{-p}) = u_p(0, s_{-p}) = 0$
(iv) $c_p(\alpha s_p, s_{-p})$, $u_p(\alpha s_p, s_{-p})$ are (strictly) increasing in $\alpha \in \mathbb{R}_+$
(v) for some $M \geq 0$, $\inf_{\hat{s} \in \mathcal{S}: \|\hat{s}_p\| \geq M} c_p(\hat{s}) > 0$.

Part (i) and (ii) of Assumption 1 are for technical reasons. Part (iii) means that using no resources has no cost and yields no utility. Part (iv) ensures sensible behavior expected of cost and utility functions in a resource allocation problem. Part (v) requires that the cost of any resource $\hat{s}_p \in \mathcal{S}_p$ with large enough intensity is uniformly higher than a certain nonzero level for all $\hat{s}_{-p} \in \mathcal{S}_{-p}$. Clearly, Assumption 1 holds in the problem of power optimization in MIMO interference systems[1].

We note that, under Assumption 1, the following are true: for all $p \in \mathcal{P}$, $(s_p, s_{-p}) \in \mathcal{S}$ with $s_p \neq 0$,

- $0 \in \mathcal{S}_p$ is the unique global minimizer of $c_p(\cdot, s_{-p})$ and $u_p(\cdot, s_{-p})$
- $\sup_{\alpha \in \mathbb{R}_+} c_p(\alpha s_p, s_{-p}) = \infty$.

Given the setup above, the objective of each player $p \in \mathcal{P}$ is to choose a strategy $s_p \in \mathcal{S}_p$ with minimal cost while achieving a certain utility level $\bar{u}_p \in \mathbb{R}_+$. More precisely, each player $p \in \mathcal{P}$ is to solve the following cost minimization problem for some target utility level $\bar{u}_p \in \mathbb{R}_+$:

$$\min_{s_p \in \mathcal{C}_p(s_{-p})} c_p(s_p, s_{-p}) \tag{2}$$

where $\mathcal{C}_p(s_{-p}) := \{\check{s} \in \mathcal{S}_p : u_p(\check{s}, s_{-p}) \geq \bar{u}_p\}$ (without the knowledge of $s_{-p} \in \mathcal{S}_{-p}$, in actuality!). Note that, for any $s_{-p} \in \mathcal{S}_{-p}$, $\mathcal{C}_p(s_{-p})$ is closed and convex; moreover, $\mathcal{C}_p(s_{-p})$ is *unbounded*, whenever it is nonempty.

Lemma 1: For any $s_{-p} \in \mathcal{S}_{-p}$, if $\mathcal{C}_p(s_{-p}) \neq \emptyset$, then the minimum in (2) is achieved by some $\hat{s}_p \in \mathcal{C}_p(s_{-p})$ such that $u_p(\hat{s}_p, s_{-p}) = \bar{u}_p$.

Proof: Clearly, if the minimum in (2) is achieved by some $\hat{s}_p \in \mathcal{C}_p(s_{-p})$, then $u_p(\hat{s}_p, s_{-p}) = \bar{u}_p$. To show that the minimum in (2) is achieved by some $\hat{s}_p \in \mathcal{C}_p(s_{-p})$, let

$$\underline{c}_p(s_{-p}) := \inf_{s_p \in \mathcal{C}_p(s_{-p})} c_p(s_p, s_{-p}).$$

We have $\underline{c}_p(s_{-p}) \in \mathbb{R}_+$. There exists a sequence $\{s_p^n\}_{n \geq 1}$ in $\mathcal{C}_p(s_{-p})$ such that $c_p(s_p^n, s_{-p}) \to \underline{c}(s_{-p})$.

Suppose that $\sup_{n \geq 1} \|s_p^n\| = \infty$. Then $\{s_p^n\}_{n \geq 1}$ has a subsequence $\{s_p^{n_k}\}_{k \geq 1}$ such that $\|s_p^{n_k}\| \geq k$. Since $c_p(\cdot, s_{-p})$ is convex in $\mathcal{S}_p$, we have

$$0 \leq c_p\left(s_p^{n_k} / \|s_p^{n_k}\|, s_{-p}\right) \leq \frac{1}{k} c_p(s_p^{n_k}, s_{-p}) \to 0.$$

The properties of c_p implies that $s_p^{n_k} / \|s_p^{n_k}\| \to 0$, which is false. Therefore, $\sup_{n \geq 1} \|s_p^n\| < \infty$.

Since $\sup_{n \geq 1} \|s_p^n\| < \infty$, $\{s_p^n\}_{n \geq 1}$ has a subsequence $\{s_p^{n_\ell}\}_{\ell \geq 1}$ converging to some $\hat{s}_p \in \mathcal{C}_p(s_{-p})$. Since c_p is continuous in $\mathcal{S}$, we have

$$c_p(s_p^{n_\ell}, s_{-p}) \to c_p(\hat{s}_p, s_{-p}) = \underline{c}(s_{-p}).$$

■

We will refer to the cost minimization game corresponding to the target utility levels $\bar{u} \in \mathbb{R}_+^P$ as $\Gamma^c(\bar{u})$. A profile of strategies $s^* \in \mathcal{S}$ that mutually solves each player's cost minimization problem (2) is called a *generalized Nash equilibrium*[2]. In other words, a profile of strategies $s^* \in \mathcal{S}$ is an equilibrium if and only if, for all $p \in \mathcal{P}$,

$$c_p(s_p^*, s_{-p}^*) = \min_{s_p \in \mathcal{C}_p(s_{-p}^*)} c_p(s_p, s_{-p}^*).$$

An equilibrium can also be regarded as a fixed point of the best response correspondence $BR^c : \mathcal{S} \rightrightarrows \mathcal{S}$, where $BR^c = (BR_1^c, \ldots, BR_P^c)$ is given by: for all $p \in \mathcal{P}$ and $s_{-p} \in \mathcal{S}_{-p}$,

$$BR_p^c(s_{-p}) := \operatorname*{argmin}_{s_p \in \mathcal{C}_p(s_{-p})} c_p(s_p, s_{-p}).$$

With this notation, a profile of strategies $s^* \in \mathcal{S}$ constitutes an equilibrium if and only if

$$s^* \in BR^c(s^*).$$

At equilibrium, no player has an incentive to unilaterally deviate to an alternative strategy. Hence, the concept of

[1] If any channel matrix $\mathrm{H}_{k,k}$ is rank deficient, then the set of strategies need to be restricted in such a way that part (iv) of Assumption 1 is satisfied.

[2] We henceforth refer to a generalized Nash equilibrium simply as an equilibrium.

equilibrium is quite relevant in situations where optimizing the overall system is not feasible. In the context of power optimization in MIMO interference systems, a group of selfish links interested in minimizing their own power consumptions may settle only at an equilibrium. Hence, it is of interest to study the properties of equilibrium, starting with its existence.

A. Existence of Equilibrium

In noncooperative games, a well-known method for showing the existence of equilibrium is to use the various fixed point theorems available in the literature. A specialization of an existence result from the literature, namely Theorem 4.3.1 in [6], which relies on Kakutani's fixed point theorem, is given below using our own notation.

Theorem 1 (A specialization of Theorem 4.3.1 in [6]): Consider a noncooperative game with the set $\mathcal{P} = \{1, ..., P\}$ of players in which each player $p \in \mathcal{P}$ chooses $x_p \in \mathcal{X}_p$ to solve $\max_{x_p \in \mathcal{F}_p(x_{-p})} f_p(x_p, x_{-p})$ where

(i) $\mathcal{X}_p$ is a nonempty convex compact subset of $\mathbb{R}^{n_p}$ ($n_p \geq 1$ is a finite integer)
(ii) $\mathcal{F}_p : \mathcal{X}_{-p} \rightrightarrows \mathcal{X}_p$ is both upper semi-continuous[3] (u.s.c.) and lower semi-continuous[4] (l.s.c.) in $\mathcal{X}_{-p}$ ($\mathcal{X}_{-p} := \bigtimes_{q \in \mathcal{P}-\{p\}} \mathcal{X}_q$)
(iii) for all $x_{-p} \in \mathcal{X}_{-p}$, $\mathcal{F}_p(x_{-p})$ is nonempty, closed, and convex
(iv) $f_p : \mathcal{X} \mapsto \mathbb{R}$ is continuous in $\mathcal{X}$ ($\mathcal{X} := \bigtimes_{p \in \mathcal{P}} \mathcal{X}_p$)
(v) for all $x_{-p} \in \mathcal{X}_{-p}$, $f_p(\cdot, x_{-p})$ is quasi-concave in $\mathcal{X}_p$.

The game described above possesses an equilibrium.

Remark 1: If, for all $p \in \mathcal{P}$, $\mathcal{F}_p \equiv \bar{\mathcal{F}}_p$ where $\bar{\mathcal{F}}_p$ is some nonempty closed convex subset of $\mathcal{X}_p$, then condition (ii) and (iii) of Theorem 1 are satisfied.

It is tempting to apply Theorem 1 to a cost minimization game $\Gamma^c(\bar{u})$ by letting

$$\mathcal{X}_p = \mathcal{S}_p, \quad f_p = -c_p, \quad \mathcal{F}_p = \mathcal{C}_p, \quad \text{for all } p \in \mathcal{P}.$$

The main difficulty is that condition (i) of Theorem 1 is not satisfied, because $\mathcal{S}_1, \ldots, \mathcal{S}_P$ are not compact (although they are nonempty, closed, and convex, by assumption). This difficulty, namely the unboundedness of $\mathcal{S}_1, \ldots, \mathcal{S}_P$, can be circumvented, if there are nonempty convex compact subsets $\bar{\mathcal{S}}_p \subset \mathcal{S}_p$, for all $p \in \mathcal{P}$, such that $\bar{\mathcal{S}} := \bigtimes_{p \in \mathcal{P}} \bar{\mathcal{S}}_p$ is stable under BR^c, i.e.,

$$BR^c(\bar{\mathcal{S}}) := \{s \in \mathcal{S} : s \in BR^c(\bar{s}), \bar{s} \in \bar{\mathcal{S}}\} \subset \bar{\mathcal{S}}.$$

If this is indeed the case and the other conditions of Theorem 1 are satisfied, then the restriction of $\Gamma^c(\bar{u})$ to $\bar{\mathcal{S}}$ would possess an equilibrium which would also be an equilibrium of $\Gamma^c(\bar{u})$.

However, finding such subsets $\bar{\mathcal{S}}_1, \ldots, \bar{\mathcal{S}}_P$ itself requires an equilibration process, which is not necessarily an easier task than establishing the existence of an equilibrium strategy profile. For instance, there is no obvious way of accomplishing this for the problem of power optimization in MIMO interference systems, in general. To overcome this obstacle, we explore a duality relation with a utility maximization game introduced in the next section.

IV. A UTILITY MAXIMIZATION GAME

Using the same setup as in the previous section, a utility maximization game is introduced as a noncooperative game in which each player $p \in \mathcal{P}$ is to maximize its utility while keeping its cost below a certain level. More precisely, each player $p \in \mathcal{P}$ is to solve the following utility maximization problem for some target cost level $\bar{c}_p \in \mathbb{R}_+$:

$$\max_{s_p \in \mathcal{U}_p(s_{-p})} u_p(s_p, s_{-p}) \tag{3}$$

where $\mathcal{U}_p(s_{-p}) := \{\check{s}_p \in \mathcal{S}_p : c_p(\check{s}_p, s_{-p}) \leq \bar{c}_p\}$. Note that $\mathcal{U}_p(s_{-p})$ is nonempty, convex, and compact. Hence, the maximum in (3) is always achieved by some $\hat{s}_p \in \mathcal{U}_p(s_{-p})$ such that $c_p(\hat{s}_p, s_{-p}) = \bar{c}_p$.

We will refer to the utility maximization game corresponding to the target cost levels $\bar{c} \in \mathbb{R}_+^P$ as $\Gamma^u(\bar{c})$. The concept of equilibrium and the best response correspondence BR^u for $\Gamma^u(\bar{c})$ are defined in a completely analogous way as in the case of a cost minimization game $\Gamma^c(\bar{u})$.

A. A Duality Relation

The relevance of a utility maximization game in the context of this paper is due to the following duality relation.

Proposition 1: Fix $\bar{s} \in \mathcal{S}$, and let

$$\bar{c} := (c_1(\bar{s}), \ldots, c_P(\bar{s})) \quad \text{and} \quad \bar{u} := (u_1(\bar{s}), \ldots, u_P(\bar{s})).$$

Let $\mathcal{E}^c(\bar{u})$ and $\mathcal{E}^u(\bar{c})$ denote the sets of equilibria for the games $\Gamma^c(\bar{u})$ and $\Gamma^u(\bar{c})$, respectively. Then,

$$\bar{s} \in \mathcal{E}^c(\bar{u}) \qquad \Leftrightarrow \qquad \bar{s} \in \mathcal{E}^u(\bar{c}).$$

Proof: Suppose that $\bar{s} \notin \mathcal{E}^u(\bar{c})$. Hence, for some $p \in \mathcal{P}$, there exists an $\hat{s}_p \in \mathcal{S}_p$ such that

$$u_p(\hat{s}_p, \bar{s}_{-p}) > u_p(\bar{s}_p, \bar{s}_{-p}) \quad \text{and} \quad c_p(\hat{s}_p, \bar{s}_{-p}) \leq \bar{c}_p.$$

Because of the strict inequality above, we must have $\hat{s}_p \neq 0$. This implies that, for some $\alpha \in (0, 1)$,

$$c_p(\alpha\hat{s}_p, \bar{s}_{-p}) < c_p(\bar{s}_p, \bar{s}_{-p}) \quad \text{and} \quad u_p(\alpha\hat{s}_p, \bar{s}_{-p}) \geq \bar{u}_p$$

which means that $\bar{s} \notin \mathcal{E}^c(\bar{u})$. Therefore, $\bar{s} \in \mathcal{E}^c(\bar{u})$ implies $\bar{s} \in \mathcal{E}^u(\bar{c})$. The proof of the reversed implication is similar. ■

This duality relation reveals that a cost minimization game $\Gamma^c(\bar{u})$ possesses an equilibrium if and only if the target utility levels $\bar{u}$ can be achieved at an equilibrium of the corresponding utility maximization game $\Gamma^u(\bar{c})$ for some

[3] $\mathcal{F}_p$ is called upper semi-continuous if for every sequence $\{x_{-p}^n\}_n$ in $\mathcal{X}_{-p}$ converging to an arbitrary $\bar{x}_{-p}$, and for every neighborhood $\mathcal{G}$ of $\mathcal{F}_p(\bar{x}_{-p})$ in $\mathcal{X}_p$, there exists n_0 such that $\mathcal{F}_p(x_{-p}^n) \subset \mathcal{G}$ for all $n \geq n_0$.

[4] $\mathcal{F}_p$ is called lower semi-continuous if for every sequence $\{x_{-p}^n\}_n$ in $\mathcal{X}_{-p}$ converging to an arbitrary $\bar{x}_{-p}$, and for every open subset $\mathcal{G}$ of $\mathcal{X}_p$ for which $\mathcal{F}_p(\bar{s}_{-p}) \cap \mathcal{G} \neq \emptyset$, there exists n_0 such that $\mathcal{F}_p(x_{-p}^n) \cap \mathcal{G} \neq \emptyset$ for all $n \geq n_0$.

target cost levels $\bar{c}$. Therefore, it is of interest to characterize the set of equilibrium utility levels defined as

$$\mathbb{U}_e := \left\{ (u_1(\bar{s}), \ldots, u_P(\bar{s})) \in \mathbb{R}_+^P : \bar{s} \in \mathcal{E}^u(\bar{c}), \bar{c} \in \mathbb{R}_+^P \right\}. \tag{4}$$

Prior to characterizing $\mathbb{U}_e$, however, we first address the issue of the existence of equilibrium in utility maximization games.

B. Existence of Equilibrium

Applying Theorem 1 to a utility minimization game to establish the existence of an equilibrium results in the same difficulty as in the case of cost minimization game $\Gamma^c(\bar{u})$, that is, $\mathcal{S}_1, \ldots, \mathcal{S}_P$ are not compact. Similarly, the unboundedness of $\mathcal{S}_1, \ldots, \mathcal{S}_P$, can be circumvented, if there are nonempty convex compact subsets $\bar{\mathcal{S}}_p \subset \mathcal{S}_p$, for all $p \in \mathcal{P}$, such that $\bar{\mathcal{S}} := \bigtimes_{p\in\mathcal{P}} \bar{\mathcal{S}}_p$ is stable under BR^u, i.e.,

$$BR^u(\bar{\mathcal{S}}) := \{ s \in \mathcal{S} : s \in BR^u(\bar{s}), \bar{s} \in \bar{\mathcal{S}} \} \subset \bar{\mathcal{S}}.$$

It turns out that the existence of such subsets $\bar{\mathcal{S}}_1, \ldots, \bar{\mathcal{S}}_P$ can be shown for any utility maximization game.

Lemma 2: For all $p \in \mathcal{P}$, $\bar{c}_p \in \mathbb{R}_+$, the set $\bar{\mathcal{S}}_p$ defined as

$$\begin{aligned} \bar{\mathcal{S}}_p &:= \mathrm{co}\left(\mathrm{cl}\left(\mathcal{U}_p\left(\mathcal{S}_{-p}\right)\right)\right) \qquad (5) \\ &= \mathrm{co}\left(\mathrm{cl}\left(\{ s_p \in \mathcal{S}_p : c_p(s_p, s_{-p}) \leq \bar{c}_p, s_{-p} \in \mathcal{S}_{-p} \}\right)\right) \end{aligned}$$

is nonempty, convex, and compact.

Proof: It suffices to show that, for all $p \in \mathcal{P}$, $\mathcal{U}_p\left(\mathcal{S}_{-p}\right)$ is bounded. Suppose that, for some $p \in \mathcal{P}$, $\mathcal{U}_p\left(\mathcal{S}_{-p}\right)$ is unbounded. There must be a sequence $\{s^n\}_{n\geq 1}$ in $\mathcal{S}$ such that, $c_p(s^n) \leq \bar{c}_p$ and $\|s_p^n\| \geq n$. Let M be as in part (v) of Assumption 1. Since $c_p(\cdot, s_{-p}^n)$ is convex in $\mathcal{S}_p$, we have, for all $n \geq M$,

$$0 \leq c_p\left(s_p^n \frac{M}{\|s_p^n\|}, s_{-p}^n\right) \leq \frac{M}{\|s_p^n\|} c_p(s^n) \to 0.$$

This implies that $\inf_{\hat{s}\in\mathcal{S}:\|\hat{s}_p\|\geq M} c_p(\hat{s}) = 0$ which contradicts part (v) of Assumption 1. ■

This leads us to the following result.

Proposition 2: For any $\bar{c} \in \mathbb{R}_+^P$, the utility maximization game $\Gamma^u(\bar{c})$ possesses an equilibrium.

Proof: If $\bar{c}_p = 0$, for any $p \in \mathcal{P}$, then $BR_p^u(\mathcal{S}_{-p}) = \mathcal{U}_p(\mathcal{S}_{-p}) = \{0\}$. Therefore, we can remove any such player $p \in \mathcal{P}$ with $\bar{c}_p = 0$ by substituting 0 into s_p throughout and obtain a reduced utility maximization game with fewer players. Hence, we only consider the case where $\bar{c} \in \mathbb{R}_{++}^P$, without loss of generality.

Let $\bar{\mathcal{S}}_p$ be as in (5), for all $p \in \mathcal{P}$. Consider the restriction $\Gamma^u(\bar{c})|_{\bar{\mathcal{S}}}$ of $\Gamma^u(\bar{c})$ to $\bar{\mathcal{S}} := \bar{\mathcal{S}}_1 \times, \ldots, \times \bar{\mathcal{S}}_P$. By definition, we have $BR^u(\bar{\mathcal{S}}) \subset \bar{\mathcal{S}}$. Hence, an equilibrium of $\Gamma^u(\bar{c})|_{\bar{\mathcal{S}}}$, if exists, is also an equilibrium of $\Gamma^u(\bar{c})$.

We now apply Theorem 1 to $\Gamma^u(\bar{c})|_{\bar{\mathcal{S}}}$ by letting

$$\mathcal{X}_p = \bar{\mathcal{S}}_p, \quad f_p = u_p|_{\bar{\mathcal{S}}}, \quad \mathcal{F}_p = \mathcal{U}_p|_{\bar{\mathcal{S}}_{-p}}, \quad \text{for all } p \in \mathcal{P}.$$

Since $\bar{\mathcal{S}}_p$ is a nonempty convex compact subset of $\mathbb{R}^{n_p}$, for all $p \in \mathcal{P}$, condition (i) of Theorem 1 is satisfied.

Condition (ii) of Theorem 1 is satisfied because $\mathcal{U}_p|_{\bar{\mathcal{S}}_{-p}}$ is both u.s.c. and l.s.c. in $\bar{\mathcal{S}}_{-p}$. To see the u.s.c. property, we note that the graph of $\mathcal{U}_p|_{\bar{\mathcal{S}}_{-p}}$

$$\left\{ (s_{-p}, s_p) \in \bar{\mathcal{S}}_{-p} \times \bar{\mathcal{S}}_p : c_p(s_p, s_{-p}) \leq \bar{c}_p \right\}$$

is closed in $\bar{\mathcal{S}}_{-p} \times \bar{\mathcal{S}}_p$. This together with the compactness of $\bar{\mathcal{S}}_p$ implies that $\mathcal{U}_p|_{\bar{\mathcal{S}}_{-p}}$ is u.s.c. in $\mathcal{S}_{-p}$; see Theorem 2.2.3 in [6]. The l.s.c. property follows from the assumed properties of c_p, $\mathcal{S}_p$, $\mathcal{S}_{-p}$, and the fact that, for each $s_{-p} \in \bar{\mathcal{S}}_{-p}$, there exists some $s_p \in \bar{\mathcal{S}}_p$ such that $c_p(s_p, s_{-p}) < \bar{c}_p$; see Theorem 12 in [7].

Finally, conditions (iii), (iv), and (v) of Theorem 1 are readily satisfied due to Assumption 1. Hence, $\Gamma^u(\bar{c})|_{\bar{\mathcal{S}}}$ possesses an equilibrium which is also an equilibrium of $\Gamma^u(\bar{c})$. ■

C. The Set of Equilibrium Utility Levels

We now deal with the issue of characterizing the set of equilibrium utility levels $\mathbb{U}_e$ defined in (4). Clearly, $\mathbb{U}_e \subset \mathbb{U}_a$ where $\mathbb{U}_a$ denotes the set of achievable utility levels, i.e.,

$$\mathbb{U}_a := \{ (u_1(s), \ldots, u_P(s)) : s \in \mathcal{S} \}.$$

The next proposition shows a simple case where $\mathbb{U}_e = \mathbb{U}_a$.

Proposition 3: If, for all $p \in \mathcal{P}$, there exists $\hat{s}_p \in \mathcal{S}_p$ such that $\mathcal{S}_p = \{ \alpha \hat{s}_p : \alpha \in \mathbb{R}_+ \}$, then $\mathbb{U}_e = \mathbb{U}_a$.

Proof: Fix $\bar{s} \in \mathcal{S}$ and let $\bar{u} = (u_1(\bar{s}), \ldots, u_P(\bar{s}))$. Then, $\bar{s} \in \mathcal{E}^u(\bar{c})$ where $\bar{c} := (c_1(\bar{s}), \ldots, c_P(\bar{s}))$. To see this, consider the problem

$$\max_{s_p \in \mathcal{S}_p : c_p(s_p, \bar{s}_{-p}) \leq \bar{c}_p} u_p(s_p, \bar{s}_{-p}).$$

Clearly, the maximum above is uniquely achieved by $\bar{s}_p$. ■

In general, however, $\mathbb{U}_e$ is a proper subset of $\mathbb{U}_a$, i.e., $\mathbb{U}_e \subsetneq \mathbb{U}_a$.

V. THE CASE OF DECOUPLED COST FUNCTIONS

In this section, we consider the special case of decoupled cost functions where each player's cost function depends only on its own strategy. In other words, we assume that, for all $p \in \mathcal{P}$, $(s_p, s_{-p}) \in \mathcal{S}$,

$$c_p(s_p, s_{-p}) = c_p(s_p)$$

where, by a slight abuse of notation, $c_p(s_p)$ denotes player p's cost for using the resource $s_p \in \mathcal{S}_p$ regardless of the strategies of the other players. The problem of power optimization in MIMO interference systems falls into this special case.

In the case of decoupled cost functions, it is possible to obtain an inner estimate of the set of equilibrium utility levels, without resorting to an equilibration process. For this, we define the set of minimax utility levels as

$$\mathbb{U}_m := \bigcup_{\bar{c} \in \mathbb{R}_+^P} \Big\{ \bar{u} \in \mathbb{R}_+^P : \text{for all } p \in \mathcal{P}, \\ \bar{u}_p < \min_{s_{-p} \in \bar{\mathcal{S}}_{-p}(\bar{c}_{-p})} \max_{s_p \in \bar{\mathcal{S}}_p(\bar{c}_p)} u_p(s_p, s_{-p}) \Big\}$$

where, for all $p \in \mathcal{P}$, $\bar{\mathcal{S}}_p(\bar{c}_p) = \{s_p \in \mathcal{S}_p : c_p(s_p) \leq \bar{c}_p\}$. Note that, for all $c_p \in \mathbb{R}_+$, $\bar{\mathcal{S}}_p(\bar{c}_p)$ is nonempty, convex, and compact.

Proposition 4: In the case of decoupled cost functions,

$$\mathbb{U}_m \subset \mathbb{U}_e.$$

Proof: Let $(\bar{u}, \bar{c}) \in \mathbb{U}_m \times \mathbb{R}^P_+$ be such that, for all $p \in \mathcal{P}$,

$$\bar{u}_p < \min_{s_{-p} \in \bar{\mathcal{S}}_{-p}(\bar{c}_{-p})} \max_{s_p \in \bar{\mathcal{S}}_p(\bar{c}_p)} u_p(s_p, s_{-p}).$$

This means that, for all $p \in \mathcal{P}$, $s_{-p} \in \bar{\mathcal{S}}_{-p}(\bar{c}_{-p})$, there exists some $s_p \in \bar{\mathcal{S}}_p(\bar{c}_p)$ such that

$$\bar{u}_p < u_p(s_p, s_{-p}) \qquad \text{and} \qquad c_p(s_p) \leq \bar{c}_p.$$

Consider the game $\Gamma^c(\bar{u})$. It is straightforward to see that

$$\begin{aligned} s_{-p} \in \bar{\mathcal{S}}_{-p}(\bar{c}_{-p}), s_p \in BR^c_p(s_{-p}) \quad &\Rightarrow \quad c_p(s_p) \leq \bar{c}_p \\ &\Rightarrow \quad s_p \in \bar{\mathcal{S}}_p(\bar{c}_p). \end{aligned}$$

Therefore, $\bar{\mathcal{S}}(\bar{c}) := \bigtimes_{p \in \mathcal{P}} \bar{\mathcal{S}}_p(\bar{c}_p)$ is stable under $BR^c(\bar{u})$. Applying Theorem 1 to the restriction $\Gamma^c(\bar{u})|_{\bar{\mathcal{S}}(\bar{c})}$ of $\Gamma^c(\bar{u})$ to $\bar{\mathcal{S}}(\bar{c})$ leads to the existence of an equilibrium of $\Gamma^c(\bar{u})$. Therefore, $\bar{u} \in \mathbb{U}_e$. ■

An immediate consequence of Proposition 4 is that all target utility levels that are sufficiently small are minimax (hence equilibrium) utility levels.

Proposition 5: If the player cost functions are decoupled, then there exists some $\hat{u} \in \mathbb{R}^P_{++}$ such that

$$\left\{\bar{u} \in \mathbb{R}^P_+ : \bar{u} \leq \hat{u} \text{ (elementwise)}\right\} \subset \mathbb{U}_m.$$

In some cases, it is possible to obtain the entire set of equilibrium utility levels through the set of minimax utility levels, i.e., $\mathbb{U}_m = \mathbb{U}_e$. However, in general, $\mathbb{U}_m$ can be a proper set of $\mathbb{U}_e$, i.e., $\mathbb{U}_m \subsetneq \mathbb{U}_e$.

A. The Case of Weakly Coupled Utility Functions

As a final application of the minimax approach, we consider the case of weakly coupled utility functions (in the context of decoupled cost functions). We formalize the concept of weak coupling in terms of a coupling coefficient $\eta \in \mathbb{R}_+$. We consider an $\eta-$parameterized family of cost minimization and utility maximization games with some cost functions of the form $c_p(s_p)$ and some utility functions of the form $u_p(s_p, \eta s_{-p})$. Thus, if $\eta = 0$, then both the utility and the cost functions are decoupled; whereas, if $\eta > 0$ is small, then the utility functions are weakly coupled and the cost functions are decoupled. We will refer to the achievable, minimax, and the equilibrium utility levels corresponding to η as $\mathbb{U}^\eta_a$, $\mathbb{U}^\eta_m$, and $\mathbb{U}^\eta_e$, respectively. The following result states that essentially all utility levels achievable in the case of complete decoupling ($\eta = 0$) are minimax (hence equilibrium) utility levels in the case of weak decoupling of the utility functions.

Proposition 6: If $0 \leq \bar{u} < \hat{u} \in \mathbb{U}^{\eta=0}_a$ (elementwise), then there exists some $\bar{\eta} > 0$ such that, for all $\eta \in [0, \bar{\eta}]$,

$$\bar{u} \in \mathbb{U}^\eta_m \subset \mathbb{U}^\eta_e.$$

Proof: Suppose that $0 \leq \bar{u} < \hat{u} \in \mathbb{U}^{\eta=0}_a$. There exists some $\hat{s} \in \mathcal{S}$ such that, for all $p \in \mathcal{P}$, $\hat{u}_p = u_p(\hat{s}_p, 0)$. For all $p \in \mathcal{P}$, let $\hat{c}_p := c_p(\hat{s}_p)$ and $\hat{S}_p := \{s_p \in \mathcal{S}_p : c_p(s_p) \leq \hat{c}_p\}$. The subsets $\hat{S}_p$, for all $p \in \mathcal{P}$, are nonempty, convex, and compact. Since u_p is continuous, for some $\bar{\eta} > 0$, for all $\eta \in [0, \bar{\eta}]$, for all $p \in \mathcal{P}$, we have $\bar{u}_p < \min_{s_{-p} \in \hat{\mathcal{S}}_{-p}} u_p(\hat{s}_p, \eta s_{-p})$ which implies that

$$\bar{u}_p < \min_{s_{-p} \in \hat{\mathcal{S}}_{-p}} \max_{s_p \in \hat{\mathcal{S}}_p} u_p(s_p, \eta s_{-p}).$$

Therefore, for all $\eta \in [0, \bar{\eta}]$, $\bar{u} \in \mathbb{U}^\eta_m \subset \mathbb{U}^\eta_e$. ■

VI. AN EXACT PENALTY APPROACH

In the cost minimization problem, each player considers the achievement of a certain utility level as a hard constraint on itself. However, if the target utility levels are not equilibrium utility levels, then the players would not be able to agree on any resource allocation profile. In a realistic cost minimization game, a particular player would not know if its target utility level together with the other players' target utility levels constitute a profile of equilibrium utility levels. If the target utility levels are not equilibrium utility levels, then players may be caught up in an everlasting process of updating their strategies with no possibility of reaching an equilibrium solution.

To alleviate this issue, we relax each player's hard constraint by incorporating a penalty term into each player's cost function, which penalizes the deviations from achieving its target utility level. More precisely, using the notation of Section III, we introduce a weighted cost minimization game, referred to as $\Gamma^w(\bar{u})$, in which each player $p \in \mathcal{P}$ is to solve

$$\min_{s_p \in \mathcal{S}_p} c_p(s_p, s_{-p}) + w_p \left(\bar{u}_p - u_p(s_p, s_{-p})\right)^+$$

where $w_p \in \mathbb{R}_+$ is player p's unit cost of not achieving the target utility level $\bar{u}_p \in \mathbb{R}_+$. It will be clear shortly that the minimum above always exists; see Proposition 7.

A strategy profile $s^* \in \mathcal{S}$ is called an equilibrium of $\Gamma^w(\bar{u})$ if and only if

$$s^* \in BR^w(s^*)$$

where $BR^w = (BR^w_1, \ldots, BR^w_P)$ denotes the best response correspondence for $\Gamma^w(\bar{u})$. We will denote the set of equilibria in a weighted cost minimization game $\Gamma^w(\bar{u})$ by $\mathcal{E}^w(\bar{u})$.

Our primary interest is in the case where $w_p \uparrow \infty$, for all $p \in \mathcal{P}$. As $w_p \uparrow \infty$, player p would be expected to achieve its target utility level, if at all possible, since otherwise player p would be penalized heavily. If a player p's target utility level is "too high" to achieve, then player p would incur a very "high cost" as $w_p \uparrow \infty$. In a practical scenario, a player who cannot achieve its target utility level despite incurring a very high cost may be encouraged to downgrade its target utility level to a more "reasonable" level. However, regardless of the target utility levels $\bar{u} \in \mathbb{R}^P_+$, a weighted cost minimization game $\Gamma^w(\bar{u})$ always possesses equilibria, whose relationship with the equilibria of the corresponding cost minimization game $\Gamma^c(\bar{u})$ is established next.

Proposition 7: For any $\bar{u} \in \mathbb{R}^P_+$, $w \in \mathbb{R}^P_+$, we have:

1) $\Gamma^w(\bar{u})$ possesses an equilibrium
2) $\{s \in \mathcal{E}^w(\bar{u}) : u_p(s) \geq \bar{u}_p, \text{ for all } p \in \mathcal{P}\} \subset \mathcal{E}^c(\bar{u})$
3) For some $\bar{w} \in \mathbb{R}^P_+$, $w \geq \bar{w} \Rightarrow \mathcal{E}^c(\bar{u}) \subset \mathcal{E}^w(\bar{u})$.

Proof:

1) Note that $s_p = 0$ achieves the cost $w_p \bar{u}_p$. Therefore, for all $p \in \mathcal{P}$, $s_{-p} \in \mathcal{S}_{-p}$, $\bar{s}_p \in BR^w_p(s_{-p})$, we have $c_p(\bar{s}_p, s_{-p}) \leq w_p \bar{u}_p$. Hence, BR^w maps $\mathcal{S}$ into the subsets of the set $\bar{\mathcal{S}} := \bigtimes_{p \in \mathcal{P}} \bar{\mathcal{S}}_p$ where, for all $p \in \mathcal{P}$, $\bar{\mathcal{S}}_p$ is given by
$$\text{co}\left(\text{cl}\left(\{s_p \in \mathcal{S}_p : c_p(s_p, s_{-p}) \leq w_p \bar{u}_p, s_{-p} \in \mathcal{S}_{-p}\}\right)\right).$$
By Lemma 2, $\bar{\mathcal{S}}$ is nonempty, convex, and compact. By Theorem 1, the restriction $\Gamma^w(\bar{u})|_{\bar{\mathcal{S}}}$ of $\Gamma^w(\bar{u})$ to $\bar{\mathcal{S}}$ possesses an equilibrium which is also an equilibrium of $\Gamma^w(\bar{u})$; see Remark 1.
2) Let $\hat{s} \in \mathcal{E}^w(\bar{u})$ be such that $u_p(\hat{s}) \geq \bar{u}$, for all $p \in \mathcal{P}$. We have, for all $p \in \mathcal{P}$, $s_p \in \mathcal{S}_p$,
$$c_p(\hat{s}) \leq c_p(s_p, \hat{s}_{-p}) + w_p \left(\bar{u}_p - u_p(s_p, \hat{s}_{-p})\right)^+.$$
Hence, we have, for all $p \in \mathcal{P}$, $s_p \in \mathcal{S}_p$,
$$u_p(s_p, \hat{s}_{-p}) \geq \bar{u}_p \quad \Rightarrow \quad c_p(\hat{s}) \leq c_p(s_p, \hat{s}_{-p}).$$
3) Let $s^* \in \mathcal{E}^c(\bar{u})$. Note that each player p's problem is convex and, for some $\bar{s}_p \in \mathcal{S}_p$, the regularity condition $\bar{u}_p - u_p(\bar{s}_p, s^*_{-p}) < 0$ is satisfied. Hence, by Theorem 1 on page 217 of [8], there exists a vector of Lagrange multipliers $\lambda^* \in \mathrm{R}^P_+$, such that, for all $p \in \mathcal{P}$,
$$c_p(s^*) = \min_{s_p \in \mathcal{S}_p} c_p(s_p, s^*_{-p}) + \lambda^*_p \left(\bar{u}_p - u_p(s_p, s^*_{-p})\right).$$
This results in, for all $p \in \mathcal{P}$ and $w_p \geq \lambda^*_p$,
$$c_p(s^*) = \min_{s_p \in \mathcal{S}_p} c_p(s_p, s^*_{-p}) + w_p \left(\bar{u}_p - u_p(s_p, s^*_{-p})\right)^+.$$
Therefore, for all $w \geq \lambda^*$, $s^* \in \mathcal{E}^w(\bar{u})$; see also [9].

■

Below, we provide a converse to part 3) of Proposition 7. Since $c_p(\cdot, s_{-p})$, $-u_p(\cdot, s_{-p})$ are convex, the one-sided directional derivatives
$$c'_p((s_p, s_{-p}); \hat{s}_p) := \lim_{\lambda \downarrow 0} \frac{c_p(s_p + \lambda \hat{s}_p, s_{-p}) - c_p(s_p, s_{-p})}{\lambda}$$
$$u'_p((s_p, s_{-p}); \hat{s}_p) := \lim_{\lambda \downarrow 0} \frac{u_p(s_p + \lambda \hat{s}_p, s_{-p}) - u_p(s_p, s_{-p})}{\lambda}$$
exist, for all $p \in \mathcal{P}$, $(s_p, s_{-p}) \in \mathcal{S}$, $\hat{s}_p \in \mathcal{S}_p$. We make the following additional assumption for the rest of the paper.

Assumption 2: For all $p \in \mathcal{P}$, $\alpha, \beta \in \mathbb{R}_+$,
$$\sup_{(s_p, s_{-p}) \in \mathcal{S} : \|s_p\| = 1, \|s_{-p}\| \leq \beta} \frac{c'_p((\alpha s_p, s_{-p}); s_p)}{u'_p((\alpha s_p, s_{-p}); s_p)} < \infty.$$

Proposition 8: Let $\{w^n\}_{n \geq 1}$ be such that, for all $p \in \mathcal{P}$, $0 \leq w^1_p \leq w^2_p \leq \cdots \to \infty$. For some $\bar{u} \in \mathbb{R}^P_+$, let $\{s^n\}_{n \geq 1}$ be such that, for all $n \geq 1$, $s^n \in \mathcal{E}^{w^n}(\bar{u})$.

1) $\sup_{n \geq 1} \|s^n\| < \infty \Rightarrow s^n \in \mathcal{E}^c(\bar{u})$, for all large $n \geq 1$.
2) $\bar{u} \notin \mathbb{U}_e \Rightarrow \sup_{n \geq 1} \|s^n\| = \infty$.

Proof:

1) For all $p \in \mathcal{P}$, let $\mathcal{N}_p := \{n \geq 1 : u_p(s^n_p, s^n_{-p}) < \bar{u}_p\}$. If $s^n_p \neq 0$, let $\hat{s}^n_p := s^n_p / \|s^n_p\|$; otherwise, let $\hat{s}^n_p \in \mathcal{S}$ be arbitrary except $\|\hat{s}^n_p\| = 1$. We have, for all $n \in \mathcal{N}_p$,
$$w^n_p \leq c'_p((\|s^n_p\| \hat{s}^n_p, s^n_{-p}); \hat{s}^n_p) / u'_p((\|s^n_p\| \hat{s}^n_p, s^n_{-p}); \hat{s}^n_p).$$
In view of Assumption 2, this implies that $\mathcal{N}_p$ is finite. Therefore, there exists an $\bar{n} \geq 1$ such that, for all $p \in \mathcal{P}$, $n \geq \bar{n}$, we have $u_p(s^n_p, s^n_{-p}) \geq \bar{u}_p$. Hence, for all $n \geq \bar{n}$, we have $s^n \in \mathcal{E}^c(\bar{u})$.
2) Assume that $\sup_{n \geq 1} \|s^n\| < \infty$. Part (a) implies that, for large enough n, $s^n \in \mathcal{E}^c(\bar{u})$. This leads to $\bar{u} \in \mathbb{U}_e$, which contradicts the hypothesis.

■

VII. CONCLUSIONS

This paper attempts to extend our previous work on power minimization in MIMO interference systems to general competitive resource allocation problems. Our extension involves a cost minimization and a utility maximization game, which are dual to each other. In the most generalized case, both the objective function and the strategy set of each player depend on the strategies of the other players. We obtain satisfactory counterparts of our previous results in our generalized setting by exploiting, in particular, a duality relation.

Obtaining less conservative estimates of the set of equilibrium utility levels is left for future work. Developing learning dynamics with a universally convergent behavior is another future research problem. Finally, improving the efficiency of equilibrium without requiring centralized coordination remains as yet another future research problem.

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