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Optimal control of LTI systems over unreliable communication links☆

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Abstract

In this paper, optimal control of linear time-invariant (LTI) systems over unreliable communication links is studied. The motivation of the problem comes from growing applications that demand remote control of objects over Internet-type or wireless networks where links are prone to failure. Depending on the availability of acknowledgment (ACK) signals, two different types of networking protocols are considered. Under a TCP structure, existence of ACK signals is assumed, unlike the UDP structure where no ACK packets are present. The objective here is to mean-square (m.s.) stabilize the system while minimizing a quadratic performance criterion when the information flow between the controller and the plant is disrupted due to link failures, or packet losses. Sufficient conditions for the existence of stabilizing optimal controllers are derived.

Keywords: Optimal control; Communication networks; Networked control systems; TCP/UDP

1. Introduction

One of the fundamental questions in control system theory and design is the effect controller-plant communication has on the performance of the control system. In this paper, the medium of communication between the several components of a control system is generically called a *communication network*. The network is jointly used by sensor, actuator, and controller nodes. The term *networked control system* (NCS) is used to describe the combined system of controllers, actuators, sensors, and the communication network that connects them together (Tipsuwan & Chow, 2003; Walsh, Hong, & Bushnell, 2002; Zhang, Branicky, & Phillips, 2001).

As illustrated in Fig. 1, in an NCS, several components of the system may communicate over the common network that connects them together. Thus, there may be communication taking place between the sensor and the controller nodes, among the sensors themselves, and the controller and the actuator nodes. The purpose of this communication is to improve the performance of the control system. The performance may be a measurable quantity defined in terms of a performance criterion, as in the case of optimal control or estimation, or it may be a qualitative measure described as a desired behavior.

The presence of a network brings in constraints in the design of the control system, as information between the various decision makers must be exchanged according to the rules and dynamics of the network. Our goal in this paper is to study communication network constraints characterized by link failures, and design the control system so as to do its best given these constraints. The basic model we introduce here focuses on the unreliable nature of the links in both directions; see also Fig. 2. Some of the most relevant papers sharing this theme are Eisenbeis (2004), Hadjicostis and Touri (2002), Imer, Yüksel, and Başar (2004), and Sadjadi (2003).

In a network, link failures cause the information flow between the controller and the plant to be disrupted, which

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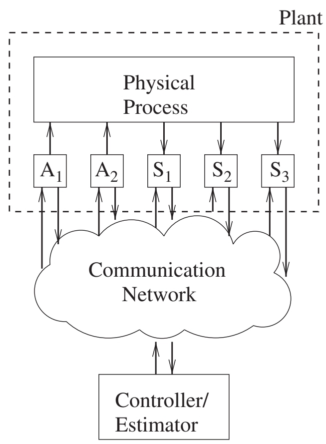

Fig. 1. A prototype NCS.

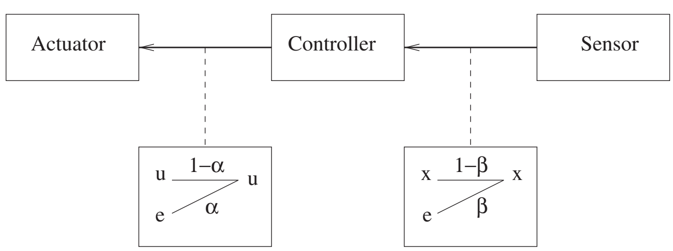

Fig. 2. An NCS with unreliable links.

results in control and/or measurement packets being lost. Packets may also be lost due to congestion. Link failures, on the other hand, may occur due to the unreliable nature of the links, such as in the case of wireless networks. Whatever the reason, this disruption of communication has a deteriorating effect on the NCS performance. Therefore, it is important to develop an understanding of how much loss the control system can tolerate before the system becomes unstable, or in the case of estimation before the estimation error becomes unbounded. Also, if the statistical description of the link failure process is given a priori, a problem of interest is to determine the optimal control and estimation policies under the link failure constraints.[1]

Note that, packet losses may occur both from the sensor to the controller, and from the controller to the actuator. In the first case, the measurement packets are lost and, therefore, the controller has access to the state *intermittently*. In the latter case, the control or actuation packets are lost, and this causes the actuator to have access to the controls intermittently. One has to define what happens if the actuator does not receive a control packet at a given time. There are two potential actions for the actuator in this case. First one is to apply "zero control", and the second one is to apply the "last available control". The latter action is equivalent to the zero-order hold (ZOH) action in continuous time, whereas the former action can be justified by observing that zero control would cost the least amount of control energy among all possible control actions. In this paper, we assume "zero control" action by the actuator in case the controller-actuator fails (Hadjicostis & Touri, 2002; Imer et al., 2004; Sadjadi, 2003).

In Section 2, we model the unreliable nature of the links by a Bernoulli process, where links fail, or packets are lost, independently. The type of communication protocol used for plant-controller communication affects the information structure of the problem. More specifically, it is important to distinguish between the case when the controller receives an acknowledgment for each control packet it sends to the actuator, and not. In the Internet, for example, since every packet in transmission control protocol (TCP) is acknowledged (Jacobson, 1988), the structure of the controller in this case is different than the case when the control packets are sent over a best-effort, or user datagram protocol (UDP) type network (Postel, 1980). With this stochastic structure, and under the induced information structures, the goal is to determine a stabilizing optimal control policy with the objective of minimizing a quadratic performance criterion.

The rest of the paper is organized as follows. We introduce the TCP and UDP information structures and the corresponding optimal control problems in Section 2, where we also review some of the most relevant work. The finite and infinite-horizon optimal controllers under TCP and UDP information structures are derived in Sections 3 and 4, respectively. We present some numerical simulation results in Section 5, and the paper ends with the concluding remarks of Section 6, where we also discuss some future research directions.

2. Problem formulation

Consider the NCS shown in Fig. 2, where the links connecting the sensor to the controller, and the controller to the actuator are prone to failure.

The plant is described by the discrete-time dynamics

$$x_{k+1} = Ax_k + \alpha_k Bu_k + w_k, \quad k = 0, 1, \ldots, \tag{1}$$

where $x_k \in \mathbb{R}^n$ is the state, $u_k \in \mathbb{R}^m$ is the control. We assume that $m \leqslant n$. The disturbances, $w_k \in \mathbb{R}^n$, are independent zero-mean second-order random vectors, also independent of $\{\alpha_k\}$, and the initial state x_0, which is a random vector with a given probability distribution P_{x_0}.

The stochastic process $\{\alpha_k\}$ models the unreliable nature of the link from the controller to the actuator. Basically, $\alpha_k = 0$ when this link fails, i.e., the control packet is lost, and $\alpha_k = 1$, otherwise. Note that, this corresponds to the "zero control" action by the actuator. We let $\{\alpha_k\}$ be an i.i.d Bernoulli process with $P[\alpha_k = 0] = \alpha$, and $P[\alpha_k = 1] = 1 - \alpha := \bar{\alpha}$.

The link from the sensor to the controller is prone to failure as well, but potentially with a different probability, β. Thus, the controller has access to the state intermittently

$$y_k = \beta_k x_k, \quad k = 0, 1, \ldots . \tag{2}$$

Here the process $\{\beta_k\}$ is an independent Bernoulli process with parameter β, i.e., $P[\beta_k = 0] = \beta$, and $P[\beta_k = 1] = 1 - \beta := \bar{\beta}$. We assume that $\{\alpha_k\}$ and $\{\beta_k\}$ are also independent of each other, and both processes are also independent of the plant noise $\{w_k\}$, and initial state x_0.

[1] The estimation problem has been studied in Nahi (1969) and Sinopoli et al. (2003). Thus, here we concentrate on the control problem.

Note that in our formulation, when the controller–actuator link fails the entire control vector is lost. Similarly, the failure of the sensor–controller link causes the entire state vector to be lost. This, although a realistic scenario in many cases, does not capture the more general scenario in which packets are lost with different probabilities on each sensor and actuator link. This is a topic that will be addressed in the future.

Also, there is no measurement noise in this basic model. The rationale for not including any measurement noise is the assumption that the communication between the sensor and the controller is taking place at the network layer, where the packets sent are either received or lost. Alternatively, one can think of the sensor and the controller connected through a binary erasure channel with infinite capacity, i.e., no quantization or encoding of the state (Tatikonda & Mitter, 2004).

Let I_k denote the information available to the controller at time k. It is important to distinguish between two scenarios. In the first one, the controller at time k knows if the control at time $k-1$, u_{k-1}, has been successfully transmitted or not. In the second case, the controller does not know if any of the previous controls has been successfully transmitted or not. In other words, α_{k-1} is part of the information set I_k of the controller in the first case, whereas no α_k belongs to any of the information sets of the controller in the second case. Over a network, the first information structure can be justified if there is a mechanism in which acknowledgment packets are generated by the actuator to signal the successful receipt of control packets. In the Internet, since every packet in TCP is acknowledged, we call this information structure I_k^{TCP}. If, on the other hand, the controller and the actuator are linked through a best-effort or UDP network, it is not possible for the controller to know if any of its past controls have been applied by the actuator or not. We denote the resulting information vector of this controller by I_k^{UDP}. We have

$$I_k^{\mathrm{UDP}} = (y_0, \ldots, y_k; u_0, \ldots, u_{k-1}; \beta_0, \ldots, \beta_k),\quad k = 1, 2, \ldots,$$

$$I_0^{\mathrm{UDP}} = (y_0, \beta_0). \tag{3}$$

In TCP, I_k^{TCP} includes $(\alpha_0, \ldots, \alpha_{k-1})$ as well, i.e.,

$$I_k^{\mathrm{TCP}} = (I_k^{\mathrm{UDP}}; \alpha_0, \ldots, \alpha_{k-1}), \quad k = 1, 2, \ldots,$$

$$I_0^{\mathrm{TCP}} = I_0^{\mathrm{UDP}}.$$

Note that β_k is included in the information set of both controllers, as the controller can identify β_k unless $x_k = 0$ in which case no control action is required. Also, in both cases we assume that the controller has access to its past actions.

Consider the class of policies consisting of a sequence of functions $\pi = \{\mu_0, \mu_1, \ldots, \mu_{N-1}\}$, where N is the decision-horizon, and each function μ_k maps the information vector I_k^{UDP} (or I_k^{TCP}) into some control space C_k, i.e., $u_k = \mu_k(I_k)$. Such policies are called *admissible*. We want to find an admissible policy that minimizes the quadratic cost function

$$J_\pi = E\left\{x_N^{\mathrm{T}} F x_N + \sum_{k=0}^{N-1} x_k^{\mathrm{T}} Q x_k + \alpha_k u_k^{\mathrm{T}} R u_k\right\} \tag{4}$$

subject to the system (1) and measurement (2) equations. We assume that $R > 0$, $Q \geqslant 0$, $F \geqslant 0$. Note that the control u_k is penalized only if it is applied to the plant by the actuator.

In what follows, we first solve this optimization problem in finite-horizon, and obtain the structure of the optimal control under both TCP and UDP information structures. Subsequently, we establish explicit conditions, and easy-to-check tests for the existence and stabilizability properties of the infinite-horizon controllers. But first, we review the relevant literature.

2.1. Relevant work

In its present form, the optimal control problem with TCP information structure resembles the optimal quadratic control of a jump linear system. Suppose, in addition to I_k^{TCP}, α_k is also known at time k. In this case, the problem can be studied in the framework of jump-linear systems (Chizeck & Ji, 1988). Solution of the jump linear quadratic Gaussian (JLQG) problem relies on the fact that the so-called *form process* is observable. The form process is the underlying Markov process that takes values in a finite set. With the inclusion of α_k into the information state, the optimal controller can be obtained directly from (Chizeck & Ji (1988)); also see Sengupta (2001). Here, however, we do not allow for any knowledge of α_k at time k, even in the TCP case.

Another way of looking at the problem is in the context of uncertainty threshold principle (Athans, Ku, & Gershwin, 1977; Katayama, 1976; Koning, 1982; Ku & Athans, 1977). In particular, if we assume perfect state measurements, i.e., $\beta = 0$, the linear system with the quadratic cost structure fits into the framework of Koning (1982). When the controller has access to the state intermittently, however, the solution of Koning (1982) cannot be used.

A more recent attempt with a similar formulation is given in Sadjadi (2003), where the information structure of the problem is I_k^{UDP}. However, rather than obtaining the optimal solution, the author proposes to separate the estimation and control to simplify the solution. The sub-optimal solution is then obtained with an intuitive construction of a controller and an estimator.

This work also relates to a recent work by Sinopoli et al. (2003), in which estimation counterpart of the problem posed here, i.e., optimal estimation of an LTI system with intermittent observations, is discussed. The problem of optimal recursive estimation with missing observations was first introduced by Nahi (1969).

3. Optimal control over TCP networks

3.1. Finite horizon optimal control

Consider the plant dynamics (1) along with the measurement equation (2). The objective is to minimize the quadratic cost

(4) over $\pi = \{\mu_0(I_0^{\mathrm{TCP}}), \ldots, \mu_{N-1}(I_{N-1}^{\mathrm{TCP}})\}$. From the dynamic-programming (DP) equation (Bertsekas, 1995), we obtain the cost to go from stage $N-1$

$$\begin{aligned} J_{N-1}(I_{N-1}^{\mathrm{TCP}}) = {} & E\{x_{N-1}^{\mathrm{T}} K_{N-1} x_{N-1} | I_{N-1}^{\mathrm{TCP}}\} \\ & + E\{e_{N-1}^{\mathrm{T}} P_{N-1} e_{N-1} | I_{N-1}^{\mathrm{TCP}}\} \\ & + E\{w_{N-1}^{\mathrm{T}} F w_{N-1}\}, \end{aligned}$$

where $e_k := x_k - \hat{x}_k$ is the state estimation error, and the estimator $\hat{x}_k$ is given by

$$\hat{x}_k = E\{x_k | I_k^{\mathrm{TCP}}\}.$$

The optimal policy for the last stage is

$$u_{N-1}^* = -(R + B^{\mathrm{T}} F B)^{-1} B^{\mathrm{T}} F A E\{x_{N-1} | I_{N-1}^{\mathrm{TCP}}\},$$

where K_{N-1} and P_{N-1} are given by

$$P_{N-1} = \bar{\alpha} A^{\mathrm{T}} F B (R + B^{\mathrm{T}} F B)^{-1} B^{\mathrm{T}} F A,$$

$$K_{N-1} = A^{\mathrm{T}} F A + Q - P_{N-1}.$$

The DP for the next stage yields

$$\begin{aligned} J_{N-2}(I_{N-2}^{\mathrm{TCP}}) = \min_{u_{N-2}} & E\{x_{N-1}^{\mathrm{T}} K_{N-1} x_{N-1} \\ & + \bar{\alpha} u_{N-2}^{\mathrm{T}} R u_{N-2} + x_{N-2}^{\mathrm{T}} Q x_{N-2} \\ & + e_{N-1}^{\mathrm{T}} P_{N-1} e_{N-1} | I_{N-2}^{\mathrm{TCP}}\} \\ & + E\{w_{N-2}^{\mathrm{T}} K_{N-1} w_{N-2}\} \\ & + E\{w_{N-1}^{\mathrm{T}} F w_{N-1}\}. \end{aligned} \tag{5}$$

Note that we can exclude the last term from the minimization with respect to u_{N-2}, as there is no dual effect of the control (Shalom & Tse, 1974), i.e., $x_k - E\{x_k | I_k^{\mathrm{TCP}}\}$ is not a function of the past control u_{k-1} due to the acknowledgments in TCP. Proceeding similarly, we obtain the optimal policy for every stage

$$u_k^* = G_k E\{x_k | I_k^{\mathrm{TCP}}\},$$

where the matrix G_k is given by

$$G_k = -(R + B^{\mathrm{T}} K_{k+1} B)^{-1} B^{\mathrm{T}} K_{k+1} A$$

with the matrices K_k given recursively by the Riccati equation (RE)

$$P_k = \bar{\alpha} A^{\mathrm{T}} K_{k+1} B (R + B^{\mathrm{T}} K_{k+1} B)^{-1} B^{\mathrm{T}} K_{k+1} A, \tag{6}$$

$$K_k = A^{\mathrm{T}} K_{k+1} A - P_k + Q \tag{7}$$

with initial conditions $K_N = F$, $P_N = 0$. Since the separation of estimation and control holds, the estimator part of the controller can be designed separately, and in our case, since there is no measurement noise, it takes the following form, where $\hat{x}_0 = E_{P_{x_0}}\{x_0\}$ if $\beta_0 = 0$, otherwise $\hat{x}_0 = x_0$

$$\hat{x}_k = \begin{cases} A\hat{x}_{k-1} + \alpha_{k-1} B u_{k-1}, & \beta_k = 0, \\ x_k, & \beta_k = 1. \end{cases}$$

3.2. Infinite horizon optimal control

Since noise is present, in order to achieve a finite cost as the number of decision stages increases indefinitely, we change the performance criterion to the infinite-horizon average cost criterion given by

$$J_\pi = \limsup_{N \to \infty} \frac{1}{N} E\left\{ \sum_{k=0}^{N-1} x_k^{\mathrm{T}} Q x_k + \alpha_k u_k^{\mathrm{T}} R u_k \right\}.$$

We first investigate the asymptotic properties of the matrix RE (6)–(7), which by substituting for P_k, and reversing the time index, can be written as

$$\begin{aligned} K_{k+1} = {} & A^{\mathrm{T}} K_k A - \bar{\alpha} A^{\mathrm{T}} K_k B (R + B^{\mathrm{T}} K_k B)^{-1} \\ & \times B^{\mathrm{T}} K_k A + Q \end{aligned}$$

In Katayama (1976), a necessary and sufficient condition for stability of this equation is given when B is invertible, which we restate here for convenience.

Lemma 1. *Let B be square and of full rank, and $(A, Q^{1/2})$ be observable. Then, $\{K_k\}$ converges to a unique positive definite steady-state solution K if and only if $\sqrt{\bar{\alpha}} A$ is asymptotically stable.*

Requiring B to be invertible is rather restrictive, as it means that the control is of the same dimension as the state, but it is useful to have an explicit condition of stability in terms of α for the special case when B *is* invertible. A weaker condition for convergence, when B is not necessarily invertible, is derived in Koning (1982), which we state in the next lemma.

Lemma 2. *Let $(A, Q^{1/2})$ be observable. Then, the RE K_k converges to a unique positive definite steady-state solution K if and only if the following RE converges from the initial condition $\Lambda_0 = I$*

$$\Lambda_{k+1} = A^{\mathrm{T}} \Lambda_k A - \bar{\alpha} A^{\mathrm{T}} \Lambda_k B (B^{\mathrm{T}} \Lambda_k B)^{-1} B^{\mathrm{T}} \Lambda_k A.$$

In the case when B is invertible, using Lemma 1, we conclude that the sequence $\{K_k\}$ generated by the RE (6)–(7) will converge if and only if

$$\max_i |\lambda_i(A)| < \frac{1}{\sqrt{\bar{\alpha}}}, \tag{8}$$

where $\lambda_i(A)$ is an eigenvalue of A. Thus, as the failure rate $\bar{\alpha}$ becomes larger, the bound becomes tighter, reaching the stability condition of A, when $\bar{\alpha} = 1$. When B is not invertible, a similar statement can be made, however there is no explicit condition that one can impose on the eigenvalues of A in the form of (8).

Next, we investigate the stability of the closed-loop system. Using the TCP estimator from Section 3.1, we first write a recursion for the state estimation error

$$e_k = \begin{cases} A e_{k-1} + w_{k-1}, & \beta_k = 0, \\ 0, & \beta_k = 1. \end{cases} \tag{9}$$

As it turns out, the mean-square (m.s.) stability of the estimation error is independent of that of the state, x_k, in this case. So, we arrive at the following:

Theorem 3. *Let* $(A, Q^{1/2})$ *be observable. Suppose*

$$\rho_m := \max_i |\lambda_i(A)| < \frac{1}{\sqrt{\beta}}$$

and (α, A, B) *are such that the RE,* Λ_k, *given in Lemma* 2 *converges from* $\Lambda_0 = I$, *or in case B is invertible eigenvalues of A are such that*

$$\rho_m < \min\left\{\frac{1}{\sqrt{\alpha}}, \frac{1}{\sqrt{\beta}}\right\}. \tag{10}$$

Then:

(a) *There exists* $K > 0$ *such that for every* $K_0 \geqslant 0$, $\lim_{k\to\infty} K_k = K$. *Furthermore K is the unique solution of the algebraic matrix equation*

$$K = A^{\mathrm{T}}KA + Q - \bar{\alpha}A^{\mathrm{T}}KB(R + B^{\mathrm{T}}KB)^{-1}B^{\mathrm{T}}KA$$

within the class of positive semidefinite matrices.

(b) *The corresponding closed-loop* (CL) *system is stable*; *that is, the* $2n$-*dimensional system* $[x_k\ e_k]^{\mathrm{T}}$ *remains bounded in the m.s. sense.*

Proof. Part (a) follows from Lemma 1 (or Lemma 2 if B is not invertible). For part (b), from (9), we see that $E\{\|e_k\|^2\}$ remains bounded if and only if the spectral radius of A is bounded by

$$\rho_m < \frac{1}{\sqrt{\beta}}.$$

Now, the closed-loop system evolves according to

$$x_{k+1} = (A + \alpha_k BG)x_k - \alpha_k BGe_k + w_k,$$

where $G = -(R + B^{\mathrm{T}}KB)^{-1}B^{\mathrm{T}}KA$. Note that, the estimation error covariance $E\{\|e_k\|^2\}$ is uniformly bounded. Thus, the state, x_k, will remain bounded, if and only if the linear system

$$\xi_{k+1} = (A + \alpha_k BG)\xi_k \tag{11}$$

with the initial condition $\xi_0 = x_0$, is stable in the m.s. sense.

By (10), and Lemma 1, the sequence generated by the RE converges. Thus by the DP equation (or direct substitution) we can verify the following useful equality:

$$\begin{aligned} K = {} & \bar{\alpha}G^{\mathrm{T}}RG + \alpha A^{\mathrm{T}}KA + \bar{\alpha}(A + BG)^{\mathrm{T}} \\ & \times K(A + BG) + Q. \end{aligned} \tag{12}$$

We will now show that the system (11) is m.s. stable, which in turn will imply that $E\{\|x_k\|^2\}$ is bounded. We have for all k, by using (12)

$$\begin{aligned} E\{\xi_{k+1}^{\mathrm{T}}K\xi_{k+1} - \xi_k^{\mathrm{T}}K\xi_k\} &= E\{\xi_k^{\mathrm{T}}(\alpha A^{\mathrm{T}}KA - K)\xi_k\} \\ &\quad + E\{\bar{\alpha}\xi_k^{\mathrm{T}}(A + BG)^{\mathrm{T}}K \\ &\quad \times (A + BG)\xi_k\} \\ &= -E\{\xi_k^{\mathrm{T}}(Q + \bar{\alpha}G^{\mathrm{T}}RG)\xi_k\}. \end{aligned}$$

Hence

$$\begin{aligned} & E\{\xi_{k+1}^{\mathrm{T}}K\xi_{k+1}\} \\ & \quad = E\{\xi_0^{\mathrm{T}}K\xi_0\} - \sum_{i=0}^{k} E\{\xi_i^{\mathrm{T}}(Q + \bar{\alpha}G^{\mathrm{T}}RG)\xi_i\}. \end{aligned}$$

Since the left-hand side of this equation is bounded below by zero

$$\lim_{k\to\infty} E\{\xi_k^{\mathrm{T}}(Q + \bar{\alpha}G^{\mathrm{T}}RG)\xi_k\} = 0.$$

Since $R > 0$, in view of the observability assumption, we must have $E\{\|\xi_k\|^2\} \to 0$. Therefore, we conclude that $E\{\|x_k\|^2\}$ is bounded as $k \to \infty$. □

4. Optimal control over UDP networks

4.1. Finite horizon optimal control

Consider the linear system dynamics (1)–(2) along with the quadratic cost structure. Now, we want to find the optimal controller under the UDP information structure, I_k^{UDP}. It is easy to see from the DP equation for the last stage that the optimal control policy for the last stage is identical to that in the TCP case:

$$\begin{aligned} J_{N-1}(I_{N-1}^{\mathrm{UDP}}) = {} & E\{x_{N-1}^{\mathrm{T}}K_{N-1}x_{N-1}|I_{N-1}^{\mathrm{UDP}}\} \\ & + E\{e_{N-1}^{\mathrm{T}}P_{N-1}e_{N-1}|I_{N-1}^{\mathrm{UDP}}\} \\ & + E\{w_{N-1}^{\mathrm{T}}Fw_{N-1}\}, \end{aligned}$$

where K_{N-1} and P_{N-1} are as given in Section 3.1, and

$$u_{N-1}^* = -(R + B^{\mathrm{T}}FB)^{-1}B^{\mathrm{T}}FAE\{x_{N-1}|I_{N-1}^{\mathrm{UDP}}\}.$$

The DP equation for period $N-2$ is identical to (5) with I_{N-2}^{TCP} replaced by I_{N-2}^{UDP}. However, this time we cannot claim that the control does not have *dual effect* (Shalom & Tse, 1974). In order to see the extent of past control u_{N-2}'s effect on the future state estimation error, we expand (5), and after some algebra we arrive at the following equation:

$$\begin{aligned} J_{N-2}(I_{N-2}^{\mathrm{UDP}}) = {} & E\{x_{N-2}^{\mathrm{T}}A^{\mathrm{T}}K_{N-1}Ax_{N-2}|I_{N-2}^{\mathrm{UDP}}\} \\ & + E\{x_{N-2}^{\mathrm{T}}Qx_{N-2}|I_{N-2}^{\mathrm{UDP}}\} \\ & + \beta E\{e_{N-2}^{\mathrm{T}}A^{\mathrm{T}}P_{N-1}Ae_{N-2}|I_{N-2}^{\mathrm{UDP}}\} \\ & + \bar{\alpha}\min_{u_{N-2}}[u_{N-2}^{\mathrm{T}}B^{\mathrm{T}}K_{N-1}Bu_{N-2} \\ & + u_{N-2}^{\mathrm{T}}(R + \alpha\beta B^{\mathrm{T}}P_{N-1}B)u_{N-2} \\ & + 2\hat{x}_{N-2}^{\mathrm{T}}A^{\mathrm{T}}K_{N-1}Bu_{N-2}] \\ & + E\{w_{N-2}^{\mathrm{T}}K_{N-1}w_{N-2}\} \\ & + E\{w_{N-1}^{\mathrm{T}}Fw_{N-1}\}. \end{aligned} \tag{13}$$

Minimization in (13) yields, with $\hat{x}_k = E\{x_k|I_k^{\mathrm{UDP}}\}$:

$$\begin{aligned} u_{N-2}^* = {} & -(R + B^{\mathrm{T}}(K_{N-1} + \alpha\beta P_{N-1})B)^{-1} \\ & \times B^{\mathrm{T}}K_{N-1}A\hat{x}_{N-2}. \end{aligned}$$

Substituting the control back, we obtain

$$J_{N-2}(I_{N-2}^{\mathrm{UDP}}) = E\{x_{N-2}^{\mathrm{T}} K_{N-2} x_{N-2} | I_{N-2}^{\mathrm{UDP}}\} + E\{e_{N-2}^{\mathrm{T}} P_{N-1} e_{N-2} | I_{N-2}^{\mathrm{UDP}}\} + E\{w_{N-2}^{\mathrm{T}} K_{N-1} w_{N-2}\} + E\{w_{N-1}^{\mathrm{T}} F w_{N-1}\}.$$

Proceeding similarly we obtain: $u_k^* = G_k \hat{x}_k$, where

$$G_k = -(R + B^{\mathrm{T}}(K_{k+1} + \alpha\beta P_{k+1})B)^{-1} B^{\mathrm{T}} K_{k+1} A \tag{14}$$

with K_k and P_k given recursively by the *coupled* REs

$$P_k = \bar{\alpha} A^{\mathrm{T}} K_{k+1} B (R + B^{\mathrm{T}}(K_{k+1} + \alpha\beta P_{k+1})B)^{-1} \times B^{\mathrm{T}} K_{k+1} A + \beta A^{\mathrm{T}} P_{k+1} A, \tag{15}$$

$$K_k = A^{\mathrm{T}} K_{k+1} A - P_k + \beta A^{\mathrm{T}} P_{k+1} A + Q \tag{16}$$

with initial conditions $K_N = F$, $P_N = 0$.

Note that, although the control has a dual effect (Shalom & Tse, 1974) under the UDP information structure, the optimal estimator can still be designed separately

$$\hat{x}_k = \begin{cases} A\hat{x}_{k-1} + \bar{\alpha} B u_{k-1}, & \beta_k = 0, \\ x_k, & \beta_k = 1, \end{cases} \tag{17}$$

where $\hat{x}_0 = E_{P_{x_0}}\{x_0\}$ if $\beta_0 = 0$, otherwise $\hat{x}_0 = x_0$.

4.2. Infinite horizon optimal control

We again replace the objective function with the infinite-horizon average cost criterion given by

$$J_\pi = \limsup_{N\to\infty} \frac{1}{N} E\left\{ \sum_{k=0}^{N-1} x_k^{\mathrm{T}} Q x_k + \alpha_k u_k^{\mathrm{T}} R u_k \right\}.$$

Let us start by investigating the asymptotic properties of the coupled REs (15)–(16). First, we present a negative result, which shows the necessity of the condition of Theorem 3.

Lemma 4. *Let $(A, Q^{1/2})$ be observable. Suppose*

$$\max_i |\lambda_i(A)| \geqslant \min\left\{ \frac{1}{\sqrt{\alpha}}, \frac{1}{\sqrt{\beta}} \right\}.$$

Then, $\{P_k\}$ and $\{K_k\}$ generated by (15)–(16) diverge as $k\to\infty$.

Proof. Reversing the time, from (15)–(16) we have

$$P_{k+1} = \beta A^{\mathrm{T}} P_k A + (1-\alpha) A^{\mathrm{T}} (K_k - M_k) A,$$

$$K_{k+1} = \alpha A^{\mathrm{T}} K_k A + (1-\alpha) A^{\mathrm{T}} M_k A + Q, \tag{18}$$

where

$$M_k = K_k - K_k B (R + B^{\mathrm{T}}(K_k + \alpha\beta P_k)B)^{-1} B^{\mathrm{T}} K_k \geqslant K_k - K_k B (R + B^{\mathrm{T}} K_k B)^{-1} B^{\mathrm{T}} K_k := \bar{M}_k.$$

It is known that under the assumption of $(A, Q^{1/2})$ observable, $\bar{M}_k \geqslant 0$. Thus,

$$K_{k+1} \geqslant \alpha A^{\mathrm{T}} K_k A + Q. \tag{19}$$

It follows from (19) that since $(A, Q^{1/2})$ is observable, if $\sqrt{\alpha}A$ is not asymptotically stable, $\{K_k\}$ diverges as $k \to \infty$. Also, since $K_k - M_k \geqslant 0$,

$$P_{k+1} \geqslant \beta A^{\mathrm{T}} P_k A$$

which indicates that unless $\sqrt{\beta}A$ is asymptotically stable, $\{P_k\}$ will diverge. □

Hence, we conclude that the condition (10) of Theorem 3 is also *necessary* for the convergence of the REs (15)–(16).

In order to find a sufficient condition for asymptotic stability of the RE (15)–(16), we proceed as in Koning (1982), where it has been shown that the m.s. stabilizability of the system (1) with a stationary control law of the form

$$u_k = G\hat{x}_k,$$

where $\hat{x}_k$ is the optimal estimator given by (17), is sufficient for the convergence of the REs (15)–(16) from $K_0 = 0$, and $P_0 = 0$. Then, one can relate the m.s. stabilizability of the system (1) to an auxiliary optimal control problem of minimizing a norm of the terminal state, $E\{\|x_N\|^2\}$, for a given $N \geqslant n$, where n is the dimension of the state. Note that the solution of this problem may not be unique, and it corresponds to the optimal controller under the UDP information structure with $Q = R = 0$, and $F = I$. Thus, we arrive at the following result.

Lemma 5. *Let $(A, Q^{1/2})$ be observable. Then, the coupled REs (15)–(16) converge if and only if the following coupled REs converge from the initial condition $\Lambda_0 = I$, $\Pi_0 = 0$*

$$\Lambda_{k+1} = -\bar{\alpha} A^{\mathrm{T}} \Lambda_k B (B^{\mathrm{T}}(\Lambda_k + \alpha\beta\Pi_k)B)^{-1} B^{\mathrm{T}} \Lambda_k A + A^{\mathrm{T}} \Lambda_k A, \tag{20}$$

$$\Pi_{k+1} = \bar{\alpha} A^{\mathrm{T}} \Lambda_k B (B^{\mathrm{T}}(\Lambda_k + \alpha\beta\Pi_k)B)^{-1} B^{\mathrm{T}} \Lambda_k A + \beta A^{\mathrm{T}} \Pi_k A. \tag{21}$$

Proof. The proof follows from Theorem 5 of Koning (1982), and the preceding discussion. □

For a given pair of failure probabilities (α, β), Lemma 5 provides a test for checking the convergence of the REs (15)–(16) from an arbitrary initial condition $K_0 = F \geqslant 0$. However, analytical calculation of the stability region is not possible due to the nonlinear nature of the REs (20)–(21). Nevertheless, as we show next, in the case when B is invertible, it is possible to find *sufficient* conditions for the convergence of these equations by bounding the recursion of (Λ_k, Π_k) from above by a linear recursion. For this purpose, we first state a rather obvious, but useful, result.

Lemma 6. *Let the matrix recursions*

$$X_{k+1} = T_1(X_k, Y_k), \quad Y_{k+1} = T_2(X_k, Y_k)$$

are given, where X, Y are symmetric matrices of the same dimension. Suppose there exist monotonically increasing functions $L_1(X,Y)$, $L_2(X,Y)$ such that for all symmetric X, Y

$$T_1(X,Y) \leqslant L_1(X,Y),$$

$$T_2(X,Y) \leqslant L_2(X,Y).$$

Then, starting with $\bar{X}_0 = X_0$, $\bar{Y}_0 = Y_0$, we have $X_k \leqslant \bar{X}_k$, $Y_k \leqslant \bar{Y}_k$ for all $k \geqslant 0$ where $\bar{X}_{k+1} = L_1(\bar{X}_k, \bar{Y}_k)$, $\bar{Y}_{k+1} = L_2(\bar{X}_k, \bar{Y}_k)$.

Proof. The proof follows by induction. Say at time k, we have $X_k \leqslant \bar{X}_k$, $Y_k \leqslant \bar{Y}_k$. Then

$$X_{k+1} = T_1(X_k, Y_k) \leqslant L_1(X_k, Y_k) \leqslant L_1(\bar{X}_k, \bar{Y}_k) = \bar{X}_{k+1},$$

$$Y_{k+1} = T_1(X_k, Y_k) \leqslant L_2(X_k, Y_k) \leqslant L_2(\bar{X}_k, \bar{Y}_k) = \bar{Y}_{k+1}. \qquad \square$$

The next lemma shows how to bound the recursions of (Λ_k, Π_k) from above by linear recursions when B is $n \times n$, and invertible.

Lemma 7. *Let B be invertible. Then, the sequence of matrices $(\bar{\Lambda}_k, \bar{\Pi}_k)$ obtained from the linear recursions*

$$\bar{\Lambda}_{k+1} = \alpha A^{\mathrm{T}} \bar{\Lambda}_k A + \alpha\bar{\alpha}\beta A^{\mathrm{T}} \bar{\Pi}_k A, \tag{22}$$

$$\bar{\Pi}_{k+1} = \bar{\alpha} A^{\mathrm{T}} \bar{\Lambda}_k A + \beta A^{\mathrm{T}} \bar{\Pi}_k A \tag{23}$$

with the initial condition $(\bar{\Lambda}_0, \bar{\Pi}_0) = (I, 0)$ are such that

$$\Lambda_k \leqslant \bar{\Lambda}_k, \quad \Pi_k \leqslant \bar{\Pi}_k \quad \text{for all } k \geqslant 0,$$

where (Λ_k, Π_k) are generated by (20)–(21).

Proof. Using the property that B is invertible, the updates (20)–(21) can be simplified to

$$\Lambda_{k+1} = A^{\mathrm{T}} \Lambda_k A - \bar{\alpha} A^{\mathrm{T}} \Lambda_k (\Lambda_k + \alpha\beta\Pi_k)^{-1} \Lambda_k A,$$

$$\Pi_{k+1} = \beta A^{\mathrm{T}} \Pi_k A + \bar{\alpha} A^{\mathrm{T}} \Lambda_k (\Lambda_k + \alpha\beta\Pi_k)^{-1} \Lambda_k A.$$

Note that, the update for Π_k can be written as

$$\begin{aligned}\Pi_{k+1} = {} & \bar{\alpha} A^{\mathrm{T}} \Lambda_k^{1/2} \Lambda_k^{1/2} (\Lambda_k + \alpha\beta\Pi_k)^{-1} \Lambda_k^{1/2} \Lambda_k^{1/2} A \\ & + \beta A^{\mathrm{T}} \Pi_k A.\end{aligned}$$

Next, we claim that for all $k \geqslant 0$

$$\Lambda_k^{1/2} (\Lambda_k + \alpha\beta\Pi_k)^{-1} \Lambda_k^{1/2} \leqslant I.$$

To see this, let $L_k^{\mathrm{T}} = \Lambda_k^{1/2}$, and rewrite the inequality as

$$L_k^{\mathrm{T}} (L_k L_k^{\mathrm{T}} + \alpha\beta\Pi_k)^{-1} L_k \leqslant I.$$

The inequality follows from the fact that $\Pi_k \geqslant 0$, $\forall k \geqslant 0$.

The update for Λ_k can similarly be written as

$$\begin{aligned}\Lambda_{k+1} = {} & -\bar{\alpha} A^{\mathrm{T}} [\Lambda_k - \Lambda_k (\Lambda_k + \alpha\beta\Pi_k)^{-1} \Lambda_k] A \\ & + \alpha A^{\mathrm{T}} \Lambda_k A.\end{aligned}$$

Now, we claim that

$$\Lambda_k - \Lambda_k (\Lambda_k + \alpha\beta\Pi_k)^{-1} \Lambda_k \leqslant \alpha\beta\Pi_k.$$

Let $\Gamma_k = \alpha\beta\Pi_k$. Then the condition is equivalent to

$$\Lambda_k - \Lambda_k (\Lambda_k + \Gamma_k)^{-1} \Lambda_k \leqslant \Gamma_k,$$

where $\Lambda_k > 0$, and $\Gamma_k \geqslant 0$. If $\Gamma_k = 0$, the inequality holds with equality. If $\Gamma_k > 0$, we use the matrix inversion lemma to rewrite the inequality as

$$\Lambda_k^{-1} + \Gamma_k^{-1} \geqslant \Gamma_k^{-1} \Rightarrow \Lambda_k^{-1} \geqslant 0$$

thus completing the proof. $\square$

Remark 8. If A is scalar, the condition for convergence of the linear recursions (22)–(23) is given by

$$\lambda\left(A^2 \begin{bmatrix} \alpha & \alpha\bar{\alpha}\beta \\ \bar{\alpha} & \beta \end{bmatrix}\right) < 1$$

which can be expressed as

$$|A| < \left(\frac{1}{\alpha^2 (2-\alpha)\beta}\right)^{1/4}, \tag{24}$$

$$\alpha^2 (2-\alpha)\beta A^4 - (\alpha+\beta) A^2 + 1 > 0. \tag{25}$$

In general, the convergence of the REs (15)–(16) is not sufficient for the optimal UDP controller to be stabilizing. However, under the observability assumption it can be shown that the closed-loop system is m.s. stable.

Theorem 9. *Let $(A, Q^{1/2})$ be observable. Suppose that the REs (20)–(21) converge from the initial condition $(I, 0)$. Then:*

(a) *There exist $K > 0$, $P > 0$ such that for every $K_0 \geqslant 0$ and $P_0 = 0$, we have*

$$\lim_{k\to\infty} K_k = K, \quad \lim_{k\to\infty} P_k = P.$$

Furthermore K, P are the unique solutions of the algebraic matrix equations

$$\begin{aligned}P = {} & \bar{\alpha} A^{\mathrm{T}} K B (R + B^{\mathrm{T}} (K + \alpha\beta P) B)^{-1} B^{\mathrm{T}} K A \\ & + \beta A^{\mathrm{T}} P A.\end{aligned}$$

$$K = A^{\mathrm{T}} K A - P + \beta A^{\mathrm{T}} P A + Q$$

within the class of positive semidefinite matrices.

(b) *The corresponding closed-loop system is stable; that is, the $2n$-dimensional system $[x_k\ e_k]^{\mathrm{T}}$ remains bounded in the m.s. sense.*

Proof. Part (a) of the proof follows from the preceding discussion. For part (b), first note that for a given $n \times n$ matrix S, we have

$$\begin{aligned}& E\{e_k^{\mathrm{T}} S x_k + x_k^{\mathrm{T}} S^{\mathrm{T}} e_k | I_k\} \\ & \quad = E\{x_k^{\mathrm{T}} (S + S^{\mathrm{T}}) x_k - \hat{x}_k^{\mathrm{T}} (S + S^{\mathrm{T}}) \hat{x}_k | I_k\} \\ & \quad = E\{(x_k - \hat{x}_k)^{\mathrm{T}} (S + S^{\mathrm{T}}) (x_k - \hat{x}_k) | I_k\} \\ & \quad = E\{e_k^{\mathrm{T}} (S + S^{\mathrm{T}}) e_k\}.\end{aligned} \tag{26}$$

Now, write

$$E\{x_{k+1}^{\mathrm{T}}Kx_{k+1} - x_k^{\mathrm{T}}Kx_k + e_{k+1}^{\mathrm{T}}Pe_{k+1} - e_k^{\mathrm{T}}Pe_k\}$$
$$= -E\{x_k^{\mathrm{T}}(Q + \bar{\alpha}G^{\mathrm{T}}RG)x_k + \bar{\alpha}e_k^{\mathrm{T}}G^{\mathrm{T}}RGe_k\}, \tag{27}$$

where we made use of (26), and the following equalities:

$$K = \alpha A^{\mathrm{T}}KA + \bar{\alpha}(A+BG)^{\mathrm{T}}K(A+BG) + \bar{\alpha}G^{\mathrm{T}}(R + \alpha\beta B^{\mathrm{T}}PB)G + Q,$$

$$P = \bar{\alpha}A^{\mathrm{T}}KA - \bar{\alpha}(A+BG)^{\mathrm{T}}K(A+BG) + \beta A^{\mathrm{T}}PA - \bar{\alpha}G^{\mathrm{T}}(R + \alpha\beta B^{\mathrm{T}}PB)G$$

which can be verified by direct substitution.

Summing (27) over k yields

$$E\{x_{k+1}^{\mathrm{T}}Kx_{k+1} + e_{k+1}^{\mathrm{T}}Pe_{k+1}\} = E\{x_0^{\mathrm{T}}Kx_0 + e_0^{\mathrm{T}}Pe_0\} - \sum_{i=0}^{k} E\{x_i^{\mathrm{T}}(Q + \bar{\alpha}G^{\mathrm{T}}RG)x_i\} + \sum_{i=0}^{k} E\{e_i^{\mathrm{T}}(\bar{\alpha}G^{\mathrm{T}}RG)e_i\}. \tag{28}$$

Since, for $Z \geqslant 0$, $\beta E\{x_k^{\mathrm{T}}Zx_k\} \geqslant E\{e_k^{\mathrm{T}}Ze_k\}$, from (28) we can write

$$E\{x_{k+1}^{\mathrm{T}}Kx_{k+1} + e_{k+1}^{\mathrm{T}}Pe_{k+1}\} \leqslant E\{x_0^{\mathrm{T}}Kx_0 + e_0^{\mathrm{T}}Pe_0\} - \sum_{i=0}^{k} E\{x_i^{\mathrm{T}}(Q + \bar{\alpha}\bar{\beta}G^{\mathrm{T}}RG)x_i\}.$$

Since the left-hand side of this inequality is bounded below by zero, it follows that

$$\lim_{k\to\infty} E\{x_k^{\mathrm{T}}(Q + \bar{\alpha}\bar{\beta}G^{\mathrm{T}}RG)x_k\} = 0.$$

Since $R > 0$, in view of the observability assumption, we must have $E\{\|x_k\|^2\}$, and $E\{\|e_k\|^2\}$ bounded unless $\alpha = 1$ or $\beta = 1$. □

Before closing our account on this section, we illustrate the range of link failure probabilities, (α, β), for which the system can be stabilized under the optimal TCP and UDP controllers. In Fig. 3 we plot the stability region in the α–β plane for a scalar plant with $A = 2$, $A = 1.4$, and $A = 1.1$. The solid lines in Fig. 3 is the condition (25) of Remark 8, and the system can be stabilized if the actual failure probabilities on the links are between this curve, and the α and β axis. Note that this curve represents only a *sufficient* condition for the optimal UDP controller to be stabilizing. The dashed lines in Fig. 3 describe the region of failure probabilities for which the TCP controller can stabilize the same plant. This condition is both *necessary* and *sufficient* for the optimal TCP controller to be stabilizing as per Theorem 3.

Note that, as the plant becomes more open-loop unstable, the stability region becomes smaller. This is an expected result, because intuitively the more open-loop unstable a plant is, the more frequently we need to observe and control it.

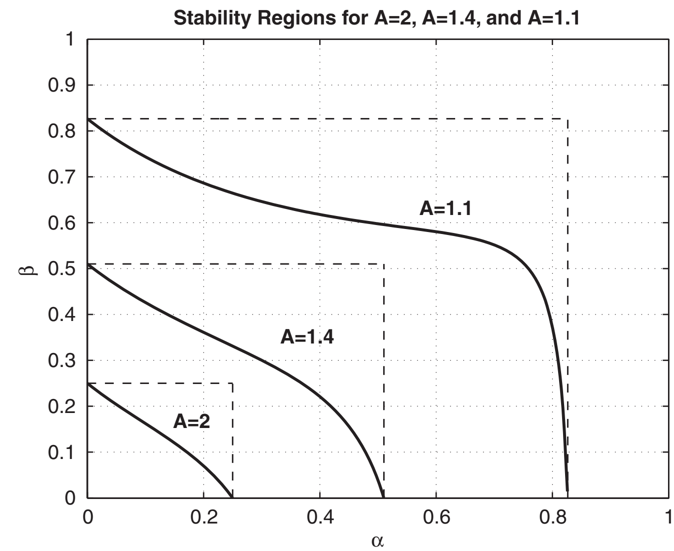

Fig. 3. Region of failure probabilities for which the optimal TCP (dashed) and UDP (solid) controllers can stabilize a plant with $A = 2$, 1.4, and 1.1.

5. A numerical example

In this section, we present some numerical results we obtained in Matlab to compare the performance of the optimal controllers in the TCP and UDP cases as the link failure probabilities, α and β, are varied. Consider the following open-loop unstable scalar plant:

$$x_{k+1} = 2x_k + \alpha_k u_k + w_k, \tag{29}$$

where $A = 2$, and $B = 1$. Let the noise process, $\{w_k\}$, be zero-mean with variance $\sigma_w^2 = 1$. The initial state is also zero-mean with variance $\sigma_{x_0}^2 = 1$.

If we let $Q = R = 1$, the RE in the TCP case is equivalent to

$$(1-4\alpha)K^2 - 4K - 1 = 0$$

with the positive solution

$$K = \frac{2 + \sqrt{4 + (1-4\alpha)}}{1-4\alpha}$$

assuming $1 - 4\alpha > 0$, i.e., $\alpha < \frac{1}{4}$. In terms of α, the gain of the infinite horizon TCP controller can be calculated as

$$G = -\frac{2K}{1+K} = \frac{4 + 2\sqrt{4 + (1-4\alpha)}}{3 - 4\alpha + \sqrt{4 + (1-4\alpha)}}.$$

In the UDP case, the coupled REs are given by

$$(1-4\beta)(1-4\alpha-4\beta+7\alpha\beta)P^2 - (1-4\beta)(14-8\alpha)P + 4(1-\alpha) = 0,$$

$$\tfrac{1}{3}((1-4\beta)P - 1) = K.$$

One can solve for (K, P) in the above equation, and substitute it into the expression for the gain of the infinite horizon UDP

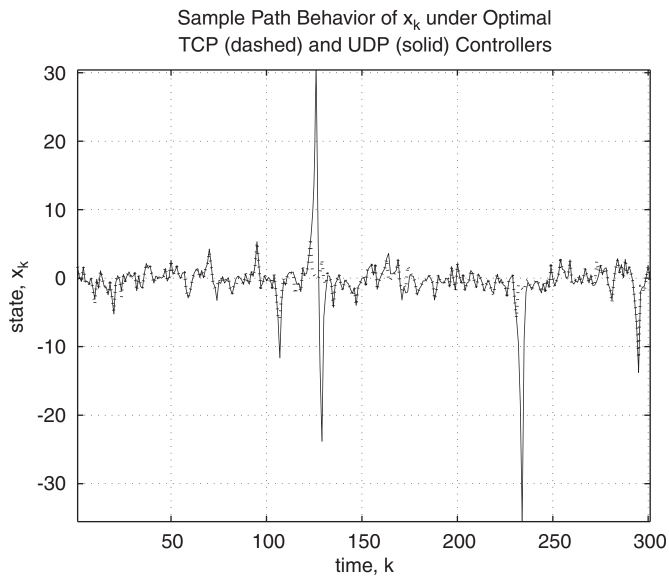

Fig. 4. Typical sample path of the plant state, x_k, under optimal TCP and UDP controllers.

controller to find the optimal stationary feedback control policy in the UDP case.

Note that for the estimator to be stable, in the TCP case, we need $|A| < 1/\sqrt{\beta}$, which for $A = 2$ implies that $\beta < \frac{1}{4}$.

We next simulate the optimal control laws for both types of protocols, as we vary α and β. The UDP controller seems to stabilize the plant only when approximately $0 \leqslant \alpha < 0.25$ and $0 \leqslant \beta < 0.25 - \alpha$, whereas the TCP controller has a larger stability region in the α-β plane, as expected.

We fix the decision horizon, N, to $N = 100$, and simulate the linear system (29) under the TCP and UDP controllers. Fig. 4 shows the typical sample path behavior of the state, x_k, under the TCP (solid curve), and UDP (dashed curve) controllers. In Fig. 4, the drop probabilities on the links are taken to be $(\alpha, \beta) = (0.15, 0.10)$.

Finally, we compare the sample path average costs under both controllers by averaging 300-stage average sample path cost over 1000 sample paths of the plant process. This yields

$$\bar{J}^{\mathrm{TCP}} \approx 15.77, \quad \bar{J}^{\mathrm{UDP}} \approx 25.44.$$

We clearly have $\bar{J}^{\mathrm{TCP}} < \bar{J}^{\mathrm{UDP}}$, since the TCP controller has access to more information than the UDP controller, resulting in a smaller average cost in the TCP case.

6. Conclusions and discussion on some extensions

In this paper, we introduced the problem of optimally controlling a linear discrete-time plant when some of the measurement and control packets are missing. We made the assumption that the packet loss processes are simple independent Bernoulli processes with control and measurement packets being lost independent across time. In this setting, we showed that the optimal control depends on the information structure of the controller, which in turn depends on the characteristics of the underlying network. Under a network structure that supports acknowledgements, we have shown that the optimal control law that minimizes a quadratic performance criterion is linear, and can be obtained by dynamic programming. Moreover, the RE that describes the evolution of the controller gain is a modified version of the standard RE with a scalar parameter that accounts for the packet loss probability on the network links. If the underlying network does not support acknowledgment packets, we have seen that the optimal control remains linear, if there is no noise in the observations. However, with no acknowledgments, the REs that describe the evolution of the controller gain become a coupled set of two matrix recursions, and we derived conditions for the convergence of these coupled REs.

There are several ways the results of this paper can be extended. We enumerate below some specific problems:

(1) One can investigate the case when the actuator applies the "last available control", as opposed to "zero control", when the control packet is lost. This extension requires extending the state–space model of the system to

$$x_{k+1} = Ax_k + \alpha_k B u_k + (1-\alpha_k)B\xi_k + w_k,$$

$$\xi_{k+1} = \alpha_k \xi_k + (1-\alpha_k)u_k,$$

where ξ_k is the state variable that keeps track of the last applied control by the actuator. Defining a new state $\bar{x}_k := [x_k \ \xi_k]^{\mathrm{T}}$, we can write the above system in state–space form as follows:

$$\bar{x}_{k+1} = \begin{bmatrix} A_{n\times n} & (1-\alpha_k)B_{n\times m} \\ 0_{m\times n} & \alpha_k I_{m\times m} \end{bmatrix} \bar{x}_k + \begin{bmatrix} \alpha_k B \\ (1-\alpha_k) I_{m\times m} \end{bmatrix} u_k + \begin{bmatrix} I_{n\times n} \\ 0_{m\times n} \end{bmatrix} w_k,$$

where $I_{m\times m}$, and $0_{m\times n}$ denote the $m \times m$ identity matrix, and $m \times n$ zero matrix, respectively. The optimal controller in the TCP case can be derived following along the lines of the derivations of Section 3.1. Since we have access to α_{k-1}, there is no dual-effect, hence the only difference between this case and the zero-control case of Section 3.1 is the randomness of the plant matrices $(A(\alpha_k), B(\alpha_k))$ through α_k. Therefore, the REs (6)–(7) describing the optimal TCP controller will remain the same with the matrices (A, B) replaced by their counterparts $(\bar{A}, \bar{B})$

$$\bar{A} = \begin{bmatrix} A & \alpha B \\ 0 & (1-\alpha) I \end{bmatrix},$$

$$\bar{B} = \begin{bmatrix} (1-\alpha)B \\ \alpha I \end{bmatrix}.$$

The UDP case is more involved due to the dual nature of the control in this case. In this case, one has to follow along the lines of the derivations of Section 4.1 by expanding the future estimation error terms in the current cost-to-go function. This expansion needs to take into account the random nature of the plant matrices $(A(\alpha_k), B(\alpha_k))$.

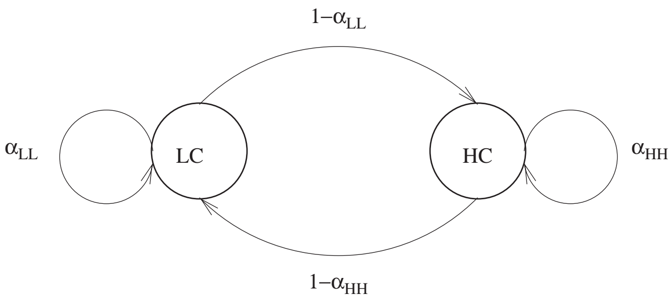

Fig. 5. Two-state Markov-chain model for packet drops.

(2) Another extension of this paper is to study the noisy measurements case, where the measurement equation (2) is replaced by

$$y_k = \beta_k(x_k + v_k)$$

where $\{v_k\}$ is a zero-mean i.i.d. Gaussian random process. Note that only the sensor measurements are noisy, not the packet drops. This problem can be easily solved under the TCP information structure, and the state estimator can be shown to be linear in the best estimate of the state because of the acknowledgments in TCP. The UDP case is more difficult due to the dual nature of the control.

(3) In general, the link failure process $\{\alpha_k\}$ can be correlated. This correlation can be modeled as a two-state Markov chain, where state HC corresponds to high congestion (HC), and state LC corresponds to low congestion (LC), as shown in Fig. 5. Now given the transition probabilities, $(\alpha_{LL}, \alpha_{HH})$, between these states, the problem is to determine the optimal controller under the TCP and UDP information structures. In the TCP case, since at time k, the controller has access to α_{k-1}, the solution can be obtained by a direct extension of the derivation of Section 3.1. If we do not have access to α_k's, on the other hand, the derivation is analogous to the one in Section 4.1. As in the derivation of the optimal UDP controller for the uncorrelated case, one needs to expand the future estimation error terms in the cost-to-go function of the current stage to see their effect on the current cost. The quadratic nature of the cost-to-go functions will be preserved, however the coupled REs describing the evolution of the cost will be more involved containing the probabilities $(\beta, \alpha_{LL}, \alpha_{HH})$.

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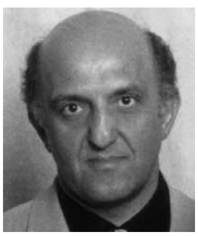

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