An Asymptotically Optimal Two-Part Fixed-Rate Coding Scheme for Networked Control

by

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Abstract

It is known that under fixed-rate information constraints, adaptive quantizers can be used to stabilize an open-loop-unstable linear system on $\mathbb{R}^n$ driven by unbounded noise. These adaptive schemes can be designed so that they have near-optimal rate, and the resulting system will be stable in the sense of having an invariant probability measure, or ergodicity, as well as boundedness of the state second moment. In addition, structural results and information theoretic bounds on the performance of encoders have also been studied. However, the performance of such adaptive quantizers beyond stabilization has not been addressed. In this thesis, we construct a two-part adaptive coding scheme that achieves state second moment convergence to the classical optimum (i.e., for the fully observed setting) under a mild moment condition on the noise process. The first part, as in prior work in this context, leads to ergodicity (via positive Harris recurrence) and the second part ensures that the state second moment converges to the classical optimum at high rates. These results are established using an intricate analysis which uses random-time Lyapunov drift conditions as a core tool.

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Chapter 1

Introduction

Networked control or *information-constrained control* refers to control systems in which the controllers, sensors, and systems (actuators/plants) are connected through communication channels or a data-rate constrained network. Thus, there may be a data link between the sensors (which collect information), the controllers (which make decisions), and the actuators (which execute the controller commands). Moreover, the sensors, controllers and the plant themselves could be geographically separated. For such information-constrained control (or networked control) systems, one needs to jointly design encoders and controllers for satisfactory performance, which may have stability or optimality as a design objective.

In cases where stability is the primary design objective, one is typically concerned with the minimum capacity above which stabilization is possible, and there are many results of this flavour in the existing literature (which we will discuss in detail shortly in Section 1.4).

This thesis is primarily concerned with optimality as the primary design objective. In this context, the notion of optimality is minimization of some cost function over a specified time horizon (which here is infinite), and for the kinds of systems we consider here, one seeks asymptotic bounds on the cost function as the data rate becomes large.

1.1 Contribution of Thesis

We study a large class of stochastic linear systems driven by unbounded noise that are to be controlled across a discrete noiseless channel of finite capacity. For such systems, we present a novel two-stage coding scheme, in which the first stage is time-adaptive and stabilizing (in the sense of positive Harris recurrence and finite limiting system moments), while the second stage is fixed in time. The first coding stage is a variation on the schemes in [1, 2, 3].

Crucially utilizing the ergodicity results of the first coding stage, we show that this two-part coding scheme attains convergence of the limiting system second moment to the classical optimum as the data rate C grows large, with explicit rate of convergence. While multi-stage quantization schemes have been studied before in the source coding literature [4], our implementation is novel in that one stage of the code is time-adaptive and stabilizing. An inspiration for this approach also comes from Berger [5] and Sahai [6].

To our knowledge, this is the first scheme proven to have this convergence property for systems of this type (in particular, with unbounded noise) in networked control.

The primary line of reasoning is relatively simple, but it requires some quite involved technical results which necessitate very careful analysis using, among other tools, random-time Lyapunov drift conditions.

1.2 Organization of Thesis

The thesis is organized as follows. Section 1.3 introduces the networked control problem we study and Section 1.4 provides a brief literature review. The remainder of Chapter 1 contains relevant preliminaries, some background on stochastic stability for general state-space Markov chains, and statements of some useful random-time Lyapunov drift theorems that are central to our analysis.

Chapter 2 presents the construction of our two-part scheme in the scalar case for simplicity, and provides a somewhat detailed proof program to guide the reader. Chapter 3 provides the construction of our two-part scheme in the vector case, in full generality. These two chapters contain our main results.

Chapter 4 is dedicated to the proof program for our ultimate result in the vector case, Theorem 3.0.1. Chapter 5 briefly addresses the case of open-loop-stable modes, and Chapter 6 presents a simulation that illustrates our two-part scheme and main results.

Finally, as many of the proofs are rather mechanical and quite involved, the appendix contains complete proofs of many key results stated in the body of the thesis.

1.3 Problem Statement

First, we introduce the system to be controlled under no information constraints. The optimal cost in this "classical" case will yield a lower bound over all information-constrained policies.

Consider control of the multi-dimensional system,

$$x_{t+1} = Ax_t + Bu_t + w_t, \tag{1.1}$$

where the state process x_t, the control u_t, and the noise process w_t live in $\mathbb{R}^n$, A and B are $n \times n$ real matrices, and we assume B to be invertible (this can be relaxed to controllability by a sampling argument for stability, though in this case the optimality results we present are not maintained).

At each time stage $t \geq 0$, the controller has access to the history $I_t = x_{[0,t]} = (x_0, \ldots, x_t)$ and may apply any control $u_t \in \mathbb{R}^n$. The noise process $\{w_t\}_{t=0}^{\infty}$ is assumed to be i.i.d., zero-mean with covariance matrix $\Sigma := E\left[w_0 w_0^T\right]$. Furthermore, we suppose that the noise process admits a pdf η with respect to Lebesgue measure on $\mathbb{R}^n$ which is positive everywhere (in particular, this implies that w_t has an unbounded support).

Consider the following idealized problem with full state measurements:

For an $n \times n$ positive definite matrix Q, the optimal control problem is to choose a policy γ which minimizes the infinite-horizon average quadratic form,

$$\limsup_{T \to \infty} \frac{1}{T} E^{\gamma} \left[\sum_{t=0}^{T-1} x_t^T Q x_t \right]. \tag{1.2}$$

Formally, an admissible policy γ is a sequence of Borel measurable mappings $\{\gamma_t, t \geq 0\}$ where $\gamma_t : I_t \to \mathbb{R}^n$ is such that it produces the control $u_t = \gamma_t(I_t)$ at each time stage.

Proposition 1.3.1. *For the fully observed setup described above, the optimal control policy is $u_t = -B^{-1} A x_t$, achieving an optimal cost of* $\operatorname{tr}(Q\Sigma)$ *where* $\operatorname{tr}(\cdot)$ *is the trace operator.*

A short proof of this result is provided in the appendix.

In contrast to the idealized fully observed setup, in this thesis we assume that the controller only has access to x_t through a discrete noiseless channel of capacity C

bits. We assume that the encoder is causal. In particular, with $\mathcal{M}$ a finite-cardinality alphabet (such that $|\mathcal{M}| \leq 2^C$), the encoder is specified by a quantization policy Π, which is a sequence of functions $\{\eta_t\}_{t=0}^{\infty}$ with $\eta_t : \mathcal{M}^t \times \mathbb{X}^{t+1} \to \mathcal{M}$. At time t, the encoder transmits the $\mathcal{M}$-valued message

$$q_t = \eta_t(I_t)$$

where $I_0 = x_0$, $I_t = (q_{[0,t-1]}, x_{[0,t]})$ for $t \geq 1$. The collection of all such zero-delay policies is called the set of admissible quantization policies and is denoted by $\mathbf{\Pi}_A$. Upon receiving q_t, the receiver generates the control u_t, also without delay. A zero-delay controller policy is a sequence of functions $\gamma = \{\gamma_t\}_{t=0}^{\infty}$ with $\gamma_t : \mathcal{M}^{t+1} \to \mathbb{U}$, where $\mathbb{U}$ is the control action space.

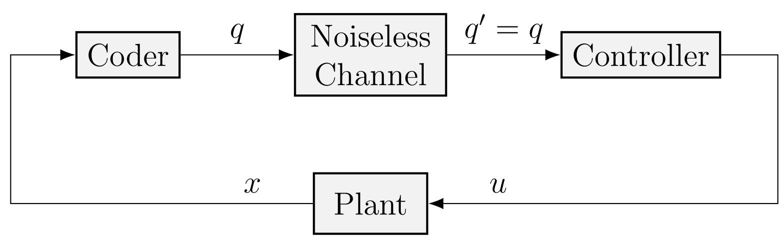

Figure 1.1: Block diagram of the communication and control loop.

Thus, the data rate is fixed, and we assume zero coding delay. In this setup, it becomes necessary to describe not just a control policy, but also a coding scheme with which to communicate information about the current state vector.

1.4 Literature Review

For systems of this nature, various authors have obtained the smallest channel capacity above which stabilization is possible, under various assumptions on the system and the admissible coders and controllers. This result is usually referred to as a *data-rate theorem* and takes the following form, with $\{\lambda_i\}$ being the eigenvalues of the system matrix A:

$$C > R_{\min} := \sum_{|\lambda_i|\geq 1} \log_2 |\lambda_i| \tag{1.3}$$

That is, the capacity must exceed the sum of the unstable eigenvalue logarithms.

Some of the earliest works in this context are [7] and [8]. More general versions of the data-rate theorem have been proven in [9] and [10]. For noisy systems and mean-square stabilization, or more generally, moment-stabilization, analogous data-rate theorems have been proven in [11] and [12], see also [13, 14].

In [1, 2], a joint fixed-rate coding and control scheme is given which, in the scalar case $n = 1$ with unstable eigenvalue $|\lambda| \geq 1$ and where w_t is Gaussian, stabilizes the system (1.1) while being nearly rate-optimal, in that the rate used satisfies only $C > \log(|\lambda| + 1)$. This is achieved using an adaptive uniform quantization scheme, where the quantizer bin sizes "zoom" in and out exponentially to track the state x_t. Here, the notion of stability is ergodicity and finiteness of all limiting system moments. By increasing a sampling period T, the achievable rate $\frac{1}{T}\log(|\lambda|^T + 1)$ gets arbitrarily close to $C > R_{\min} = |\lambda|$ [3, Theorem 2.3]. This scheme can be generalized to one which stabilizes the multi-dimensional system (1.1) (where the noise is more general than Gaussian) using a similar approach [3]. Furthermore, this leads to a closed loop system which is positive Harris recurrent (and hence, ergodic) and admits finite limiting system second moment [3, Theorem 2.2]. For related recent fixed-rate constructions, we refer the reader to [15] and [16].

Despite being near rate-optimal for achieving stability (i.e., *finite* system moments), the schemes in [1, 2, 3] have not been shown to yield second moment convergence to the classical optimum $\mathrm{tr}\,(Q\Sigma)$ as the data rate C grows large.

With regard to high-rates, among papers that are most relevant here is [17] where causal coding under a high rate assumption for stationary sources and individual sequences was studied. Linder and Zamir [17], among many other results, established asymptotic quantitative relations between the differential entropy rate and the entropy rate of a memoryless and uniformly quantized stationary process, and through this analysis established the near optimality of uniform quantizers in the low distortion regime. This study, however, assumed stationarity where adaptation is not needed. Furthermore, control is not present.

In the literature, information theoretic relaxations of the problem noted have been studied. These typically replace number of bins with entropy of the quantization symbols, or the latter with mutual information bounds. Yet another common method is via dithering which "uniformizes" the noise even for low rates under common randomness (see e.g., [18], [19], [20, 21, 22], [23], [24], [25]).

A particularly important approach is to minimize the directed information. This leads to an information theoretic lower bound to the optimal estimation error subject to an information rate constraint. There has been a surge of research activity on this problem since [26] where explicit solutions, bounds, as well as convex analytic numerical solutions (including via semi-definite programming) have been presented in [27, 28, 20, 29, 30, 31, 25, 32, 33, 34] and [35] (see also [22]).

See [36] and [37] for a detailed review on structural results for optimal coding of controlled linear systems. Infinite horizon zero-delay coding for linear systems are reported in [38, 39].

1.5 Definitions and Conventions

We will denote the nonnegative and strictly positive reals by $\mathbb{R}_+$ and $\mathbb{R}_{++}$, respectively.

For $x \in \mathbb{R}^n$ and $p \in [1, \infty)$ we denote

$$\|x\|_p := \left(\sum_{i=1}^{n} \left| x^i \right|^p \right)^{\frac{1}{p}}$$

and for $p = \infty$,

$$\|x\|_\infty := \max_{1 \le i \le n} |x^i|.$$

It is well-known that $\|\cdot\|_p$ is a norm on $\mathbb{R}^n$ for all $p \in [1, \infty]$. The following pair of inequalities is also well-known (e.g., see [40, Exercise 3.5(a)]):

Proposition 1.5.2. *Suppose $1 \le p \le q \le \infty$. Then for any $x \in \mathbb{R}^n$,*

$$\|x\|_q \le \|x\|_p \le n^{\frac{1}{p} - \frac{1}{q}} \|x\|_q . \tag{1.4}$$

where $\frac{1}{\infty}$ is taken to be 0 by convention.

The first inequality follows by a simple renormalization argument, while the second follows from Hölder's inequality.

For a matrix $V \in \mathbb{R}^{m \times n}$, we will refer to the (i, j)-th component either as V_{ij} or as $[V]_{ij}$. The latter notation will be used when V takes on an expression involving square brackets (e.g., an expectation) so as to avoid ambiguity.

For a matrix $A \in \mathbb{R}^{m\times n}$ we define its ∞-norm as

$$\|A\|_\infty := \max_{1\le i\le m} \sum_{j=1}^{n} |A_{ij}| .$$

This norm is consistent with the vector ∞-norm in the following sense. Let $v \in \mathbb{R}^n$ and $A \in \mathbb{R}^{m\times n}$. Then,

$$\|Av\|_\infty \le \|A\|_\infty \|v\|_\infty . \tag{1.5}$$

We will find it useful to use Landau notation for comparing function asymptotics. For two functions $f, g : [a, \infty) \to \mathbb{R}_+$, where $a \in \mathbb{R}$, we say that $f = \mathcal{O}_u(g)$ if

$$\limsup_{u\to\infty} \frac{f(u)}{g(u)} < \infty,$$

i.e., for all u sufficiently large one has $f(u) \le cg(u)$ for some constant $c > 0$.

Finally, for a vector-valued random variable X we denote its "tail function" as

$$T_X(u) := P(\|X\|_\infty > u), \quad u \ge 0.$$

1.6 Stochastic Stability

In this section we provide some brief background on stochastic stability, particularly that which will be relevant to the stabilizing properties of the two-stage scheme we present (namely, positive Harris recurrence).

Suppose $\{\phi_t\}_{t=0}^{\infty}$ is a Markov chain with state space $\mathbb{X}$, where $\mathbb{X}$ is a complete separable metric space that is locally compact; its Borel sigma algebra is denoted $\mathcal{B}(\mathbb{X})$. The transition probability is denoted by P, so that for any $\phi \in \mathbb{X}$ and $A \in$

$\mathcal{B}(\mathbb{X})$, the probability of moving in one step from state ϕ to the set A is given by $P(\phi_{t+1} \in A \mid \phi_t = \phi) = P(\phi, A)$. The n-step transitions are obtained in the usual way, $P(\phi_{t+n} \in A \mid \phi_t = \phi) = P^n(\phi, A)$ for any $n \geq 1$. The transition law acts on measurable functions $f : \mathbb{X} \to \mathbb{R}$ and measures μ on $\mathcal{B}(\mathbb{X})$ via

$$Pf(\phi) = \int_{\mathbb{X}} P(\phi, dy) f(y), \quad \text{for all } \phi \in \mathbb{X}$$

and

$$\mu P(A) = \int_{\mathbb{X}} \mu(d\phi) P(\phi, A), \quad \text{for all } A \in \mathcal{B}(\mathbb{X}).$$

A probability measure π on $\mathcal{B}(\mathbb{X})$ is called invariant if $\pi P = \pi$, that is:

$$\int_{\mathbb{X}} \pi(d\phi) P(\phi, A) = \pi(A), \quad \text{for all } A \in \mathcal{B}(\mathbb{X}).$$

For any initial probability measure ν on $\mathcal{B}(\mathbb{X})$ we can construct a stochastic process with transition law P and $\phi_0 \sim \nu$. We let P_ν denote the resulting probability measure on the sample space, with the usual convention that $\nu = \delta_\phi$ (i.e., $\nu(\{\phi\}) = 1$) when the initial state is $\phi \in \mathbb{X}$. When $\nu = \pi$ is invariant, the resulting process is stationary.

There is at most one stationary solution under the following irreducibility assumption. For a set $A \in \mathcal{B}(\mathbb{X})$ we denote,

$$\tau_A := \min \{t \geq 1 : \phi_t \in A\}. \tag{1.6}$$

Definition 1.6.1. Let φ denote a σ-finite measure on $\mathcal{B}(\mathbb{X})$.

(i) The Markov chain is called *φ-irreducible* if for any $\phi \in \mathbb{X}$ and $B \in \mathcal{B}(\mathbb{X})$ satisfying

$\varphi(B) > 0$ we have

$$P_\phi(\tau_B < \infty) > 0.$$

(ii) A φ-irreducible Markov chain is *aperiodic* if for any $\phi \in \mathbb{X}$ and any $B \in \mathcal{B}(\mathbb{X})$ satisfying $\varphi(B) > 0$, there exists $n_0 = n_0(\phi, B)$ such that for all $n \geq n_0$,

$$P^n(\phi, B) > 0.$$

(iii) A φ-irreducible Markov chain is *Harris recurrent* if $P_\phi(\tau_B < \infty) = 1$ for any $\phi \in \mathbb{X}$ and any $B \in \mathcal{B}(\mathbb{X})$ satisfying $\varphi(B) > 0$. It is *positive Harris recurrent* if in addition there is an invariant probability measure π.

The notion of full and absorbing sets will be useful to us.

Definition 1.6.2. For a φ-irreducible Markov chain $\{\phi_t\}_{t=0}^{\infty}$, a set $A \in \mathcal{B}(\mathbb{X})$ is called full if $\varphi(A^C) = 0$.

Definition 1.6.3. A set $A \in \mathcal{B}(\mathbb{X})$ is called absorbing if $P(x, A) = 1$ for all $x \in A$.

Finally, we define the notion of small sets for a Markov chain.

Definition 1.6.4. A set $C \in \mathcal{B}(\mathbb{X})$ is (m, δ, ν)-small on $(\mathbb{X}, \mathcal{B}(\mathbb{X}))$ (for integer $m \geq 1$, $\delta \in (0, 1]$ and a probability measure ν on $(\mathbb{X}, \mathcal{B}(\mathbb{X}))$) if for all $x \in C$ and $B \in \mathcal{B}(\mathbb{X})$,

$$P^m(x, B) \geq \delta\nu(B).$$

A set is called small (or sometimes m-small) if it is (m, δ, ν)-small for some (m, δ, ν).

Briefly we provide an intuition for small sets. Suppose $C \in \mathcal{B}(\mathbb{X})$ is (m, δ, ν)-small. Whenever the state ϕ_t happens to visit the set C, it forgets its entire past with probability at least $\delta > 0$ and transitions according to the probability measure ν over the next m time stages. In this way, the small set C acts as a "regenerative set" from which the process can forget its history. This line of investigation leads one to Nummelin's splitting technique [41, 42], which is a key tool in many stability results for irreducible Markov chains.

1.7 Lyapunov Drift Conditions for Stability

In this section, we define a drift condition and use it to show two stability results which will be crucial to our analysis. As in Section 1.6, we consider a general Markov chain $\{\phi_t\}_{t=0}^{\infty}$. We consider a sequence of stopping times $\{\mathcal{T}_z\}_{z=0}^{\infty}$ which are strictly increasing with $\mathcal{T}_0 = 0$.

The following is a condition on the general Markov chain $\{\phi_t\}_{t=0}^{\infty}$ which appears in [2].

Condition 1.7.1 (Random-Time Lyapunov Drift)**.** For a measurable function $V : \mathbb{X} \to (0, \infty)$, measurable functions $f, d : \mathbb{X} \to [0, \infty)$, a constant $b \in \mathbb{R}$ and a set $C \in \mathcal{B}(\mathbb{X})$, we say that $\{\phi_t\}_{t=0}^{\infty}$ satisfies the random-time Lyapunov drift condition at $\phi \in \mathbb{X}$ if for all $z = 0, 1, 2...$

$$E\left[V(\phi_{\mathcal{T}_{z+1}}) \mid \mathcal{F}_{\mathcal{T}_z}\right] \leq V(\phi_{\mathcal{T}_z}) - d(\phi_{\mathcal{T}_z}) + b\mathbb{1}_{\{\phi_{\mathcal{T}_z} \in C\}},$$

and

$$E\left[\sum_{t=\mathcal{T}_z}^{\mathcal{T}_{z+1}-1} f(\phi_t) \;\Big|\; \mathcal{F}_{\mathcal{T}_z}\right] \leq d(\phi_{\mathcal{T}_z}). \tag{1.7}$$

when $\phi_0 = \phi$.

Remark. Suppose that the stopping times $\mathcal{T}_z$ are the sequential return times to some set $\Lambda \in \mathcal{B}(\mathbb{X})$, that is $\mathcal{T}_0 = 0$ and

$$\mathcal{T}_{z+1} = \min\left\{t > \mathcal{T}_z : \phi_t \in \Lambda\right\}.$$

In this case, if one is able to verify for all $\phi \in \Lambda$ that

$$E_\phi\left[V(\phi_{\tau_\Lambda})\right] \leq V(\phi) - d(\phi) + b\mathbb{1}_{\{\phi \in C\}},$$

and

$$E_\phi\left[\sum_{t=0}^{\tau_\Lambda - 1} f(\phi_t)\right] \leq d(\phi) \tag{1.8}$$

then it follows automatically that Condition 1.7.1 holds at every $\phi \in \Lambda$. Notably, if $C \subseteq \Lambda$ then Condition 1.7.1 holds at every $\phi \in C$.

This drift condition, in combination with different assumptions on the functions and sets involved, can lead to many useful results on stability (e.g., [2, Theorem 2.1]). We present two such results here, the proofs of which can be found in the appendix.

Remark. The results presented here are variations of [2, Theorem 2.1], presented in the form most useful for our application. The proofs of these results draw heavily from the proof program in [2].

Lemma 1.7.3. *Suppose $\{\phi_t\}_{t=0}^\infty$ is φ-irreducible and satisfies Condition 1.7.1 at all $\phi \in C$, with the restrictions that C is a small set, $\sup_{\phi \in C} V(\phi) < \infty$, $f \equiv 1$ and $d(\phi) \geq 1$. Then the set*

$$\mathcal{X} := \{\phi \in \mathbb{X} : P_\phi\left(\tau_C < \infty\right) = 1\}$$

is full and absorbing, and the restriction of $\{\phi_t\}_{t=0}^\infty$ to $\mathcal{X}$ is positive Harris recurrent.

If in addition one can show that $P_\phi(\tau_C < \infty) = 1$ *for all* $\phi \in \mathbb{X}$ *(i.e.,* $\mathcal{X} = \mathbb{X}$*), then* $\{\phi_t\}_{t=0}^{\infty}$ *is positive Harris recurrent.*

Lemma 1.7.4. *Suppose that* $\{\phi_t\}_{t=0}^{\infty}$ *is positive Harris recurrent with invariant measure* π *and satisfies Condition 1.7.1 at some* $\phi \in \mathbb{X}$*. Then,*

$$E_\pi\left[f(\phi_t)\right] \leq b$$

and for any function $g : \mathbb{X} \to [0, \infty)$ *which is bounded by* f *in that* $g(\cdot) \leq cf(\cdot)$ *for some constant* $c > 0$*, we have the following ergodic theorem for* g*,*

$$\lim_{n\to\infty} \frac{1}{n} E_{\phi_0}\left[\sum_{t=0}^{n-1} g(\phi_t)\right] = E_\pi\left[g(\phi_t)\right] \tag{1.9}$$

for every $\phi_0 \in \mathbb{X}$*.*

Chapter 2

Scalar Linear Systems

We begin with a discussion of the scalar case for simplicity. Here we consider control of the scalar system,

$$x_{t+1} = ax_t + bu_t + w_t \tag{2.1}$$

where $b \neq 0$ and the noise process w_t is assumed to be i.i.d., zero-mean with finite second moment $\sigma^2 = E\left[w_0^2\right]$. Furthermore, we suppose the noise process admits a pdf η with respect to Lebesgue measure on $\mathbb{R}$ which is positive everywhere. Here we will also suppose that $|a| \geq 1$ so that the system is open-loop-unstable.

Then the optimal control problem we study is to choose a policy γ which minimizes the infinite-horizon quadratic cost,

$$\limsup_{T\to\infty} \frac{1}{T} E\left[\sum_{t=0}^{T-1} x_t^2\right]. \tag{2.2}$$

By Proposition 1.3.1, in the fully observed setup the optimal control policy is $u_t = -\frac{a}{b}x_t$ which achieves an optimal cost of σ^2. We suppose now that the controller only has access to x_t through a discrete noiseless channel of capacity C bits.

Our goal here is to minimize $\lim_{T\to\infty} \frac{1}{T} E\left[\sum_{t=0}^{T-1} x_t^2\right]$ which we know is bounded below by σ^2. As the state will only be partially observed by the controller, we seek to arrive at asymptotic bounds on the "optimality gap" $\lim_{T\to\infty} E\left[\sum_{t=0}^{T-1} x_t^2\right] - \sigma^2$ in terms of the channel capacity C.

We now make this precise. As the state is observed through a channel with finite capacity C, it is necessary to specify a coding scheme alongside a control policy γ (which now must depend on C via the coding scheme). We will do this jointly and in particular, we are interested in the limiting "high-rate" case $C \to \infty$.

We say that ϕ_C is a joint coding and control scheme for $C > 0$ if ϕ_C specifies both a coding scheme for communicating over the channel of capacity C as well as a corresponding control scheme at the channel receiver.

The following result demonstrates an ultimate limit on achievable rates of convergence.

Lemma 2.0.1. *Under any joint coding and control scheme ϕ_C we have*

$$\limsup_{T\to\infty} \frac{1}{T} E\left[\sum_{t=0}^{T-1} x_t^2\right] - \sigma^2 \geq \frac{a^2\sigma^2}{2^{2C} - a^2}. \tag{2.3}$$

Therefore, for any "convergence rate" function $r : \mathbb{R}_+ \to \mathbb{R}_+$ such that the optimality gap is $\mathcal{O}_C\left(r(C)\right)$ it must be the case that $\limsup_{C\to\infty} 2^{2C} r(C) > 0$.

Sketch of Proof. By following the core arguments in the proof of [43, Theorem 11.3.2] we are able to provide a lower bound on the channel capacity in terms of the ergodic system second moment and some differential entropy terms. Through a careful analysis this simplifies and yields (2.3). The entropy-power inequality is a crucial tool in the proof. A complete proof is provided in the appendix. □

Intuitively, the above lemma implies that the fastest rate of convergence (of the optimality gap to zero) one can hope for is 2^{-2C}. Motivated by this, we normalize potential rate functions by 2^{2C} and make the following definition.

Definition 2.0.1. A joint coding and control scheme ϕ_C achieves second moment convergence with rate function $r : \mathbb{R}_+ \to \mathbb{R}_+$ if,

$$\limsup_{T\to\infty} \frac{1}{T} E\left[\sum_{t=0}^{T-1} x_t^2\right] - \sigma^2 = \mathcal{O}_C\left(\frac{r(C)}{2^{2C}}\right).$$

For brevity we may say simply that ϕ_C achieves the rate function $r(C)$.

Intuitively, one seeks to achieve a rate function $r(C)$ which grows slowly in order to maximize the rate of convergence. Lemma 2.0.1 implies that the best achievable rate function is the constant function, so any achievable rate function satisfies $\limsup_{C\to\infty} r(C) > 0$.

We will impose the following extremely mild condition on the noise process.

Condition 2.0.1. For some $\beta > 2$, $E\left[|w_0|^\beta\right] < \infty$. That is, the noise process has finite βth moment.

In the scalar case, our ultimate result is the following.

Theorem 2.0.2. *Supposing Condition 2.0.1 holds with $\beta > 2$, for any $\varepsilon \in (0, \beta - 2)$ there exists a joint coding and control scheme, denoted Scheme $P(\beta, \varepsilon)$, which achieves the exponential rate function $r(C) = 2^{\frac{4}{\beta-\varepsilon}C}$.*

Intuitively, with only the condition that the noise admits finite βth moment, we are able to construct schemes that nearly (as $\varepsilon \to 0$) achieve convergence of the optimality gap like $2^{(-2+\frac{4}{\beta})C}$. If β is quite large, then the convergence becomes much closer to 2^{-2C} and, in the extreme case where w_t admits finite moments of all orders, one can construct schemes achieving convergence of the optimality gap like $2^{(-2+\delta)C}$ for any $\delta > 0$.

The rest of Chapter 2 is dedicated to the construction of Scheme $\mathrm{P}(\beta, \varepsilon)$ and a high-level proof program of Theorem 2.0.2.

2.1 Two-Part Code with Uniform Quantization

In this section we describe the joint coding and control scheme of Theorem 2.0.2.

The coding scheme is in two parts where the first part is adaptive in time and the second is fixed. The adaptive part will yield stability, and the fixed part will yield an optimal rate of convergence via simple iterated expectation arguments.

To communicate over a finite capacity channel, it is necessary to employ some kind of quantization scheme. We will work solely with uniform quantizers.

Let $M \geq 2$ be an even integer and $\Delta > 0$ be a scalar "bin size". With $\lfloor \cdot \rfloor$ the usual floor function, we define the *scalar modified uniform quantizer* Q_M^Δ by,

$$Q_M^\Delta(x) = \begin{cases} \Delta \left\lfloor \frac{x}{\Delta} \right\rfloor + \frac{\Delta}{2}, & \text{if } x \in \left[-\frac{M}{2}\Delta, \frac{M}{2}\Delta\right) \\ \frac{M}{2}\Delta - \frac{\Delta}{2}, & \text{if } x = \frac{M}{2}\Delta \\ 0, & \text{if } |x| > \frac{M}{2}\Delta. \end{cases} \tag{2.4}$$

This quantizer uniformly quantizes $x \in \left[-\frac{M}{2}\Delta, \frac{M}{2}\Delta\right]$ into M bins of size Δ and maps all larger x to zero. This requires $M+1$ output levels.

We will use this quantizer for two different purposes.

(i) The first is to use adaptive bin sizes which vary with time to achieve stability. Let $K \geq 2$ be an even integer, and suppose that $\{\Delta_t\}_{t=0}^\infty$ is a sequence of strictly positive "bin sizes" varying with time. We will make use of the quantizer $Q_K^{\Delta_t}$.

(ii) Secondly, we will use this quantizer to achieve optimal convergence. For a given even number of bins $N \geq 2$, let $\Delta_{(N)}$ be a bin size which is a function of N. We will make use of the quantizer $U_N^{\Delta_{(N)}}$. For brevity, we denote $U_N := U_N^{\Delta_{(N)}}$ and where necessary, specify the dependence of $\Delta_{(N)}$ on N. Note that this quantizer is fixed in time, in contrast to $Q_K^{\Delta_t}$.

Suppose that the sequence of bin sizes $\{\Delta_t\}_{t=0}^{\infty}$ is such that Δ_{t+1} is a function of only Δ_t and the indicator random variable $\mathbb{1}_{\left\{|x_t| \leq \frac{K}{2}\Delta_t\right\}}$. Also assume that both the encoder and decoder (controller) know Δ_0. Then so long as $Q_K^{\Delta_t}(x_t)$ is sent over the channel, it is possible to synchronize knowledge of Δ_t between the quantizer and the controller since $|x_t| \leq \frac{K}{2}\Delta_t$ if and only if $Q_K^{\Delta_t}(x_t) \neq 0$.

The coding scheme is as follows. For $\{\Delta_t\}_{t=0}^{\infty}$ as above, we calculate the adaptive quantizer output $Q_K^{\Delta_t}(x_t)$ and the adaptive system error $e_t := x_t - Q_K^{\Delta_t}(x_t)$. Then for integer $N \geq 2$ we use a fixed quantizer U_N with bin size $\Delta_{(N)}$ as mentioned above to calculate the fixed quantizer output $U_N(e_t)$. We then send $Q_K^{\Delta_t}(x_t)$ and $U_N(e_t)$ across the noiseless channel where the channel capacity is at least,

$$C = \log_2(K+1) + \log_2(N+1). \tag{2.5}$$

We estimate the state x_t as $\hat{x}_t := Q_K^{\Delta_t}(x_t) + U_N(e_t)$. To mirror the fully observed case, the controller applies the control

$$u_t = -\frac{a}{b}(Q_K^{\Delta_t}(x_t) + U_N(e_t)) = -\frac{a}{b}\hat{x}_t.$$

The scheme is illustrated in Figure 2.1.

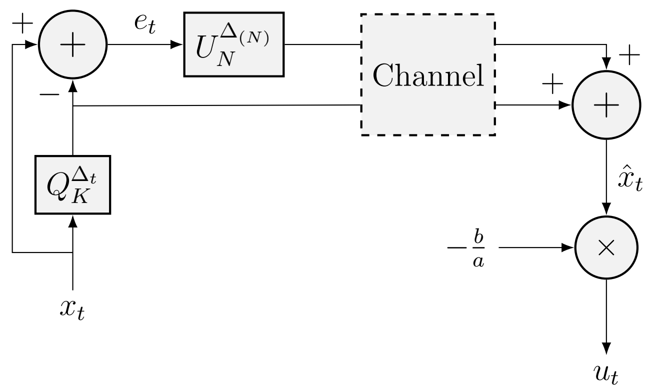

Figure 2.1: Block diagram of the two-stage coding and control scheme in the scalar case.

The controlled system dynamics resulting from this scheme are

$$\begin{aligned} x_{t+1} &= a(x_t - \hat{x}_t) + w_t \\ &= a(e_t - U_N(e_t)) + w_t. \end{aligned} \tag{2.6}$$

Finally, we describe the adaptive bin size update dynamics where, as in prior work in this context [1, 2], a simple *zooming* scheme is employed. We assume that $K \geq 2$ is even and large enough that $K > |a|$ and choose scalars α, ρ and L such that $\frac{|a|}{K} < \alpha < 1$, $\rho > |a|$ and $L > 0$. We also assume that $\rho \geq K\alpha$. Choose $\Delta_0 \geq L$ arbitrarily, then for $t \geq 1$ the bin update is

$$\Delta_{t+1} = \begin{cases} \rho\Delta_t, & \text{if } |x_t| > \frac{K}{2}\Delta_t \\ \alpha\Delta_t, & \text{if } |x_t| \leq \frac{K}{2}\Delta_t, \Delta_t \geq L \\ \Delta_t, & \text{if } |x_t| \leq \frac{K}{2}\Delta_t, \Delta_t < L. \end{cases} \tag{2.7}$$

Proposition 2.1.3. *With dynamics* (2.6) *and* (2.7)*, the process* $\{(x_t, \Delta_t)\}_{t=0}^{\infty}$ *is a*

Markov chain.

The motivation for this scheme is that the adaptive part leads to stability in the sense of positive Harris recurrence, while the fixed quantizer U_N leads to order-optimal convergence of the ergodic second moment (2.2) as the fixed quantization rate N grows large.

The state space for the process $\{(x_t, \Delta_t)\}_{t=0}^{\infty}$ highly depends on the following "countability condition".

Condition 2.1.2. There exist relatively prime integers $j, k \geq 1$ such that $\alpha^j \rho^k = 1$. Equivalently, $\log_\alpha \rho$ is rational.

If this condition holds, then starting from an arbitrary $\Delta_0 > 0$ there exists $\kappa, b \in \mathbb{R}$ such that $\log \Delta_t$ always belongs to a subset of $\mathbb{Z}\kappa + b = \{n\kappa + b : n \in \mathbb{Z}\}$ (see e.g. [2, Theorem 3.1]). If the condition fails, then starting from any fixed Δ_0 the set of reachable bin sizes is a dense but countable subset of $\mathbb{R}_{++}$.

We restrict our analysis to the case where Condition 2.1.2 holds. This is not restrictive; it can be shown that for any arbitrary (α, ρ) there exists (α', ρ') arbitrarily close that satisfy Condition 2.1.2. We let the state space for Δ_t be

$$\Omega_\Delta \coloneqq \left\{ \alpha^j \rho^k \Delta_0 : j, k \in \mathbb{Z}_{\geq 0} \right\}.$$

The state space for the Markov chain $\{(x_t, \Delta_t)\}_{t=0}^{\infty}$ is then $\mathbb{R} \times \Omega_\Delta$. What remains is to specify additional constraints which complete our proposed scheme.

We assume that Condition 2.0.1 holds for some $\beta > 2$. For any $\varepsilon \in (0, \beta - 2)$ we finish our construction of Scheme $\mathrm{P}(\beta, \varepsilon)$ by requiring that $\rho > |a|^{\frac{\beta}{\varepsilon}}$ and specifying the dependence of $\Delta_{(N)}$ on N as $\Delta_{(N)} = 2N^{-1+\frac{2}{\beta-\varepsilon}}$.

Remark. In this scheme, the constant multiplying $\Delta_{(N)}$ is arbitrary for convergence purposes.

We also impose the following condition.

Condition 2.1.3. The minimum adaptive bin size is at least,

$$\alpha L > \frac{|a|}{K\alpha - |a|}\Delta_{(N)}.$$

This may place an implicit dependence of L on the number of fixed quantization bins $N \geq 2$, though if all other parameters remain fixed as N increases then one can ensure Condition 2.1.3 holds by ensuring that it holds for $N = 2$ (since $\Delta_{(N)}$ as specified above is monotone decreasing in N).

Finally, as $C \to \infty$ we fix K and let $N \to \infty$ to take advantage of fixed quantization results at high rates, to be presented shortly.

Since the proof is rather tedious, we present a somewhat detailed proof program to guide the reader for the scalar setup.

We note that while the proof method for stabilization of our scheme builds on the random-time Lyapunov drift approach introduced in [1, 2], the coupling between the two parts of the coding scheme significantly complicates the analysis. Furthermore, we consider performance bounds as the data rate grows without bound. Altogether, this requires a cautious analysis between moments and high-rate quantization coupled with random-time drift criteria.

2.2 High-level Proof Program

In this section we outline the high-level proof program for Theorem 2.0.2, i.e., the proof that Scheme $\mathrm{P}(\beta, \varepsilon)$ achieves the exponential rate function $r(C) = 2^{\frac{4}{\beta-\varepsilon}C}$. Results without complete proofs admit more general sister results in the vector case, which is discussed in detail in Chapter 4.

Theorem 2.2.4. *Under Scheme $P(\beta, \varepsilon)$ with $K > |a|$, $\{(x_t, \Delta_t)\}_{t=0}^{\infty}$ is positive Harris recurrent for every even $N \geq 2$ (i.e., as C grows without bound). Therefore, for every even $N \geq 2$, Scheme $P(\beta, \varepsilon)$ yields a unique invariant measure π_N for the process $\{(x_t, \Delta_t)\}_{t=0}^{\infty}$.*

Sketch of Proof. We establish φ-irreducibility and aperiodicity for $\{(x_t, \Delta_t)\}_{t=0}^{\infty}$ where φ is the product of the Lebesgue and discrete measures on $\mathbb{R} \times \Omega_\Delta$. The logarithmic function $V(x, \Delta) = c \log_\alpha \Delta$ is shown to satisfy Condition 1.7.1 with $d(x, \Delta)$ constant and $f \equiv 1$ (i.e., in the form required by Lemma 1.7.3), leading to positive Harris recurrence. For a complete proof, please see the proof of Theorem 4.2.2. $\square$

Remark. The analysis here is highly similar to that of the proof of [2, Theorem 3.1]. The only major change is that the upper bound of [2, Lemma 5.2] for the tail probabilities $P_{x_0,\Delta_0}(\tau_\Lambda \geq k)$ must be re-derived in some form, as the out-of-view state dynamics change significantly due to the fixed quantization stage. We address this by providing a similar bound in Lemma A.5 (compare to equation (26) in [2]), which decays suitably fast for summability in the proof program under Condition 2.0.1.

We denote $(x_{*,N}, \Delta_{*,N}) \sim \pi_N$ as the state under invariant measure. This will also induce an invariant measure for the system adaptive error e_t, which we denote by $e_{*,N} \sim \pi_N^{\mathrm{err}}$.

We have the following ergodicity result.

Proposition 2.2.5. *The infinite-horizon second moment and the invariant second moment agree, that is*

$$\lim_{T\to\infty} \frac{1}{T} E\left[\sum_{t=0}^{T-1} x_t^2\right] = E\left[(x_{*,N})^2\right].$$

Sketch of Proof. We are able to show using the drift conditions of Section 1.7 that functions $g(x, \Delta)$ that are bounded asymptotically by $|x|^{\beta-\varepsilon}$ satisfy the above ergodicity condition. Since $\varepsilon < \beta - 2$, $g(x, \Delta) = x^2$ is bounded by $|x|^{\beta-\varepsilon}$ and the result follows. For a complete proof, please see the proof of Proposition 4.2.3. □

Finally, we use a simple iterated expectation argument. Suppose that $(x_0, \Delta_0) \sim \pi_N$. Let $e_0 = x_0 - Q_K^{\Delta}(x_0)$ and $x_1 = a(e_0 - U_N(e_0)) + Z$ where $Z \sim \eta$ (recall η is the distribution of w_t). Since we have applied the one-step transition kernel and π_N is the invariant measure, the marginal distributions of x_0 and x_1 are identical.

For brevity, we denote $s_0 := e_0 - U_N(e_0)$ so that $x_1 = as_0 + Z$. Supposing that $E\left[(x_{*,N})^2\right] < \infty$ (this will be shown as part of the full proof program), we then have by invariance and iterated expectations that

$$\begin{aligned} E\left[x_0^2\right] &= E\left[x_1^2\right] = E\left[E\left[x_1^2 \mid s_0\right]\right] = E\left[E\left[(as_0 + Z)^2 \mid s_0\right]\right] \\ &= E\left[a^2 s_0^2 + 2as_0 E[Z] + E\left[Z^2\right]\right] = a^2 E\left[s_0^2\right] + \sigma^2 \\ &= a^2 E\left[(e_0 - U_N(e_0))^2\right] + \sigma^2. \end{aligned} \tag{2.8}$$

Let $e_{*,N}$ denote the system adaptive error under invariant measure, i.e., $e_{*,N} = x_{*,N} -$

$Q_K^{\Delta_{*,N}}(x_{*,N})$. Rearranging (2.8), we find that the optimality gap is given as

$$E\left[(x_{*,N})^2\right] - \sigma^2 = a^2 E\left[(e_{*,N} - U_N(e_{*,N}))^2\right] \tag{2.9}$$

which is (up to a constant) the distortion of the fixed quantizer U_N applied to the random variable $e_{*,N}$.

With this in mind, we state the following result for high-rate distortion of U_N on sequences of suitably well-behaved random variables. The proof builds on balancing the trade-off between distortion due to the high-rate granular region and the overflow region.

Lemma 2.2.6. *Let $\{x_N\}_{N=2}^{\infty}$ be a sequence of random variables that satisfy,*

$$\sup_{N\geq 2} E\left[|x_N|^m\right] =: B_m < \infty \tag{2.10}$$

for some $m > 2$ (not necessarily integer). Set the bin size for the quantizer U_N as $\Delta_{(N)} = 2N^{-1+\frac{2}{m}}$. Then we have,

$$E\left[(x_N - U_N(x_N))^2\right] = \mathcal{O}_N\left(N^{-2+\frac{4}{m}}\right).$$

The proof is mostly mechanical. For a proof, see the proof of Lemma 4.1.1 in the appendix.

In light of (2.9), we would like to apply Lemma 2.2.6 to the sequence of random variables $\{e_{*,N}\}_{N=2}^{\infty}$. To do this, we need to establish (2.10). This is ensured by the following result.

Lemma 2.2.7. *Under Scheme $P(\beta, \varepsilon)$, the invariant system error has finite $(\beta - \varepsilon)$th* *moment uniformly in* $N \geq 2$. *That is,*

$$\sup_{N \geq 2} E\left[|e_{*,N}|^{\beta-\varepsilon}\right] < \infty. \tag{2.11}$$

Sketch of Proof. With Lyapunov functions $V(x, \Delta)$ and $d(x, \Delta)$ proportional to $\Delta^{\beta-\varepsilon}$ and appropriate set C and constant b, we show that these functions satisfy the drift condition (1.8). In particular, we do this for f proportional to $|x|^{\beta-\varepsilon}$ and f proportional to $\Delta^{\beta-\varepsilon}$ which, with the Lyapunov parameters independent of N leads to results of the form (2.11) by Lemma 1.7.4 for the invariant state and adaptive bin size. A simple invariance argument finishes the result. Condition 2.0.1 is crucial to the proof of this result. For a complete proof, please see the proof of Lemma 4.2.5 in the appendix. □

Therefore, in light of (2.9) and the above lemma, we find that the optimality gap decays asymptotically at the rate $\mathcal{O}_N\left(N^{-2+\frac{4}{\beta-\varepsilon}}\right)$. Since K is fixed and 2^C is linearly proportional to N, we find that the optimality gap is asymptotically in C,

$$E\left[(x_{*,N})^2\right] - \sigma^2 = \mathcal{O}_C\left(\frac{2^{\frac{4}{\beta-\varepsilon}C}}{2^{2C}}\right)$$

which establishes Theorem 2.0.2.

Chapter 3

The Multi-Dimensional Case

We now present the more general vector case. Recall that we consider control of the multi-dimensional system (1.1) over a discrete noiseless channel of capacity C bits and the aim is to minimize the infinite-horizon average quadratic form (1.2).

As stated earlier, in the fully observed setup the optimal control policy is $u_t = -B^{-1}Ax_t$ which achieves an optimal cost of $\mathrm{tr}\,(Q\Sigma)$. In light of this, we seek to arrive at asymptotic bounds on the optimality gap,

$$\limsup_{T\to\infty} \frac{1}{T} E\left[\sum_{t=0}^{T-1} x_t^T Q x_t\right] - \mathrm{tr}\,(Q\Sigma)$$

in terms of the channel capacity C.

It is again necessary to specify joint coding and control schemes. First, we generalize Definition 2.0.1 to the vector case.

Definition 3.0.1. A joint coding and control scheme ϕ_C achieves second moment convergence with rate function $r : \mathbb{R}_+ \to \mathbb{R}_+$ if,

$$\limsup_{T\to\infty} \frac{1}{T} E\left[\sum_{t=0}^{T-1} x_t^T Q x_t\right] - \mathrm{tr}\,(Q\Sigma) = \mathcal{O}_C\left(\frac{r(C)}{2^{\frac{2}{n}C}}\right).$$

For brevity we may say simply that ϕ_C achieves the rate function $r(C)$.

Remark. In the vector case, the extra factor of $\frac{1}{n}$ in the above definition comes intuitively from the fact that to tile a hypercube in $\mathbb{R}^n$ of width L it takes $(L/\Delta)^n$ bins of width Δ which grows exponentially in n.

As in the scalar case, one seeks to achieve a rate function $r(C)$ which grows slowly to maximize the rate of convergence.

We will impose the following condition on the noise process, analogous to Condition 2.0.1.

Condition 3.0.1. For some $\beta > 2$, $E\left[\|w_0\|_\infty^\beta\right] < \infty$. That is, the noise process has finite βth moment.

The following generalization of Theorem 2.0.2 is our ultimate result.

Theorem 3.0.1. *Supposing Condition 3.0.1 holds with $\beta > 2$, for any $\varepsilon \in (0, \beta - 2)$ there exists a joint coding and control scheme Scheme $P(\beta, \varepsilon)$ which achieves the exponential rate function $r(C) = 2^{\left(\frac{4}{\beta-\varepsilon}\right)\frac{1}{n}C}$.*

The rest of Chapter 3 is dedicated to proving Theorem 3.0.1.

3.1 Vector Quantization

In the multi-dimensional case, we will make use of vector quantization. We will use scalar uniform quantizers to define two types of cubic lattice vector quantizers.

Let $M \geq 2$ be an even integer and $\Delta > 0$ be a scalar "bin size". With $\lfloor \cdot \rfloor$ the usual floor function, we define the partial uniform quantizer $u_M^\Delta : [-\frac{M}{2}\Delta, \frac{M}{2}\Delta] \to \mathbb{R}$

as

$$u_M^\Delta(x) = \begin{cases} \Delta \left\lfloor \frac{x}{\Delta} \right\rfloor + \frac{\Delta}{2}, & \text{if } x \in \left[-\frac{M}{2}\Delta, \frac{M}{2}\Delta\right) \\ \frac{M}{2}\Delta - \frac{\Delta}{2}, & \text{if } x = \frac{M}{2}\Delta. \end{cases}$$

Note that $0 \notin \mathrm{range}(u_M^\Delta)$. Then, we define the type I vector quantizer $Q_M^\Delta : \mathbb{R}^n \to \mathbb{R}^n$ as

$$Q_M^\Delta(x) = \begin{cases} \left(u_M^\Delta(x^i)\right)_{i=1}^n, & \text{if } \|x\|_\infty \leq \frac{M}{2}\Delta \\ 0, & \text{if } |x^i| > \frac{M}{2}\Delta^i \text{ for some } 1 \leq i \leq n \end{cases} \tag{3.1}$$

and the type II vector quantizer $U_M^\Delta : \mathbb{R}^n \to \mathbb{R}^n$ component-wise as

$$\left(U_M^\Delta(x)\right)^i = \begin{cases} u_M^\Delta(x^i), & \text{if } |x^i| \leq \frac{M}{2}\Delta \\ 0, & \text{otherwise.} \end{cases}$$

Remark. If $n = 1$ then the above definitions both correspond to the scalar quantizer (2.4). For this reason and because the ∞-norm is a generalization of the scalar absolute value, the scheme we present for the vector case is a direct generalization of the scalar scheme.

As in the scalar case, we will use these vector quantizers for two different purposes. The type I quantizer will be used with adaptive bin sizes to achieve stability. Let $K \geq 2$ be an even integer, and suppose that $\{\Delta_t\}_{t=0}^\infty$ is some sequence of strictly positive bin sizes varying with time. We will make use of the quantizer $Q_K^{\Delta_t}$.

The type II quantizer will be used to achieve optimal convergence. For a given even number of bins $N \geq 2$, let $\Delta_{(N)}$ be a bin size which is a function of N. We will make use of the quantizer $U_N^{\Delta_{(N)}}$. For brevity we denote $U_N := U_N^{\Delta_{(N)}}$ and where necessary, specify the dependence of $\Delta_{(N)}$ on N.

3.2 System in a Jordan Form

Briefly, we discuss a reduction of the system matrix A into its separate modes. This will have the useful side effect of making $\|A\|_\infty$ very close to the absolute eigenvalue of each mode.

Let $\{\lambda_i\}_{i=1}^n$ be the (possibly repeated) eigenvalues of the system matrix A. Without loss of generality, we assume that A is in real Jordan normal form. To see why this is without loss, note that for any matrix A there exists an invertible matrix P such that $\tilde{A} := P^{-1}AP$ is the real Jordan normal form of A [44, Theorem 3.4.1.5]. Let $P\tilde{x}_t = x_t$ and left-multiply the system (1.1) by P^{-1}. Defining $\tilde{B} = P^{-1}B$ and $\tilde{w}_t = P^{-1}w_t$, the system dynamics become

$$\tilde{x}_{t+1} = \tilde{A}\tilde{x}_t + \tilde{B}u_t + \tilde{w}_t$$

which is in the same form as (1.1) but with the system matrix in real Jordan normal form.

For the purposes of controlling this system, it suffices to consider each of the Jordan blocks of A individually. Since the matrix B is invertible, the control of each of these blocks will be identical to the control problem for the full system (1.1). Therefore, by a slight abuse of notation, we will consider the control of a single mode or Jordan block $A \in \mathbb{R}^{n\times n}$ (which may now be part of a larger system) with the single repeated eigenvalue $\lambda \in \mathbb{C}$. By the real Jordan normal form [44, Theorem 3.4.1.5], we know then that A takes the form

$$\begin{bmatrix} \lambda & 1 & & \\ & \lambda & \ddots & \\ & & \ddots & 1 \\ & & & \lambda \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} D & I & & \\ & D & \ddots & \\ & & \ddots & I \\ & & & D \end{bmatrix} \tag{3.2}$$

where $\lambda \in \mathbb{R}$ and $\lambda \in \mathbb{C} \setminus \mathbb{R}$ respectively. For $\lambda = a + bi \in \mathbb{C} \setminus \mathbb{R}$ above, I is the 2×2 identity matrix and D is the 2×2 matrix

$$D = \begin{bmatrix} a & b \\ -b & a \end{bmatrix}.$$

We note that for A in either of the forms above, we can describe $\|A\|_\infty$ quite easily:

Proposition 3.2.2. *There are four cases to consider, which are:*

- *If $\lambda \in \mathbb{R}$ and $n = 1$ then $\|A\|_\infty = |\lambda|$.*
- *If $\lambda \in \mathbb{R}$ and $n > 1$ then $\|A\|_\infty = |\lambda| + 1$.*
- *If $\lambda = a + bi \in \mathbb{C} \setminus \mathbb{R}$ and $n = 2$ then $\|A\|_\infty = |a| + |b| \leq \sqrt{2}\,|\lambda|$.*
- *If $\lambda = a + bi \in \mathbb{C} \setminus \mathbb{R}$ and $n > 2$ then $\|A\|_\infty = |a| + |b| + 1 \leq \sqrt{2}\,|\lambda| + 1$.*

The equalities follow just by observing the rows of A under each assumption. In the complex case, the upper bound follows by Cauchy-Schwarz inequality in $\mathbb{R}^2$. The largest upper bound is $\sqrt{2}\,|\lambda| + 1$, so in view of (1.5) we may always write that

$$\|Ax\|_\infty \leq (\sqrt{2}\,|\lambda| + 1)\, \|x\|_\infty . \tag{3.3}$$

Briefly, we remark that the above bound can be improved in the case $n > 1$ by applying an invertible transform S. That is, as before we let $\tilde{A} = S^{-1}AS$ and apply the same "change-of-view" transformations $\tilde{x}_t = Sx_t$, $\tilde{B} = S^{-1}B$ and $\tilde{w}_t = S^{-1}w_t$. By left-multiplying the system (1.1) by S^{-1} we arrive at an identical control problem now with the system matrix $\tilde{A} = S^{-1}AS$.

From here it suffices to determine how small $\|S^{-1}AS\|_\infty$ can be made over all invertible transformations S. At least in the case $\lambda \in \mathbb{R}$ we demonstrate that the infimum is at most $|\lambda|$ using the following construction.

First, suppose A is a real Jordan block, i.e. that of (3.2) for $\lambda \in \mathbb{R}$. For any $\varepsilon > 0$ we let $S_\varepsilon := \mathrm{diag}(1, \varepsilon, \varepsilon^2, ..., \varepsilon^{n-1})$. which has inverse $S_\varepsilon^{-1} = \mathrm{diag}(1, \varepsilon^{-1}, \varepsilon^{-2}, ..., \varepsilon^{-(n-1)})$. Then,

$$S_\varepsilon^{-1} A S_\varepsilon = \begin{bmatrix} \lambda & \varepsilon & & \\ & \lambda & \ddots & \\ & & \ddots & \varepsilon \\ & & & \lambda \end{bmatrix}.$$

Therefore, $\|S_\varepsilon^{-1} A S_\varepsilon\|_\infty = |\lambda| + \varepsilon$. By taking $\varepsilon \to 0$ this value is arbitrarily close to $|\lambda|$. If one allows for complex vector and matrix entries, an identical argument can be made in the case $\lambda \in \mathbb{C} \setminus \mathbb{R}$.

The minimization of $\|A\|_\infty$ will be relevant to our scheme in terms of the minimum capacity required for stabilization.

We will need to distinguish between the cases $|\lambda| < 1$ and $|\lambda| \geq 1$. In the former case, stability of the process is simple to ensure since the process is open-loop-stable. In the latter case, the process is open-loop-unstable and achieving stability under the information constraints is a subtle issue which we will address carefully. Therefore, we postpone discussion of the stable modes and assume moving forward that $|\lambda| \geq 1$.

3.3 Scheme P(β, ε) in the Vector Case

In this section we describe the joint coding and control scheme of Theorem 3.0.1. The scheme presented is very similar to that of the scalar case and is in fact a direct generalization of the joint scheme of Theorem 2.0.2.

Let $K \geq 2$ be an even integer and suppose $\{\Delta_t\}_{t=0}^{\infty}$ is a sequence of positive "bin sizes" such that Δ_{t+1} is a function of only Δ_t and the indicator random variable $\mathbb{1}_{\left\{\|x_t\|_\infty \leq \frac{K}{2}\Delta_t\right\}}$. Also assume that both the encoder and decoder (controller) know Δ_0. Then so long as the type I quantization $Q_K^{\Delta_t}(x_t)$ is sent over the channel, it is possible to synchronize knowledge of Δ_t between the quantizer and the controller since $\|x_t\|_\infty \leq \frac{K}{2}\Delta_t$ if and only if $Q_K^{\Delta_t}(x_t) \neq 0$.

The coding scheme is as follows. For $\{\Delta_t\}_{t=0}^{\infty}$ as above, we calculate the adaptive quantizer output $Q_K^{\Delta_t}(x_t)$ and the adaptive system error $e_t \coloneqq x_t - Q_K^{\Delta_t}(x_t)$. Then for integer $N \geq 2$ we use a fixed type II quantizer U_N with bin size $\Delta_{(N)}$ as in Section 3.1 to calculate the fixed quantizer output $U_N(e_t)$. We then send $Q_K^{\Delta_t}(x_t)$ and $U_N(e_t)$ across the noiseless channel where the channel capacity is at least,

$$C = \log_2\left(K^n + 1\right) + \log_2\left((N+1)^n\right). \tag{3.4}$$

We estimate the state x_t as $\hat{x}_t \coloneqq Q_K^{\Delta_t}(x_t) + U_N(e_t)$. To mirror the fully observed case, the controller applies the control

$$u_t = -B^{-1}A\left(Q_K^{\Delta_t}(x_t) + U_N(e_t)\right) = -B^{-1}A\hat{x}_t.$$

The scheme is illustrated in Figure 3.1.

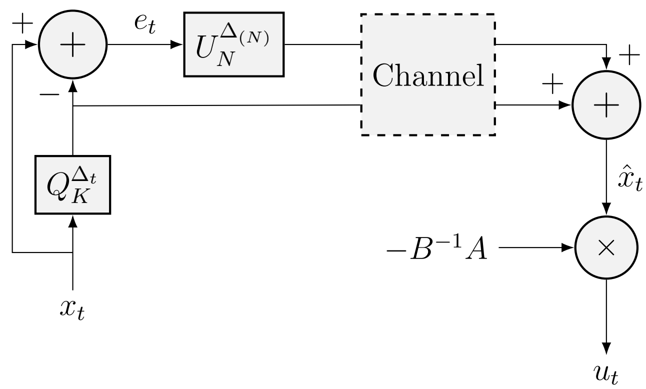

Figure 3.1: Block diagram of the two-stage coding and control scheme.

The controlled system dynamics resulting from this scheme are

$$\begin{aligned} x_{t+1} &= A(x_t - \hat{x}_t) + w_t \\ &= A(e_t - U_N(e_t)) + w_t. \end{aligned} \tag{3.5}$$

The update dynamics for $\{\Delta_t\}_{t=0}^{\infty}$ are nearly identical to the scalar case. We assume that $K > \|A\|_\infty$ and choose scalars $\frac{\|A\|_\infty}{K} < \alpha < 1$, $\rho > \|A\|_\infty$ and $L > 0$. We assume again that $\rho \geq K\alpha$. Choose $\Delta_0 \geq L$ arbitrarily, then for $t \geq 1$ the bin update is

$$\Delta_{t+1} = \begin{cases} \rho\Delta_t, & \text{if } \|x_t\|_\infty > \frac{K}{2}\Delta_t \\ \alpha\Delta_t, & \text{if } \|x_t\|_\infty \leq \frac{K}{2}\Delta_t, \Delta_t \geq L \\ \Delta_t, & \text{if } \|x_t\|_\infty \leq \frac{K}{2}\Delta_t, \Delta_t < L. \end{cases} \tag{3.6}$$

Proposition 3.3.3. *With dynamics* (3.5) *and* (3.6)*, the process* $\{(x_t, \Delta_t)\}_{t=0}^{\infty}$ *is a Markov chain.*

The motivation for this scheme is as in the scalar case. The adaptive type I quantizer $Q_K^{\Delta_t}$ will lead to stability in the sense of positive Harris recurrence, and the fixed type II quantizer U_N will lead to order-optimal convergence of the system quadratic form $x^T Qx$ under invariant measure as the fixed quantization rate N grows large.

As in the scalar case, we impose Condition 2.1.2 so that the state space for Δ_t is countable. The state space for Δ_t is

$$\Omega_\Delta := \left\{\alpha^j \rho^k \Delta_0 : j, k \in \mathbb{Z}_{\geq 0}\right\}.$$

and the state space for the Markov chain $\{(x_t, \Delta_t)\}_{t=0}^{\infty}$ is $\mathbb{R}^n \times \Omega_\Delta$. What remains is to specify additional constraints which complete our proposed scheme.

We assume that Condition 3.0.1 holds for some $\beta > 2$. For any $\varepsilon \in (0, \beta - 2)$ we finish our construction of Scheme P(β, ε) by requiring that $\rho > (\|A\|_\infty)^{\frac{\beta}{\varepsilon}}$ and specifying the dependence of $\Delta_{(N)}$ on N as $\Delta_{(N)} = 2N^{-1+\frac{2}{\beta-\varepsilon}}$ (again, the constant 2 here is arbitrary).

We impose the following condition which generalizes Condition 2.1.3 to the vector case.

Condition 3.3.2. The minimum adaptive bin size is at least,

$$\alpha L > \frac{\|A\|_\infty}{K\alpha - \|A\|_\infty} \Delta_{(N)}.$$

Finally, as in the scalar case, as $C \to \infty$ we keep K fixed and let $N \to \infty$ to take advantage of fixed quantization results at high rates, to be presented shortly.

Chapter 4

Proof Program for Stability and Convergence

In this section, we outline the proof program for Theorem 3.0.1, i.e. that Scheme $\mathrm{P}(\beta, \varepsilon)$ achieves the exponential rate function $r(C) = 2^{\left(\frac{4}{\beta-\varepsilon}\right)\frac{1}{n}C}$. Many of the proofs of intermediate results are tedious and largely mechanical, so we have relegated these to the appendix.

We first give an intermediate result on high-rate quantizer distortion and then present a high-level proof program of Theorem 3.0.1 with similar key arguments as those in Section 2.2.

4.1 A Supporting Result on Uniform Quantization at High Rates

The following lemma bounds distortion of uniform quantization at high rates under a mild moment condition. Let $\{X_N\}_{N=2}^{\infty}$ be a sequence of random vectors on $\mathbb{R}^n$ and consider the quantizer U_N described in Section 3.1. We define for $N \geq 2$ the error vectors $Y_N \coloneqq X_N - U_N(X_N)$.

Lemma 4.1.1. *Suppose that*

$$\sup_{N \geq 2} E\left[\|X_N\|_\infty^m\right] =: B_m < \infty$$

for some $m > 2$ (not necessarily integer). Set the bin size for the type II quantizer U_N as $\Delta_{(N)} = 2N^{-1+\frac{2}{m}}$. Then for any positive semidefinite matrix $V \in \mathbb{R}^{n \times n}$ we have

$$\operatorname{tr}\left(VE\left[Y_N Y_N^T\right]\right) = \mathcal{O}_N\left(N^{-2+\frac{4}{m}}\right).$$

The proof is mostly mechanical and can be found in the appendix.

4.2 System Stability and Moment Results

We denote the "in-view" set Λ as,

$$\Lambda := \left\{(x, \Delta) \in \mathbb{R}^n \times \Omega_\Delta : \|x\|_\infty \leq \tfrac{K}{2}\Delta\right\},$$

i.e., the set of states (x, Δ) such that the adaptive quantizer $Q_K^\Delta(x)$ is non-zero.

We will show in Lemma A.5 that states beginning in this set return to it extremely quickly, in that for constants $h > 0$ and $\xi > 1$ we have

$$P_{x,\Delta}\left(\tau_\Lambda \geq k+1\right) \leq kT_{w_0}\left(\frac{\Delta}{2}\left(\frac{h\xi^k - 1}{k} - \frac{\Delta_{(N)}}{\alpha L}\right)\right)$$

which decays incredibly fast in integer $k \geq 1$. Essentially all of the following stability results will follow from repeated application of this inequality and Condition 3.0.1 on the noise process.

Theorem 4.2.2. *Under Scheme $P(\beta, \varepsilon)$ with $K > \|A\|_\infty$, $\{(x_t, \Delta_t)\}_{t=0}^{\infty}$ is positive Harris recurrent for every even $N \geq 2$ (i.e., as C grows without bound). Therefore, for every even $N \geq 2$, Scheme $P(\beta, \varepsilon)$ yields a unique invariant measure π_N for the process $\{(x_t, \Delta_t)\}_{t=0}^{\infty}$.*

Sketch of Proof. We establish φ-irreducibility and aperiodicity for $\{(x_t, \Delta_t)\}_{t=0}^{\infty}$ where φ is the product of the Lebesgue and discrete measures on $\mathbb{R}^n \times \Omega_\Delta$. The logarithmic function $V(x, \Delta) = c \log_\alpha \Delta$ is shown to satisfy Condition 1.7.1 with $d(x, \Delta)$ constant and $f \equiv 1$ (i.e., in the form required by Lemma 1.7.3), leading to positive Harris recurrence. A complete proof is provided in the appendix.

We denote $(x_{*,N}, \Delta_{*,N}) \sim \pi_N$ as the state under invariant measure. This will also induce an invariant measure for the system adaptive error e_t, which we denote by $e_{*,N} \sim \pi_N^{\text{err}}$.

We have the following ergodicity result, similar to the scalar case.

Proposition 4.2.3. *The infinite-horizon second moment and the invariant second moment agree, that is*

$$\lim_{T \to \infty} \frac{1}{T} E\left[\sum_{t=0}^{T-1} x_t^T Q x_t\right] = E\left[(x_{*,N})^T Q (x_{*,N})\right].$$

Sketch of Proof. We are able to show using the drift conditions of Section 1.7 that functions $g(x, \Delta)$ which are bounded by $\|x\|_\infty^{\beta-\varepsilon}$ satisfy the above ergodicity condition. The quadratic form $x^T Q x$ is of the order $\|x\|_\infty^2$ and hence bounded by $\|x\|_\infty^{\beta-\varepsilon}$, since $\varepsilon < \beta - 2$. This will establish the result. A complete proof is provided in the appendix. □

The following proposition is crucial to our proof program.

Proposition 4.2.4. *We have the following characterization of the invariant second moment.*

$$E\left[(x_{*,N})^T Q (x_{*,N})\right] - \operatorname{tr}(Q\Sigma) = \operatorname{tr}\left(A^T Q A \cdot E\left[(e_{*,N} - U_N(e_{*,N}))(e_{*,N} - U_N(e_{*,N}))^T\right]\right).$$

Proof. The proof is by iterated expectations. Suppose that $(x_0, \Delta_0) \sim \pi_N$. Let $e_0 = x_0 - Q_K^{\Delta_0}(x_0)$ and $x_1 = A(e_0 - U_N(e_0)) + Z$ where $Z \sim \eta$ (recall η is the distribution of w_t). Since we have applied the one-step transition kernel and π_N is the invariant measure, the marginal distributions of x_0 and x_1 will be identical.

For brevity we denote $s_0 \coloneqq e_0 - U_N(e_0)$ so that $x_1 = As_0 + Z$. Supposing that $E\left[(x_{*,N})^T Q (x_{*,N})\right] < \infty$ (this follows from system moment results that will be stated shortly), we then have by invariance and iterated expectations that

$$\begin{aligned}
E\left[x_0^T Q x_0\right] &= E\left[x_1^T Q x_1\right] = E\left[E\left[x_1^T Q x_1 \mid s_0\right]\right] \\
&= E\left[E\left[(As_0 + Z)^T Q (As_0 + Z) \mid s_0\right]\right] \\
&= E\left[s_0^T A^T Q A s_0 + 2E\left[Z\right]^T QAs_0 + Z^T Q Z\right] \\
&= E\left[s_0^T A^T Q A s_0\right] + E\left[Z^T Q Z\right] \\
&= \operatorname{tr}\left(A^T Q A \cdot E\left[s_0 s_0^T\right]\right) + \operatorname{tr}\left(QE\left[ZZ^T\right]\right) \\
&= \operatorname{tr}\left(A^T Q A \cdot E\left[(e_0 - U_N(e_0))(e_0 - U_N(e_0))^T\right]\right) + \operatorname{tr}(Q\Sigma).
\end{aligned}$$

Rearranging the above equality completes the proof. Note that above we used the property that $Z \sim \eta$ is zero-mean. $\square$

Remark. For the type of scheme we present here (that is, using two-stage adaptive and fixed uniform quantization), if one is able to establish that

$$\operatorname{tr}\left(A^T QA \cdot E\left[(e_{*,N} - U_N(e_{*,N}))(e_{*,N} - U_N(e_{*,N}))^T\right]\right) = \mathcal{O}_N\left(\frac{R(N)}{N^2}\right)$$

for some function $R(N)$ then it follows from the above two propositions (and the fact that N is a linear function of $2^{\frac{1}{n}C}$, recalling (3.4) and that K is fixed) that the scheme in question achieves the rate function

$$r(C) = R\left((K^n + 1)^{-\frac{1}{n}} 2^{\frac{1}{n}C} - 1\right).$$

That is, we have

$$\lim_{T\to\infty} \frac{1}{T} E\left[\sum_{t=0}^{T-1} x_t^T Q x_t\right] - \operatorname{tr}(Q\Sigma) = \mathcal{O}_C\left(\frac{R\left((K^n+1)^{-\frac{1}{n}} 2^{\frac{1}{n}C} - 1\right)}{2^{\frac{2}{n}C}}\right). \tag{4.1}$$

Depending on the specific function $R(N)$ in question, this expression can be simplified (this will be the case in our analysis).

We have the following uniform stability result under invariant measure.

Lemma 4.2.5. *Under Scheme $P(\beta, \varepsilon)$, the invariant system error $e_{*,N}$ has finite $(\beta - \varepsilon)$-th moment uniformly in $N \geq 2$. That is,*

$$\sup_{N\geq 2} E\left[\|e_{*,N}\|_\infty^{\beta-\varepsilon}\right] < \infty. \tag{4.2}$$

Sketch of Proof. With Lyapunov functions $V(x, \Delta)$ and $d(x, \Delta)$ proportional to $\Delta^{\beta-\varepsilon}$, an appropriate "in-view" set C and constant b we show that these functions satisfy the drift condition (1.8). In particular, we do this for f proportional to $\|x\|_\infty^{\beta-\varepsilon}$ and f

proportional to $\Delta^{\beta-\varepsilon}$ which, with the Lyapunov parameters independent of N leads to results of the form (4.2) by Lemma 1.7.4 for the invariant state and adaptive bin size. A simple invariance argument finishes the result. A detailed proof is provided in the appendix. □

Briefly, we note that $A^T QA$ is positive semidefinite (by positive definiteness of Q). This allows us to use Lemma 4.1.1.

Finally, we prove our ultimate result.

Proof of Theorem 3.0.1. Lemma 4.2.5 allows us to use Lemma 4.1.1 with the sequence of random vectors $\{e_{*,N}\}_{N=2}^{\infty}$ from which we obtain,

$$\operatorname{tr}\left(A^T QA \cdot E\left[(e_{*,N} - U_N(e_{*,N}))(e_{*,N} - U_N(e_{*,N}))^T\right]\right) = \mathcal{O}_N\left(\frac{N^{\frac{4}{\beta-\varepsilon}}}{N^2}\right)$$

and so by the earlier remark with $R(N) = N^{\frac{4}{\beta-\varepsilon}}$, Scheme P$(\beta, \varepsilon)$ achieves the rate function

$$r(C) = \left((K^n + 1)^{-\frac{1}{n}} 2^{\frac{1}{n}C} - 1\right)^{\frac{4}{\beta-\varepsilon}} = \mathcal{O}_C\left(2^{\left(\frac{4}{\beta-\varepsilon}\right)\frac{1}{n}C}\right).$$

This proves Theorem 3.0.1 and completes our proof program. □

Chapter 5

The Case of Open-Loop-Stable Systems

Here, we consider again A to be a single Jordan block on $\mathbb{R}^{n\times n}$ with single repeated eigenvalue $\lambda \in \mathbb{C}$ where now $|\lambda| < 1$. In this case, the system with dynamics (1.1) is open-loop-stable. In contrast to the case of unstable modes, we do not need to employ adaptive quantization for stability.

We suppose the capacity allocated for this mode is at least $C = \log_2((N+1)^n)$ for some $N \geq 2$. Supposing that Condition 3.0.1 holds for the noise process w_t for some $\beta > 2$ we will employ the fixed quantizer U_N with bin size $\Delta_{(N)} = 2N^{-1+\frac{2}{\beta}}$ to quantize the state x_t and send $U_N(x_t)$ across the channel at each time stage. The controller will apply the control

$$u_t = -B^{-1}AU_N(x_t)$$

which will lead to the state dynamics,

$$x_{t+1} = A(x_t - U_N(x_t)) + w_t. \tag{5.1}$$

This process is a Markov chain. We state the following major result with only a proof sketch for brevity. The proof program is extremely mechanical and largely similar to that of the unstable case, though the theorems used can be slightly weaker which we address in the proof sketch.

Theorem 5.0.1. *The state process $\{x_t\}_{t=0}^{\infty}$ is positive Harris recurrent. Denoting the invariant distribution as $x_{*,N}$, we have that*

$$\sup_{N\geq 2} E\left[\|x_{*,N}\|_{\infty}^{\beta}\right] < \infty. \tag{5.2}$$

Furthermore, the infinite-horizon second moment and invariant second moment agree, that is

$$\lim_{T\to\infty} \frac{1}{T} E\left[\sum_{t=0}^{T-1} x_t^T Q x_t\right] = E\left[(x_{*,N})^T Q (x_{*,N})\right]. \tag{5.3}$$

Sketch of Proof. The use of random-time Lyapunov drift conditions is not needed here, and one can apply the typical one-stage Foster-Lyapunov drift theorems, with appropriate Lyapunov functions, to show each of these results. Loosely, it is easy to demonstrate stability in this case because $|\lambda| < 1$ implies that the process is "forgetful" of the past with geometric rate, in view of the dynamics (5.1). □

As in the unstable case, we have a characterization of the system second moment via iterated expectations.

Proposition 5.0.2. *We have the following characterization of the system second moment.*

$$E\left[(x_{*,N})^T Q (x_{*,N})\right] - \mathrm{tr}\left(Q\Sigma\right) = \mathrm{tr}\left(A^T Q A \cdot E\left[(x_{*,N} - U_N(x_{*,N}))^T (x_{*,N} - U_N(x_{*,N}))\right]\right).$$

Sketch of Proof. The proof is by iterated expectations and is almost identical to that of Proposition 4.2.4. □

It follows from (5.2) that we may apply Lemma 4.1.1, which from the above proposition, (5.3) and similar reasoning to the remark preceding Lemma 4.2.5 tells us that the scheme presented here achieves the rate function

$$r(C) = \mathcal{O}_C\left(2^{\frac{4}{\beta}\cdot\frac{1}{n}C}\right). \tag{5.4}$$

Chapter 6

Simulation Results

In this section, we provide an example simulation to illustrate our results.

6.1 A Heavy-Tailed Distribution

We will say that a random variable Z on $\mathbb{R}$ has a "symmetric Pareto" distribution with shape parameter $\alpha > 0$ and scale parameter $\sigma > 0$ if Z has the probability density function,

$$f_Z(u) = \frac{\alpha}{2\sigma}\left(1 + \frac{|u|}{\sigma}\right)^{-(\alpha+1)}, \qquad u \in \mathbb{R}.$$

For brevity we denote $Z \sim \mathrm{SP}(\alpha, \sigma)$. These distributions are related to the typical Pareto distribution in that if $Z \sim \mathrm{SP}(\alpha, \sigma)$ then $|Z| + \sigma \sim \mathrm{Pareto}(\sigma, \alpha)$.

Such a distribution admits finite βth moment for all $\beta < \alpha$ in the sense of Condition 3.0.1 whereas all moments $\beta \geq \alpha$ are infinite. The moments less than α are given explicitly by,

$$E\left[|Z|^\beta\right] = \sigma^\beta \cdot \frac{\Gamma(\alpha - \beta)\Gamma(\beta + 1)}{\Gamma(\alpha)}.$$

The fact that such random variables admit finite βth moment only for $\beta < \alpha$ means they are badly behaved in the sense of having heavy tails. Despite this, it is clear that if $Z \sim \mathrm{SP}(\alpha, \sigma)$ with $\alpha > 2$ then Z satisfies Condition 3.0.1 for all $\beta \in (2, \alpha)$.

For this reason, such distributions are well-suited to illustrate the generality of our results. We will use these distributions as the base for our system noise process.

6.2 An Example System

Consider a linear system in $\mathbb{R}^3$,

$$x_{t+1} = Ax_t + u_t + w_t$$

where the system matrix is,

$$A = \begin{bmatrix} 1 & 1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & \frac{1}{2} \end{bmatrix}.$$

Thus the system matrix is already in real Jordan normal form with eigenvalues $\lambda_{1,2} = 1 \pm i$ and $\lambda_3 = \frac{1}{2}$. The first Jordan block is open-loop-unstable while the second block is open-loop-stable.

The process $\{w_t\}_{t=0}^{\infty}$ is given component-wise by $w_t = (w_t^1, w_t^2, w_t^3)$ where the components are independent of one another and identically distributed in time with distributions,

$$w_t^1 \sim \mathrm{SP}(3, 2), \qquad w_t^2 \sim \mathrm{SP}(3, 3), \qquad w_t^3 \sim \mathrm{SP}(3, 4).$$

Thus, the covariance matrix Σ is given by $\Sigma = \mathrm{diag}(4, 9, 16)$. For simplicity, we let $Q = I_3$ be the identity matrix so that we are interested in the minimization of

$$\limsup_{T\to\infty} \frac{1}{T} E\left[\sum_{t=0}^{T-1} x_t^T x_t\right]$$

across a noiseless discrete channel of capacity C which has a lower bound of $\mathrm{tr}\,(Q\Sigma) = \mathrm{tr}\,(\mathrm{diag}(4,9,16)) = 29$.

Let $K = 4$ (the least even integer such that $K > \|A_{1,2}\|_\infty = 2$), then for even $N \geq 2$ we let the capacity of our channel be

$$C = C_{1,2} + C_3 = \log_2 17 + 3\log_2(N+1).$$

where $C_{1,2} = \log_2(K^2+1) + 2\log_2(N+1)$ and $C_3 = \log_2(N+1)$ are the channel capacities allocated for the respective Jordan blocks.

Let $\alpha = \frac{3}{4}$, $\rho = \left(\frac{4}{3}\right)^{14}$ and $L = 2$ be the adaptive quantization parameters for the unstable Jordan block $A_{1,2}$. Note that $\alpha^{14}\rho = 1$ so that Condition 2.1.2 is satisfied. Let $\beta^{1,2} = 2.9$, $\varepsilon = 0.5$ and $\beta^3 = 2.4$ so that $\beta^3 = \beta^{1,2} - \varepsilon$. Finally, we let $\Delta_{(N)} = 2N^{-1+\frac{2}{\beta^{1,2}-\varepsilon}} = 2N^{-1+\frac{2}{\beta^3}} = 2N^{-\frac{1}{6}}$ be the bin size for the fixed quantizer U_N in both Jordan blocks of the system, for any even $N \geq 2$. Note that $\rho = \left(\frac{4}{3}\right)^{14} > 2^{5.8} = \|A_{1,2}\|_\infty^{\frac{\beta^{1,2}}{\varepsilon}}$ and that $\rho \geq K\alpha = 3$. Also note that Condition 3.3.2 is satisfied for $N \geq 500$ since $\alpha L = \frac{3}{2} > \frac{\|A_{1,2}\|_\infty}{K\alpha - \|A_{1,2}\|_\infty}\Delta_{(500)} = \frac{2}{3-2}\Delta_{(500)} = 4(500)^{-\frac{1}{6}} \approx 1.4198$.

Therefore, with $\beta^{1,2} - \varepsilon = \beta^3 = 2.4$, by employing Scheme $\mathrm{P}(\beta^{1,2},\varepsilon)$ on the unstable Jordan block $A_{1,2}$ and the scheme of Section 5 on the stable Jordan block $A_3 = \frac{1}{2}$, we are guaranteed by Theorem 3.0.1 and (5.4) that,

$$\limsup_{T\to\infty} \frac{1}{T} E\left[\sum_{t=0}^{T-1} x_t^T x_t\right] - 29 = \mathcal{O}_{C_{1,2}}\left(2^{-\frac{1}{3}\cdot\frac{1}{2}C_{1,2}}\right) + \mathcal{O}_{C_3}\left(2^{-\frac{1}{3}C_3}\right)$$

and since asymptotically we have $\frac{1}{2}C_{1,2} \sim \frac{1}{3}C$ and $C_3 \sim \frac{1}{3}C$ it follows that,

$$\limsup_{T\to\infty} \frac{1}{T} E\left[\sum_{t=0}^{T-1} x_t^T x_t\right] - 29 = \mathcal{O}_C\left(2^{-\frac{1}{9}C}\right).$$

Then with $n = 3$ this establishes that the scheme we have provided here achieves the rate function $r(C) = 2^{\frac{5}{9}C}$, in the sense of Definition 3.0.1.

With the parameters above, the system was run for all $N \in \{500, 550, ... 5000\}$ and the average second moment was recorded. The second moment achieved in each trial is plotted against the capacity $C = \log_2 17 + 3\log_2(N+1)$ below in Figure 6.1.

Since we expect that the optimality gap will converge to zero at a rate like $2^{-\delta C}$ for some $\delta > 0$, it is reasonable to expect that the logarithm of the optimality gap is approximately linear in C with slope $-\delta$. Therefore, an estimate of the order of convergence was obtained by performing a linear regression between C and the logarithm of the optimality gap. This estimate on the order of convergence can be seen in Figure 6.1. Notably, a much better convergence rate was observed in this simulation than is promised by our main results.

Finally, a sample path for the Markov chain $\{(x_t, \Delta_t)\}_{t=0}^{\infty}$ is displayed in Figure 6.2 for the case $N = 3000$.

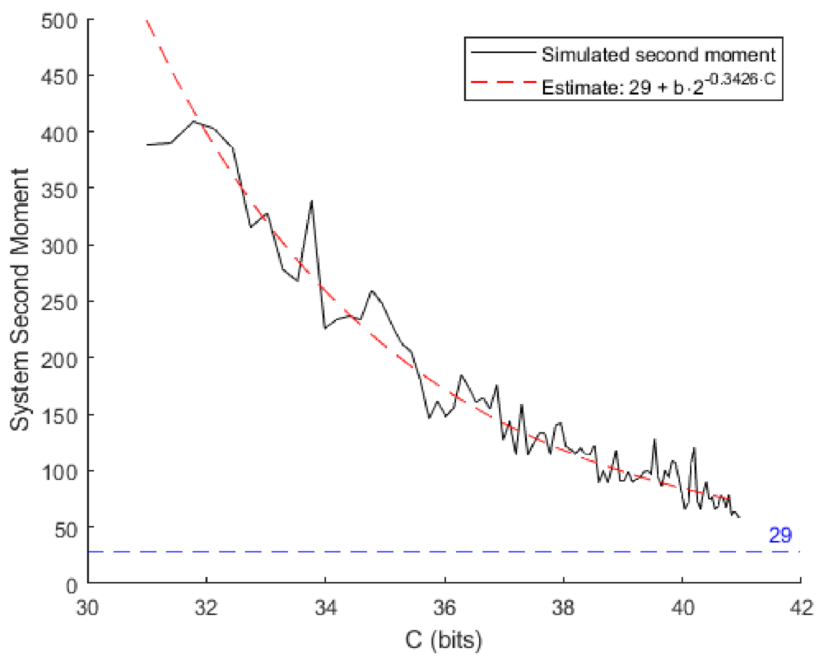

Figure 6.1: Convergence to the optimum $\operatorname{tr}(Q\Sigma) = 29$. The order of convergence is approximately $\mathcal{O}_C\left(2^{-0.3426C}\right)$, which is much better than the expected convergence $\mathcal{O}_C\left(2^{-\frac{1}{9}C}\right)$. The constant b is on the order of $2^{19.5}$.

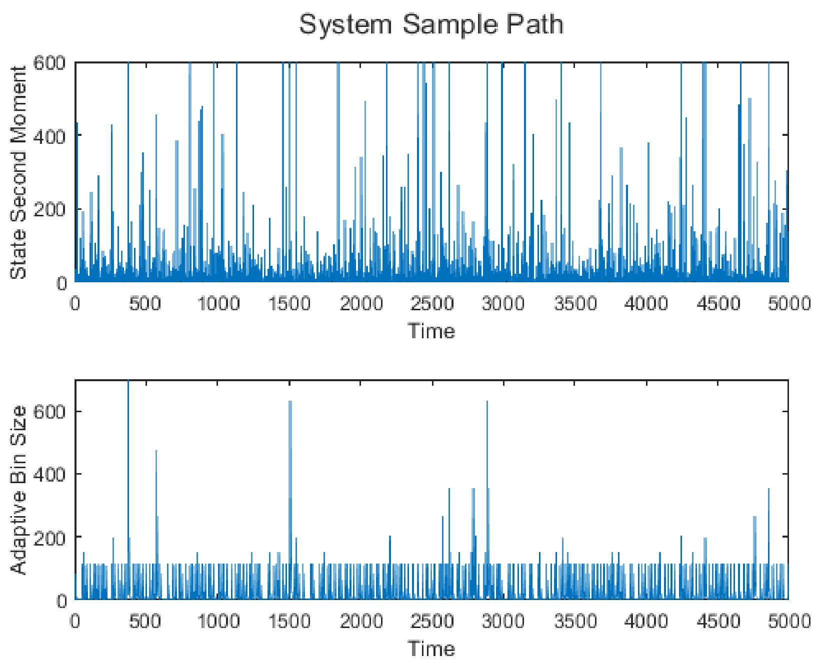

Figure 6.2: A sample path for the system with $N = 3000$ fixed quantizer bins. The system second moment and adaptive bin size are displayed over time.

Chapter 7

Conclusion and Future Work

In conclusion, this thesis has established the existence of a joint coding and control scheme for networked control systems of the form (1.1) which is asymptotically optimal in that, as the data rate grows without bound, the system second moment converges to the classical optimum with an explicit rate of convergence. The techniques in this thesis build on prior work in this context, in particular by the use of random-time Lyapunov drift conditions to establish key stability results.

There are several potential directions for future work. The first and most important for practical applications is the relaxation of the invertibility assumption on the control matrix B. In general, it is possible to achieve stability in the sense of positive Harris recurrence and *finite* system moments with just the assumption that the pair (A, B) is controllable [3, Theorem 2.2]. Controllability is a natural relaxation of the invertibility assumption to pursue for the kinds of linear systems considered here.

A second possibility is to consider stricter conditions on the noise process than Condition 3.0.1. For instance, one might ask that the noise tails are dominated by exponential decay (so-called "sub-exponential" random variables), and seek to construct schemes which, in the sense of Definition 3.0.1, achieve rate functions that are much better than exponential (e.g., polynomial in C). The two-part coding scheme presented here is well suited to generalizations of this type, because the bound of Lemma A.5 holds quite generally (e.g., with few assumptions on the noise process) and repeated use of this bound and Condition 3.0.1 leads to most of our key results. Thus, using this bound with a stricter condition on the noise process may lead to stronger stability and optimality results.

Finally, it would be interesting to explore optimality of schemes for *non-linear* systems. The two-stage scheme approach seems as though it would be fruitful here, supposing that one can establish stability and ergodicity results by the adaptive part of the code.

Appendix A

Proofs

Proof of Proposition 1.3.1. We show that in the fully observed setup, the optimal control policy which minimizes (1.2) is $u_t = -B^{-1}Ax_t$, achieving an optimal cost of $\mathrm{tr}\left(Q\Sigma\right)$ where $\mathrm{tr}\left(\cdot\right)$ is the trace operator. Let $v_t \coloneqq Ax_t + Bu_t$ so that $x_{t+1} = v_t + w_t$, then under any policy γ we have

$$\begin{aligned}
\limsup_{T\to\infty} \frac{1}{T} E\left[\sum_{t=0}^{T-1} x_t^T Q x_t\right] &= \limsup_{T\to\infty} \frac{1}{T} E\left[\sum_{t=0}^{T-1} x_{t+1}^T Q x_{t+1}\right] \\
&= \limsup_{T\to\infty} \frac{1}{T} E\left[\sum_{t=0}^{T-1} (v_t + w_t)^T Q (v_t + w_t)\right] \\
&= \limsup_{T\to\infty} \frac{1}{T} E\left[\sum_{t=0}^{T-1} v_t^T Q v_t + w_t^T Q w_t\right] \\
&\geq \limsup_{T\to\infty} \frac{1}{T} E\left[\sum_{t=0}^{T-1} w_t^T Q w_t\right] \qquad \text{(A.1)} \\
&= E\left[w_0^T Q w_0\right] = \mathrm{tr}\left(Q E\left[w_0 w_0^T\right]\right) = \mathrm{tr}\left(Q\Sigma\right)
\end{aligned}$$

by positive definiteness of Q and that w_t is i.i.d. zero-mean. In the case $u_t = -B^{-1}Ax_t$ we have $v_t = 0$ and so (A.1) is an equality, establishing optimality. $\square$

For the proof of both Lemmas 1.7.3 and 1.7.4 we will utilize supermartingale arguments. Suppose that $\{\phi_t\}_{t=0}^{\infty}$ satisfies Condition 1.7.1 at some $\phi \in \mathbb{X}$. Define the sequence of random variables $\{M_z\}_{z\geq 0}$ by $M_0 = V(\phi)$, and for any $z \geq 0$,

$$M_{z+1} = V(\phi_{\mathcal{T}_{z+1}}) + \sum_{t=0}^{\mathcal{T}_{z+1}-1} f(\phi_t) - b\sum_{k=0}^{z} \mathbb{1}_{\left\{\phi_{\mathcal{T}_k}\in C\right\}}.$$

Then the supermartingale property for M_z (i.e., $E\left[M_{z+1} \mid \mathcal{F}_{\mathcal{T}_z}\right] \leq M_z$) follows from (1.7):

$$\begin{aligned}
E\left[M_{z+1} \mid \mathcal{F}_{\mathcal{T}_z}\right] - M_z &= E\left[V(\phi_{\mathcal{T}_{z+1}}) \mid \mathcal{F}_{\mathcal{T}_z}\right] - V(\phi_{\mathcal{T}_z}) \\
&\qquad + \sum_{t=\mathcal{T}_z}^{\mathcal{T}_{z+1}-1} f(\phi_t) - b\mathbb{1}_{\{\phi_{\mathcal{T}_z}\in C\}} \\
&\leq -d(\phi_{\mathcal{T}_z}) + b\mathbb{1}_{\{\phi_{\mathcal{T}_z}\in C\}} + d(\phi_{\mathcal{T}_z}) - b\mathbb{1}_{\{\phi_{\mathcal{T}_z}\in C\}} \\
&= 0.
\end{aligned}$$

Thus, for any $z \geq 0$ we have $E_\phi\left[M_z\right] \leq V(\phi)$.

Remark. The proof of Lemma 1.7.3 largely follows that of [2, Theorem 2.1(i)] and is included here for completeness.

Proof of Lemma 1.7.3. Let $\phi \in C$ be arbitrary. We define the following two stopping times,

$$\zeta_C \coloneqq \min\left\{z \geq 1 : \phi_{\mathcal{T}_z} \in C\right\}, \qquad \zeta_C^n \coloneqq \min(n, \zeta_C)$$

where ζ_C^n is uniformly bounded. Therefore we may apply Doob's optional sampling theorem [45, Theorem 10.10] for the supermartingale sequence $\{M_z\}_{z\geq 0}$ to find that

$E_\phi\left[M_{\zeta_C^n}\right] \le V(\phi)$. This expands as

$$E_\phi\left[V(\phi_{\mathcal{T}_{\zeta_C^n}}) + \sum_{t=0}^{\mathcal{T}_{\zeta_C^n}-1} f(\phi_t) - b\sum_{k=0}^{\zeta_C^n-1} \mathbb{1}_{\left\{\phi_{\mathcal{T}_k}\in C\right\}}\right] \le V(\phi).$$

We note that for $1 \le k \le \zeta_C^n - 1$ one has by construction that $\phi_{\mathcal{T}_k} \notin C$. In combination with this and the facts that $V(\cdot) > 0$ and $f \equiv 1$ the above bound relaxes to

$$E_\phi\left[\mathcal{T}_{\zeta_C^n}\right] \le V(\phi) + b. \tag{A.2}$$

Note that $\tau_C \le \mathcal{T}_{\zeta_C}$ by construction. Since ζ_C^n converges monotonically to ζ_C from below, it follows from the monotone convergence theorem that

$$E_\phi\left[\tau_C\right] \le E_\phi\left[\mathcal{T}_{\zeta_C}\right] = E_\phi\left[\lim_{n\to\infty} \mathcal{T}_{\zeta_C^n}\right] = \lim_{n\to\infty} E_\phi\left[\mathcal{T}_{\zeta_C^n}\right] \le V(\phi) + b. \tag{A.3}$$

Recall that $\phi \in C$ is arbitrary and that V is uniformly bounded over C. It follows then that

$$\sup_{\phi\in C} E_\phi\left[\tau_C\right] < \infty. \tag{A.4}$$

This allows us to show that $\mathcal{X}$ is full and absorbing.

First, suppose $\mathcal{X}$ is not full, i.e. that $\varphi(\mathcal{X}^{\mathcal{C}}) > 0$. Since $\{\phi_t\}_{t=0}^\infty$ is φ-irreducible it then follows that for arbitrary $\phi \in C$ we have $P_\phi\left(\tau_{\mathcal{X}^C} < \infty\right) > 0$, so there are sample paths starting in C which visit $\mathcal{X}^C$ in finite time with positive probability. From here, the defining property of $\psi \in \mathcal{X}^C$ is that $P_\psi\left(\tau_C = \infty\right) > 0$, so in total there exists sample paths of positive probability starting in C which do not return to C in finite time. This contradicts (A.4), so $\mathcal{X}$ must be full.

Now suppose $\mathcal{X}$ is not absorbing, i.e. that for some $\phi \in \mathcal{X}$ we have $P(\phi, \mathcal{X}^C) > 0$.

It follows by similar arguments as above that there exists sample paths of positive probability which start at $\phi \in \mathcal{X}$, visit $\mathcal{X}^C$ in finite time, and then do not return to C in finite time. This violates the defining property of $\mathcal{X}$ which is a contradiction.

Finally, consider the restriction of $\{\phi_t\}_{t=0}^{\infty}$ to $\mathcal{X}$. By the defining property of $\mathcal{X}$, this restricted chain satisfies $P_\phi(\tau_C < \infty) = 1$ for all $\phi \in \mathcal{X}$, where C is a small set satisfying (A.4). It then follows from [46, Theorem 4.1(ii)] (where smallness implies petiteness) that $\{\phi_t\}_{t=0}^{\infty}$ restricted to $\mathcal{X}$ is positive Harris recurrent.

If $\mathcal{X} = \mathbb{X}$ (i.e., $P_\phi(\tau_C < \infty) = 1$ for all $\phi \in \mathbb{X}$) then the restriction of $\{\phi_t\}_{t=0}^{\infty}$ to $\mathcal{X}$ is $\{\phi_t\}_{t=0}^{\infty}$ itself, so $\{\phi_t\}_{t=0}^{\infty}$ is positive Harris recurrent. □

Proof of Lemma 1.7.4. The supermartingale property for M_n expands as

$$E_\phi\left[V(\phi_{\mathcal{T}_n}) + \sum_{t=0}^{\mathcal{T}_n-1} f(\phi_t) - b\sum_{k=0}^{n-1} \mathbb{1}_{\left\{\phi_{\mathcal{T}_k}\in C\right\}}\right] \le V(\phi).$$

Since V is strictly positive and $\sum_{k=0}^{n-1} \mathbb{1}_{\left\{\phi_{\mathcal{T}_k}\in C\right\}} \le n$, we can relax the above to

$$E_\phi\left[\sum_{t=0}^{\mathcal{T}_n-1} f(\phi_t)\right] \le V(\phi) + bn,$$

i.e.,

$$\frac{1}{n}E_\phi\left[\sum_{t=0}^{\mathcal{T}_n-1} f(\phi_t)\right] \le \frac{1}{n}V(\phi) + b. \tag{A.5}$$

Finally, recall that the sequence of stopping times $\mathcal{T}_k$ is strictly increasing with $\mathcal{T}_0 = 0$.

It follows that $\mathcal{T}_k \geq k$ for all $k \geq 0$, so that $n \leq \mathcal{T}_n$. In light of (A.5) this gives us:

$$\begin{aligned}\frac{1}{n}E_\phi\left[\sum_{t=0}^{n-1} f(\phi_t)\right] &\leq \frac{1}{n}E_\phi\left[\sum_{t=0}^{\mathcal{T}_n-1} f(\phi_t)\right] \\ &\leq \frac{1}{n}V(\phi) + b. \end{aligned} \tag{A.6}$$

Now let g be a function bounded by f, that is $g(\cdot) \leq cf(\cdot)$ for some $c > 0$. For integer $k \geq 1$, let $g_k := \min(g, k)$. Since g_k is bounded, it follows from the individual ergodic theorem (and Fatou's lemma) that for *every* $\phi_0 \in \mathbb{X}$ we have

$$E_\pi\left[g_k(\phi_t)\right] = \lim_{n\to\infty} \frac{1}{n}E_{\phi_0}\left[\sum_{t=0}^{n-1} g_k(\phi_t)\right]$$

so with $\phi_0 = \phi$ we have by (A.6) that,

$$E_\pi\left[g_k(\phi_t)\right] = \lim_{n\to\infty} \frac{1}{n}E_{\phi}\left[\sum_{t=0}^{n-1} g_k(\phi_t)\right] \leq c \lim_{n\to\infty} \frac{1}{n}E_{\phi}\left[\sum_{t=0}^{n-1} f(\phi_t)\right] \leq cb.$$

Therefore, since g_k converges monotonically to g from below, it follows from the monotone convergence theorem that

$$E_\pi\left[g(\phi_t)\right] = E_\pi\left[\lim_{k\to\infty} g_k(\phi_t)\right] = \lim_{k\to\infty} E_\pi\left[g_k(\phi_t)\right] \leq cb.$$

Here we remark that if $g \equiv f$ then $c = 1$ and this proves the first claim. Next we show that for arbitrary g the claimed ergodicitiy result holds. We have that for any

$\phi_0 \in \mathbb{X}$ that

$$\begin{aligned}
\lim_{n\to\infty} \frac{1}{n} E_{\phi_0}\left[\sum_{t=0}^{n-1} g(\phi_t)\right] &= \lim_{n\to\infty} \frac{1}{n} E_{\phi_0}\left[\sum_{t=0}^{n-1} \lim_{k\to\infty} g_k(\phi_t)\right] \\
&= \lim_{n\to\infty} \lim_{k\to\infty} \frac{1}{n} E_{\phi_0}\left[\sum_{t=0}^{n-1} g_k(\phi_t)\right] \\
&= \lim_{k\to\infty} \lim_{n\to\infty} \frac{1}{n} E_{\phi_0}\left[\sum_{t=0}^{n-1} g_k(\phi_t)\right] \qquad \text{(A.7)} \\
&= \lim_{k\to\infty} E_\pi\left[g_k(\phi_t)\right] \qquad \text{(A.8)} \\
&= E_\pi\left[\lim_{k\to\infty} g_k(\phi_t)\right] = E_\pi\left[g(\phi_t)\right]. \qquad \text{(A.9)}
\end{aligned}$$

Above, (A.7) holds since $\{\phi_t\}_{t=0}^{\infty}$ is positive Harris recurrent and $g \in L^1(\pi)$. As before, (A.8) holds by the individual ergodic theorem and (A.9) holds by the monotone convergence theorem. □

Proof of Lemma 2.0.1. Suppose that $b = 1$ without loss of generality so that we are considering control of the following system,

$$x_{t+1} = ax_t + u_t + w_t$$

across a discrete noiseless channel of capacity C bits using an arbitrary joint coding and control scheme ϕ_C. First, note we may assume that for all C sufficiently large,

$$\limsup_{T\to\infty} \frac{1}{T} E\left[\sum_{t=0}^{T-1} x_t^2\right] < \infty. \qquad \text{(A.10)}$$

If this fails to be true for the joint scheme ϕ_C then (2.3) will trivially hold. Note that

(A.10) implies that

$$\liminf_{T\to\infty} E\left[x_T^2\right] < \infty, \tag{A.11}$$

since otherwise $\lim_{T\to\infty} \frac{1}{T} E\left[\sum_{t=0}^{T-1} x_t^2\right] = \infty$.

We will follow the core arguments in the proof of [43, Theorem 11.3.2] with the key difference of not assuming that $\lim_{t\to\infty} E\left[x_t^2\right]$ exists.

Denote the received channel output at time $t \geq 0$ as q'_t. Let $D_t \coloneqq E\left[(x_t + \frac{1}{a}u_t)^2\right]$, $d_t \coloneqq E\left[x_t^2\right]$ and note that the system update gives us $D_t = \frac{1}{a^2}(d_{t+1} - \sigma^2)$.

By entropy-power inequality [43, Lemma 5.3.2], we are able to arrive at the following bound on the channel capacity,

$$\begin{aligned}
C &\geq \limsup_{T\to\infty} \left(\frac{1}{2}\log_2 \left(a^2 + \frac{\sigma^2}{\frac{1}{T}\sum_{t=0}^{T-1} D_t} \right) + \frac{1}{T}\left(h(x_0) - h(ax_{T-1} + w_{T-1} \mid q'_{[0,T-1]}) \right) \right) \\
&\geq \liminf_{T\to\infty} \frac{1}{2}\log_2 \left(a^2 + \frac{\sigma^2}{\frac{1}{T}\sum_{t=0}^{T-1} D_t} \right) + \limsup_{T\to\infty} \frac{1}{T}\left(h(x_0) - h(ax_{T-1} + w_{T-1} \mid q'_{[0,T-1]}) \right)
\end{aligned}$$

where $h(\cdot)$ and $h(\cdot \mid \cdot)$ are the regular and conditional differential entropy, respectively.

We will consider the two limits above separately. Let

$$d \coloneqq \limsup_{T\to\infty} \frac{1}{T}\sum_{t=0}^{T-1} d_t = \limsup_{T\to\infty} \frac{1}{T} E\left[\sum_{t=0}^{T-1} x_t^2\right] < \infty$$

and note that since $D_t = \frac{1}{a^2}(d_{t+1} - \sigma^2)$ we have that

$$\limsup_{T\to\infty} \frac{1}{T}\sum_{t=0}^{T-1} D_t = \frac{1}{a^2}(d - \sigma^2).$$

Then since $\log\left(1+\frac{1}{x}\right)$ is continuous and monotone decreasing in $x>0$ we have

$$\begin{aligned}
\liminf_{T\to\infty}\frac{1}{2}\log_2\left(a^2+\frac{\sigma^2}{\frac{1}{T}\sum_{t=0}^{T-1}D_t}\right) &= \frac{1}{2}\log_2\left(a^2+\frac{\sigma^2}{\limsup_{T\to\infty}\frac{1}{T}\sum_{t=0}^{T-1}D_t}\right)\\
&= \frac{1}{2}\log_2\left(a^2+\frac{a^2\sigma^2}{d-\sigma^2}\right)\\
&= \frac{1}{2}\log_2\left(\frac{a^2 d}{d-\sigma^2}\right).
\end{aligned}$$

Therefore, we have that

$$C \geq \frac{1}{2}\log_2\left(\frac{a^2 d}{d-\sigma^2}\right)+\limsup_{T\to\infty}\frac{1}{T}\left(h(x_0)-h(ax_{T-1}+w_{T-1}\mid q'_{[0,T-1]})\right) \tag{A.12}$$

and what remains is to bound this limit supremum. We then have,

$$\begin{aligned}
&\limsup_{T\to\infty}\frac{1}{T}\left(h(x_0)-h(ax_{T-1}+w_{T-1}\mid q'_{[0,T-1]})\right)\\
&\qquad= -\liminf_{T\to\infty}\frac{1}{T}h(ax_{T-1}+w_{T-1}\mid q'_{[0,T-1]})\\
&\qquad= -\liminf_{T\to\infty}\frac{1}{T}h(x_T-bu_{T-1}\mid q'_{[0,T-1]})\\
&\qquad= -\liminf_{T\to\infty}\frac{1}{T}h(x_T\mid q'_{[0,T-1]}) \qquad\qquad (\text{A.13})\\
&\qquad\geq -\liminf_{T\to\infty}\frac{1}{T}h(x_T). \qquad\qquad (\text{A.14})
\end{aligned}$$

Above, (A.13) follows since u_t is constant given $q'_{[0,t]}$ and differential entropy is translation invariant. (A.14) follows since conditioning reduces differential entropy.

We now show that $\liminf_{T\to\infty}\frac{1}{T}h(x_T)\leq 0$.

Note that since the noise process is added to the state at every time stage and has a pdf η which is positive everywhere on $\mathbb{R}$, the state x_t will also have a positive-everywhere density. Furthermore, the state x_t will also have finite variance $E\left[x_t^2\right] - E\left[x_t\right]^2$ at each time stage.

It is known that for a distribution over $\mathbb{R}$ with specified variance S, the Gaussian distribution maximizes differential entropy at $\frac{1}{2}\log_2\left(2\pi e S\right)$. Therefore we have,

$$\begin{aligned}\liminf_{T\to\infty}\frac{1}{T}h(x_T) &\leq \liminf_{T\to\infty}\frac{1}{2T}\log_2\left(2\pi e\left(E\left[x_T^2\right] - E\left[x_T\right]^2\right)\right)\\ &\leq \liminf_{T\to\infty}\frac{1}{2T}\log_2\left(2\pi e E\left[x_T^2\right]\right)\\ &\leq \liminf_{T\to\infty}\frac{1}{T}\log_2\left(E\left[x_T^2\right]\right) = 0,\end{aligned}$$

where the final limit is zero by (A.11).

Therefore, we have the following ultimate bound on the channel capacity,

$$C \geq \frac{1}{2}\log_2\left(\frac{a^2 d}{d-\sigma^2}\right).$$

Recall that $d = \limsup_{T\to\infty}\frac{1}{T}E\left[\sum_{t=0}^{T-1}x_t^2\right]$. Rearranging the above inequality for the optimality gap $d-\sigma^2$ we find that

$$\limsup_{T\to\infty}\frac{1}{T}E\left[\sum_{t=0}^{T-1}x_t^2\right] - \sigma^2 \geq \frac{a^2\sigma^2}{2^{2C}-a^2}$$

which completes the proof. □

Proof of Lemma 4.1.1. Define the event F by

$$F = \left\{\left\|X_N\right\|_\infty \leq \tfrac{1}{2}N\Delta_{(N)}\right\}.$$

To start, the (i,j)-th component of $Y_N Y_N^T$ is $Y_N^i Y_N^j$. Therefore,

$$\begin{aligned}\left|\left[E\left[Y_N Y_N^T\right]\right]_{ij}\right| &= \left|E\left[Y_N^i Y_N^j\right]\right| = \left|E\left[Y_N^i Y_N^j \mathbb{1}_F\right] + E\left[Y_N^i Y_N^j \mathbb{1}_{F^C}\right]\right| \\ &\leq \left|E\left[Y_N^i Y_N^j \mathbb{1}_F\right]\right| + \left|E\left[Y_N^i Y_N^j \mathbb{1}_{F^C}\right]\right| \\ &\leq E\left[\left|Y_N^i Y_N^j\right| \mathbb{1}_F\right] + E\left[\left|Y_N^i Y_N^j\right| \mathbb{1}_{F^C}\right] \end{aligned} \tag{A.15}$$

by triangle inequality and Jensen's inequality. Since F is such that $\left|X_N^k\right| \leq \frac{1}{2} N \Delta_{(N)}$ for all $1 \leq k \leq n$, it follows by construction of U_N that $\left|Y_N^k\right| \leq \frac{1}{2}\Delta_{(N)}$. Therefore,

$$E\left[\left|Y_N^i Y_N^j\right| \mathbb{1}_F\right] \leq E\left[\left(\tfrac{1}{2}\Delta_{(N)}\right)^2 \mathbb{1}_F\right] \leq \left(\tfrac{1}{2}\Delta_{(N)}\right)^2 = \tfrac{1}{4}\Delta_{(N)}^2. \tag{A.16}$$

Note that $\left|Y_N^k\right| \leq \left|X_N^k\right|$, then for the second expectation of (A.15) we have by Hölder's inequality that

$$\begin{aligned} E\left[\left|Y_N^i Y_N^j\right| \mathbb{1}_{F^C}\right] &\leq E\left[\left|X_N^i X_N^j\right| \mathbb{1}_{F^C}\right] \leq E\left[\left|X_N^i X_N^j\right|^{\frac{m}{2}}\right]^{\frac{2}{m}} E\left[\mathbb{1}_{F^C}\right]^{1-\frac{2}{m}} \\ &= E\left[\left|X_N^i X_N^j\right|^{\frac{m}{2}}\right]^{\frac{2}{m}} P\left(\|X_N\|_\infty > \tfrac{1}{2} N \Delta_{(N)}\right)^{\frac{m-2}{m}}. \end{aligned} \tag{A.17}$$

Recall that we suppose $\sup_{N \geq 2} E\left[\|X_N\|_\infty^m\right] =: B_m < \infty$. It follows by Cauchy-Schwarz inequality that the expectation in (A.17) is bounded as

$$\begin{aligned} E\left[\left|X_N^i X_N^j\right|^{\frac{m}{2}}\right]^{\frac{2}{m}} &\leq E\left[\left|X_N^i\right|^m\right]^{\frac{1}{m}} E\left[\left|X_N^j\right|^m\right]^{\frac{1}{m}} \\ &\leq E\left[\|X_N\|_\infty^m\right]^{\frac{2}{m}} \leq (B_m)^{\frac{2}{m}}. \end{aligned} \tag{A.18}$$

Note that by Markov's inequality, we have for $u > 0$ that

$$P\left(\|X_N\|_\infty > u\right) = P\left(\|X_N\|_\infty^m > u^m\right) \leq E\left[\|X_N\|_\infty^m\right] u^{-m} \leq B_m u^{-m}$$

and so since $\frac{1}{2} N \Delta_{(N)} = N^{\frac{2}{m}}$, the tail probability in (A.17) can be bounded as,

$$\begin{aligned} P\left(\|X_N\|_\infty > \tfrac{1}{2} N \Delta_{(N)}\right)^{\frac{m-2}{m}} &\leq \left(B_m \left(N^{\frac{2}{m}}\right)^{-m}\right)^{\frac{m-2}{m}} \\ &= \left(B_m\right)^{\frac{m-2}{m}} N^{-2+\frac{4}{m}}. \end{aligned}$$

Combining this with (A.17) and (A.18) we find that

$$E\left[\left|Y_N^i Y_N^j\right| \mathbb{1}_{\{F^C\}}\right] \leq \left(B_m\right)^{\frac{2}{m}+\frac{m-2}{m}} N^{-2+\frac{4}{m}} = B_m N^{-2+\frac{4}{m}} = \mathcal{O}_N\left(\Delta_{(N)}^2\right)$$

which, combined with (A.16) yields that

$$\left|\left[E\left[Y_N Y_N^T\right]\right]_{ij}\right| = \mathcal{O}_N\left(\Delta_{(N)}^2\right).$$

Now that we have demonstrated that each component of $E\left[Y_N Y_N^T\right]$ satisfies the desired bound, the proof completes as follows. For any matrix $V \in \mathbb{R}^{n \times n}$ we have by triangle inequality and Jensen's inequality that

$$\begin{aligned}
\left|\operatorname{tr}\left(VE\left[Y_N Y_N^T\right]\right)\right| &= \left|\sum_{i=1}^{n}\sum_{j=1}^{n} V_{ij}\left[E\left[Y_N Y_N^T\right]\right]_{ji}\right| = \left|\sum_{i=1}^{n}\sum_{j=1}^{n} V_{ij} E\left[Y_N^i Y_N^j\right]\right| \\
&\leq \sum_{i=1}^{n}\sum_{j=1}^{n} \left|V_{ij}\right| \left|E\left[Y_N^i Y_N^j\right]\right| \\
&\leq \left(\max_{k,l} |V_{kl}|\right) \sum_{i=1}^{n}\sum_{j=1}^{n} \left|E\left[Y_N^i Y_N^j\right]\right| = \mathcal{O}_N\left(\Delta_{(N)}^2\right). \qquad \text{(A.19)}
\end{aligned}$$

The proof can be completed by noting that for any two positive semidefinite matrices P, Q we have $\operatorname{tr}(PQ) \geq 0$. Note that for any random vector X, $E\left[XX^T\right]$ is positive semidefinite. Therefore, for any positive semidefinite matrix V it follows that

$$\operatorname{tr}\left(VE\left[Y_N Y_N^T\right]\right) = \left|\operatorname{tr}\left(VE\left[Y_N Y_N^T\right]\right)\right|$$

and so the proof concludes in view of (A.19). □

We now proceed with proving stability of $\{(x_t, \Delta_t)\}_{t=0}^{\infty}$ in the sense of both positive Harris recurrence as well as moment stability. To show positive Harris recurrence, we will need to demonstrate that our chain is irreducible and that an appropriate class of sets are small.

Proposition A.1. *The process $\{(x_t, \Delta_t)\}_{t=0}^{\infty}$ is φ-irreducible and aperiodic where φ is the product of the Lebesgue and discrete measures on $\mathbb{R}^n \times \Omega_\Delta$.*

Proof. First, we note that the condition for aperiodicity is strictly stronger than that for φ-irreducibility. That is, if we can show that for any $(x_0, \Delta_0) \in \mathbb{R}^n \times \Omega_\Delta$ and $B \in \mathcal{B}(\mathbb{R}^n \times \Omega_\Delta)$ with $\varphi(B) > 0$ that there exists $n_0 = n_0((x_0, \Delta_0), B)$ so that for all

$n \geq n_0$,

$$P^n((x_0, \Delta_0), B) > 0 \tag{A.20}$$

then it follows that $P_{x_0,\Delta_0}(\tau_B < \infty) > 0$ (since $P_{x_0,\Delta_0}(\tau_B = n) > 0$ for all $n \geq n_0$). Thus, we focus strictly on showing that (A.20) holds. This will establish both φ-irreducibility and aperiodicity at once.

We let μ be Lebesgue measure on $\mathbb{R}^n$ so that φ is the product measure of μ and the discrete measure on Ω_Δ. We consider an arbitrary initial state (x_0, Δ_0) and a set $B \in \mathcal{B}(\mathbb{R}^n \times \Omega_\Delta)$ with $\varphi(B) > 0$. Such a set must contain a subset of the form $C \times \{D\}$ where $\mu(C) > 0$. Naturally we have for any integer k that

$$P^k((x_0, \Delta_0), B) \geq P^k((x_0, \Delta_0), C \times \{D\})$$

so it suffices to show that for some n_0 we have $P^n((x_0, \Delta_0), C \times \{D\}) > 0$ for all $n \geq n_0$. Plainly, the way we will do this is to construct a sequence of events which occur with positive probability which drive the system to $C \times \{D\}$. The fact that we can do this in any specified number of time stages sufficiently large will follow from the fact that we may "hold" the bin size constant when $\Delta_t < L$.

From the known initial state (x_0, Δ_0) it follows that Δ_1 will be known deterministically (either Δ_0, $\alpha\Delta_0$ or $\rho\Delta_0$) and since the state transition kernel adds noise with a pdf η positive everywhere, the state x_1 will admit a pdf which is positive everywhere. We remark that because Δ_1 is known we may write the in-view event $\{(x_1, \Delta_1) \in \Lambda\}$ as an event of the form $\{x_1 \in L_1\}$ where $L_1 \in \mathcal{B}(\mathbb{R}^n)$ is some set of Lebesgue positive measure (i.e. that $\mu(L_1) > 0$). This event occurs with nonzero probability since x_1 admits a positive-everywhere pdf, and an identical argument can be made for the event $\{(x_1, \Delta_1) \in \Lambda^C\}$.

Thus we are able to consider a sample path in which either $(x_1, \Delta_1) \in \Lambda$ or $(x_1, \Delta_1) \in \Lambda^C$ where either occurs with positive probability. If we condition on one of these paths, we find that Δ_2 is known deterministically and x_2 again will admit some pdf which is positive everywhere. In this way, we may repeat the arguments above and are able to consider any arbitrary finite sequence of in-view/out-of-view events which must occur with positive probability.

Now, we proceed with constructing the sequence of events that drives the system from (x_0, Δ_0) to $C \times \{D\}$. We define integer $m \geq 0$ by,

$$m = \begin{cases} 0, & \text{if } \Delta_1 < L \\ \left\lfloor \log_\alpha\left(\frac{L}{\Delta_1}\right) \right\rfloor + 1, & \text{if } \Delta_1 \geq L \end{cases}$$

so that exactly m successive in-view events take Δ_1 to be less than L. Since the bin sizes each communicate with one another, there must exist integer $p, q \geq 0$ such that $\alpha^{m+p}\rho^q\Delta_1 = D$. That is, after m successive in-view stages from Δ_1, we may reach the bin size D by taking q successive out-of-view stages followed by p successive in-view stages. We let $n_0 = 1 + m + p + q$, and for arbitrary $n \geq n_0$ we propose the following sequence of events.

(1) For $1 \leq t \leq m$, $(x_t, \Delta_t) \in \Lambda$. (If $m = 0$, this is vacuously true)

(2) For $1 + m \leq t \leq m + (n - n_0)$, $(x_t, \Delta_t) \in \Lambda$.

(3) For $1 + m + (n - n_0) \leq t \leq m + q + (n - n_0)$, $(x_t, \Delta_t) \in \Lambda^C$.

(4) For $1 + m + q + (n - n_0) \leq t \leq m + q + p + (n - n_0)$, $(x_t, \Delta_t) \in \Lambda$.

(5) $x_n \in C$.

The above sequence of events drives the bin size from Δ_0 to D in exactly n time stages by construction, and by (5) we also have $x_n \in C$. By our previous discussion, the sequence of in-view/out-of-view events happens with nonzero probability, so what remains is to justify that conditioned on those events, $\{x_n \in C\}$ occurs with nonzero probability. Again, the reasoning here is that events (1) through (4) will determine Δ_{n-1} exactly and impart some prior on x_{n-1}, but the one-step transition kernel will convolve the state marginal with the positive-everywhere pdf η. Since C is of positive Lebesgue measure $\mu(C) > 0$, the event $x_n \in C$ must occur with positive probability.

Hence, for any $n \geq n_0$ we have established the existence of a sequence of events that drives the system from (x_0, Δ_0) to $C \times \{D\} \subseteq B$ in exactly n time stages with positive probability. This establishes (A.20) and so establishes φ-irreducibility and aperiodicity. $\square$

We denote the in-view set as

$$\Lambda := \left\{(x, \Delta) \in \mathbb{R}^n \times \Omega_\Delta : \|x\|_\infty \leq \tfrac{K}{2}\Delta\right\} \tag{A.21}$$

and in light of Section 1.7 we will develop results concerning the return time τ_Λ.

Proposition A.2. *For the Markov chain $\{(x_t, \Delta_t)\}_{t=0}^{\infty}$, bounded subsets of Λ are small.*

Proof. We begin by showing that subsets of Λ containing only one bin size are 1-small. That is, let $B_0 \in \mathcal{B}(\mathbb{R})$ and $D \in \Omega_\Delta$ be such that $B_0 \times \{D\}$ is a subset of Λ.

For arbitrary $(x_0, \Delta_0) \in B_0 \times \{D\}$ we then have that $\Delta_0 = D$ and $\Delta_1 = cD$ where c is constant but depends on our specific D ($c = 1$ if $D \geq L$, else $c = \alpha$). In any case, we know that $\Delta_1 = cD$ with probability one.

Since $B_0 \times \{D\}$ is bounded, we also know that $\|e_0\|_\infty$ is bounded. It follows then that the set

$$\mathcal{Y} \coloneqq \{y : y = A(e_0 - U_N(e_0)) \; \exists (x_0, \Delta_0) \in B_0 \times \{D\}\}$$

is bounded, and hence its closure $\mathrm{cl}(\mathcal{Y})$ is compact.

Recall that w_0 admits a pdf η which is positive everywhere in $\mathbb{R}^n$ and continuous with respect to Lebesgue measure, then define the pdf ζ on $\mathbb{R}^n$ by

$$\zeta(u) \coloneqq \frac{1}{\delta} \cdot \min_{y \in \mathrm{cl}(\mathcal{Y})} \eta(u - y)$$

where $\delta > 0$ is chosen to normalize ζ to a pdf. Since $\mathrm{cl}(\mathcal{Y})$ is compact and $\eta(u - y)$ is continuous on $\mathrm{cl}(\mathcal{Y})$ for every fixed u, it follows by the measurable selection theorem [47] that ζ is measurable.

Furthermore, since $\mathrm{cl}(\mathcal{Y})$ is compact and η is positive everywhere, this minimum is well defined and the pdf ζ is strictly positive everywhere.

Now let ν be the product measure of ζ and the Dirac measure δ_{cD} on $\mathbb{R}^n \times \Omega_\Delta$. We have for arbitrary $B \in \mathcal{B}(\mathbb{R}^n \times \Omega_\Delta)$ that $\nu(B) = 0$ if cD is not a bin size in B.

Now suppose that cD is a bin size in B with corresponding Borel set $B_1 \in \mathcal{B}(\mathbb{R}^n)$.

It follows then that for $(x_0, \Delta_0) \in B_0 \times \{D\}$ that,

$$\begin{aligned} P((x_0, \Delta_0), B) &= P_{x_0,\Delta_0}(x_1 \in B_1) = P_{x_0,\Delta_0}(w_0 + A(e_0 - U_N(e_0)) \in B_1) \\ &= \int_{B_1} \eta(u - A(e_0 - U_N(e_0)))du \\ &\geq \delta \int_{B_1} \zeta(u)du = \delta\nu(B_1 \times \{cD\}) = \delta\nu(B). \end{aligned} \tag{A.22}$$

Therefore, $B_0 \times \{D\}$ is $(1, \delta, \nu)$-small.

Now that we have shown sets of the form $B_0 \times \{D\} \subset \Lambda$ are small, it is easy to show that arbitrary bounded subsets of Λ are small. Indeed, we have that any such set is the finite union of sets of the above type.

Without reference to an exact definition of "petiteness" for brevity, we note that by [46, Theorem 5.5.7], small sets and petite sets are identical since our Markov chain is irreducible and aperiodic. Furthermore, [46, Proposition 5.5.5(ii)] yields that the finite union of petite sets is petite (and therefore small). It follows that arbitrary bounded subsets of Λ are the finite union of small sets and thus are themselves small (though generally not 1-small). $\square$

The following proposition will prove to be remarkably useful for the remainder of our proof program.

Proposition A.3. *Let $\{z_t\}_{t=0}^{\infty}$ be an i.i.d. sequence of nonnegative random variables. For any $b > 0$ and integer $k \geq 1$ we have*

$$P\left(\sum_{t=0}^{k-1} z_t > b\right) \leq kP\left(z_0 > \tfrac{b}{k}\right).$$

Proof. Since $\{z_t\}_{t=0}^{\infty}$ is identically distributed, we have for $k \geq 1$ that

$$P\left(\sum_{t=0}^{k-1} z_t > b\right) \leq P\left(\bigcup_{t=0}^{k-1} \left\{z_t > \tfrac{b}{k}\right\}\right) \leq \sum_{t=0}^{k-1} P\left(z_t > \tfrac{b}{k}\right) = kP\left(z_0 > \tfrac{b}{k}\right). \qquad \square$$

Corollary A.4. *Let $\{z_t\}_{t=0}^{\infty}$ be an i.i.d. sequence of nonnegative random variables. Then for any (real) $m > 0$ and integer $k \geq 1$ we have*

$$E\left[\left(\sum_{t=0}^{k-1} z_t\right)^m\right] \leq k^{m+1} E\left[z_0^m\right].$$

Proof. Using the tail formula for expectation of a nonnegative random variable and the previous proposition, we have

$$\begin{aligned} E\left[\left(\sum_{t=0}^{k-1} z_t\right)^m\right] &= \int_0^{\infty} P\left(\left(\sum_{t=0}^{k-1} z_t\right)^m > u\right) du = \int_0^{\infty} P\left(\sum_{t=0}^{k-1} z_t > u^{\frac{1}{m}}\right) du \\ &\leq k \int_0^{\infty} P\left(z_0 > \frac{u^{\frac{1}{m}}}{k}\right) = k \int_0^{\infty} P\left((kz_0)^m > u\right) du \\ &= kE\left[(kz_0)^m\right] = k^{m+1} E\left[z_0^m\right]. \end{aligned} \qquad \square$$

Lemma A.5. *Define the constants*

$$\xi := \frac{\rho}{\|A\|_{\infty}}, \quad h := \frac{K\alpha}{\rho}.$$

For $(x, \Delta) \in \Lambda$ we have for any $k \geq 1$ that

$$P_{x,\Delta}\left(\tau_{\Lambda} \geq k+1\right) \leq kT_{w_0}\left(\frac{\Delta}{2}\left(\frac{h\xi^k - 1}{k} - \frac{\Delta_{(N)}}{\alpha L}\right)\right). \tag{A.23}$$

Proof. Starting from $(x, \Delta) \in \Lambda$, let $e_0 := x - Q_K^{\Delta}(x)$. We define the map $S : \mathbb{R}^n \to \mathbb{R}^n$

by

$$S(u) \coloneqq u - U_N(u).$$

Note that $\|S(u)\|_\infty \leq \|u\|_\infty + \frac{\Delta_{(N)}}{2}$ for all $u \in \mathbb{R}^n$. The correction term $\frac{\Delta_{(N)}}{2}$ accounts for the possibility that $\|u\|_\infty < \frac{\Delta_{(N)}}{2}$.

We construct the following "zoom-out" process y_t. Let $y_0 = e_0$, and for $t \geq 0$ let

$$y_{t+1} = AS(y_t) + w_t. \tag{A.24}$$

Then the following holds for all $1 \leq t \leq \tau_\Lambda$:

$$y_t = x_t, \quad \Delta_t = \rho^{t-1}\alpha\Delta.$$

Therefore, it follows that for $k \geq 1$,

$$\begin{aligned}
\{\tau_\Lambda \geq k+1\} \;&=\; \bigcap_{t=1}^{k} \{(x_t, \Delta_t) \notin \Lambda\} \;=\; \bigcap_{t=1}^{k} \left\{\|x_t\|_\infty > \tfrac{K}{2}\Delta_t\right\} \\
&=\; \bigcap_{t=1}^{k} \left\{\|y_t\|_\infty > \tfrac{\Delta}{2} K\alpha\rho^{t-1}\right\} \\
&\subseteq\; \left\{\|y_k\|_\infty > \tfrac{\Delta}{2} K\alpha\rho^{k-1}\right\} \;=\; \left\{\|y_k\|_\infty > \tfrac{\Delta}{2} h\rho^{k}\right\}.
\end{aligned}$$

We may further relax this event as

$$\begin{aligned}
\left\{\|y_k\|_\infty > \tfrac{\Delta}{2}h\rho^k\right\} &= \left\{\|AS(y_{k-1}) + w_{k-1}\|_\infty > \tfrac{\Delta}{2}h\rho^k\right\} \\
&\subseteq \left\{\|AS(y_{k-1})\|_\infty + \|w_{k-1}\|_\infty > \tfrac{\Delta}{2}h\rho^k\right\} && \text{(A.25)} \\
&\subseteq \left\{\|A\|_\infty \|S(y_{k-1})\|_\infty + \|w_{k-1}\|_\infty > \tfrac{\Delta}{2}h\rho^k\right\} && \text{(A.26)} \\
&= \left\{\|S(y_{k-1})\|_\infty + \|A\|_\infty^{-1} \|w_{k-1}\|_\infty > \tfrac{\Delta}{2}h\xi\rho^{k-1}\right\} \\
&\subseteq \left\{\|S(y_{k-1})\|_\infty + \|w_{k-1}\|_\infty > \tfrac{\Delta}{2}h\xi\rho^{k-1}\right\} && \text{(A.27)} \\
&\subseteq \left\{\|y_{k-1}\|_\infty + \|w_{k-1}\|_\infty + \tfrac{\Delta_{(N)}}{2} > \tfrac{\Delta}{2}h\xi\rho^{k-1}\right\}. && \text{(A.28)}
\end{aligned}$$

Above, (A.25) follows by triangle inequality, (A.26) holds in light of (3.3), (A.27) holds since $\|A\|_\infty \geq |\lambda| \geq 1$ and (A.28) holds since $\|S(u)\|_\infty \leq \|u\|_\infty + \frac{\Delta_{(N)}}{2}$.

The steps (A.25) through (A.27) can be repeated $k-1$ more times to ultimately obtain that the event (A.27) implies

$$\left\{\|y_0\|_\infty + \sum_{t=0}^{k-1}\left(\|w_t\|_\infty + \tfrac{\Delta_{(N)}}{2}\right) > \tfrac{\Delta}{2}h\xi^k\right\}. \tag{A.29}$$

Using the fact that $y_0 = e_0$ and that starting in-view, $\|e_0\|_\infty \leq \frac{\Delta}{2}$, we find that

$$\{\tau_\Lambda \geq k+1\} \subseteq \left\{\sum_{t=0}^{k-1}\left(\|w_t\|_\infty + \tfrac{\Delta_{(N)}}{2}\right) > \tfrac{\Delta}{2}\left(h\xi^k - 1\right)\right\}.$$

Therefore,

$$P_{x,\Delta}\left(\tau_\Lambda \geq k+1\right) \leq P_{x,\Delta}\left(\sum_{t=0}^{k-1}\left(\|w_t\|_\infty + \tfrac{\Delta_{(N)}}{2}\right) > \tfrac{\Delta}{2}\left(h\xi^k - 1\right)\right).$$

Note that since $\alpha > \frac{\|A\|_\infty}{K}$ we have $h\xi^k > 1$ for $k \geq 1$. Therefore, $\frac{\Delta}{2}\left(h\xi^k - 1\right) >$

0 so we may apply Proposition A.3 to the sequence of i.i.d. random variables $\left\{ \|w_t\|_\infty + \frac{\Delta_{(N)}}{2} \right\}_{t=0}^{\infty}$ to find that

$$\begin{aligned} P_{x,\Delta}\left(\tau_\Lambda \geq k+1\right) &\leq kP_{x,\Delta}\left(\|w_0\|_\infty + \tfrac{\Delta_{(N)}}{2} > \frac{\frac{\Delta}{2}\cdot\left(h\xi^k - 1\right)}{k} \right) \\ &= kP_{x,\Delta}\left(\|w_0\|_\infty > \frac{\Delta}{2}\cdot\frac{h\xi^k-1}{k} - \frac{\Delta_{(N)}}{2} \right) \\ &= kT_{w_0}\left(\frac{\Delta}{2}\cdot\frac{h\xi^k-1}{k} - \frac{\Delta_{(N)}}{2} \right). \end{aligned}$$

Finally, note that with $\Delta \geq \alpha L$ we have,

$$\frac{\Delta}{2}\cdot\frac{h\xi^k-1}{k} - \frac{\Delta_{(N)}}{2} = \frac{\Delta}{2}\left(\frac{h\xi^k-1}{k} - \frac{\Delta_{(N)}}{\Delta}\right) \geq \frac{\Delta}{2}\left(\frac{h\xi^k-1}{k} - \frac{\Delta_{(N)}}{\alpha L}\right)$$

which, since $T_{w_0}(\cdot)$ is nonincreasing, proves the claimed bound (A.23).

Briefly, we justify that the argument to $T_{w_0}(\cdot)$ in (A.23) is *strictly positive* for all $k \geq 1$. First, note that Condition 3.3.2 is equivalent to,

$$\frac{\Delta_{(N)}}{\alpha L} < h\xi - 1. \tag{A.30}$$

Since $\alpha > \frac{\|A\|_\infty}{K}$ and $\rho \geq K\alpha$ we have $h \in (\frac{1}{\xi}, 1]$. It is relatively straightforward (if tedious) to verify that for $\xi > 1$ and $h \in (\frac{1}{\xi}, 1]$ that the function

$$s \mapsto \frac{h\xi^s - 1}{s}$$

is monotone increasing for real $s > 0$ (e.g. by careful analysis of its derivative). In particular, it follows that $\frac{h\xi^k-1}{k} \geq h\xi - 1$ for integer $k \geq 1$. Therefore, it follows from

(A.30) (i.e., from Condition 3.3.2) that

$$\frac{\Delta}{2}\left(\frac{h\xi^k - 1}{k} - \frac{\Delta_{(N)}}{\alpha L}\right) > \frac{\Delta}{2}\left(\frac{h\xi^k - 1}{k} - (h\xi - 1)\right) \geq 0. \qquad \square$$

Corollary A.6. *For* $(x, \Delta) \in \Lambda$*, the in-view return time satisfies*

$$\sup_{(x,\Delta)\in\Lambda} E_{x,\Delta}\left[\tau_\Lambda\right] < \infty$$

$$\lim_{\Delta\to\infty} E_{x,\Delta}\left[\tau_\Lambda - 1\right] = 0.$$

Proof. Note that by Condition 3.0.1 , we have $E\left[\|w_0\|_\infty^2\right] < \infty$ (this follows since $\beta > 2$). Therefore by Markov's inequality we have for $u > 0$ that

$$T_{w_0}(u) \leq E\left[\|w_0\|_\infty^2\right] u^{-2}.$$

We use the tail formula for the expectation of a nonnegative discrete random variable and Lemma A.5 to then find that for $(x, \Delta) \in \Lambda$,

$$\begin{aligned} E_{x,\Delta}\left[\tau_\Lambda\right] &= \sum_{k=1}^{\infty} P_{x,\Delta}\left(\tau_\Lambda \geq k\right) = 1 + \sum_{k=1}^{\infty} P_{x,\Delta}\left(\tau_\Lambda \geq k+1\right) \\ &\leq 1 + \sum_{k=1}^{\infty} kT_{w_0}\left(\frac{\Delta}{2}\left(\frac{h\xi^k - 1}{k} - \frac{\Delta_{(N)}}{\alpha L}\right)\right) \\ &\leq 1 + \sum_{k=1}^{\infty} kE\left[\|w_0\|_\infty^2\right]\left(\frac{\Delta}{2}\left(\frac{h\xi^k - 1}{k} - \frac{\Delta_{(N)}}{\alpha L}\right)\right)^{-2} \\ &= 1 + \Delta^{-2} \cdot \sum_{k=1}^{\infty} 4E\left[\|w_0\|_\infty^2\right] k^3 \left(h\xi^k - 1 - \frac{\Delta_{(N)}}{\alpha L} k\right)^{-2} \end{aligned}$$

where the series converges because each term is $\mathcal{O}_k\left(k^3\xi^{-2k}\right)$ for $\xi > 1$. We remark that $\Delta \geq \alpha L$ implies that the above is uniformly bounded (when one replaces Δ by αL) and that as $\Delta \to \infty$ the above is $1 + \mathcal{O}_\Delta\left(\Delta^{-2}\right)$, which proves both claims. $\square$

Finally, we prove positive Harris recurrence.

Proof of Theorem 4.2.2. Following the remark after Condition 1.7.1, we will satisfy the drift condition (1.8) for Lemma 1.7.3. In this case, we set the following.

$$\begin{aligned}
&V(x,\Delta) = c\log_\rho \Delta,\\
&d(x,\Delta) \equiv d = \sup_{(x,\Delta)\in\Lambda} E_{x,\Delta}\left[\tau_\Lambda\right],\\
&f(x,\Delta) \equiv 1,\\
&C = \Lambda \cap \left\{(x,\Delta) \in \mathbb{R}^n \times \Omega_\Delta : \Delta \leq D\right\},\\
&b = c(d-1) + c\log_\rho \alpha + d,
\end{aligned}$$

where d is finite by Corollary A.6, $c \geq \frac{2d}{\log_\rho(\frac{1}{\alpha})}$ and D is such that $\Delta > D$ implies that $E_{x,\Delta}\left[\tau_\Lambda - 1\right] \leq \frac{1}{2}\log_\rho(\frac{1}{\alpha})$ (such a D exists by Corollary A.6). We note that C is a small set by Proposition A.2.

First, notice that by construction we have for any $(x,\Delta) \in \Lambda$ that $E_{x,\Delta}\left[\tau_\Lambda\right] \leq d$, so the second inequality of (1.8) is satisfied. We now show that the first inequality also holds, which in this case is that for any $(x,\Delta) \in \Lambda$,

$$E_{x,\Delta}\left[V(x_{\tau_\Lambda}, \Delta_{\tau_\Lambda})\right] - V(x,\Delta) \leq -d + b\mathbb{1}_{\{(x,\Delta)\in C\}}. \tag{A.31}$$

First, note that for $1 \leq t \leq \tau_\Lambda$ we have that $\Delta_t = \rho^{t-1}\alpha\Delta$, so,

$$\begin{aligned} V(x_{\tau_\Lambda}, \Delta_{\tau_\Lambda}) &= c \log_\rho \Delta_{\tau_\Lambda} = c \log_\rho \left(\rho^{\tau_\Lambda - 1}\alpha\Delta\right) \\ &= c(\tau_\Lambda - 1) + c \log_\rho \alpha + c \log_\rho \Delta \\ &= c(\tau_\Lambda - 1) + c \log_\rho \alpha + V(x, \Delta). \end{aligned}$$

Thus we have that

$$E_{x,\Delta}\left[V(x_{\tau_\Lambda}, \Delta_{\tau_\Lambda})\right] - V(x, \Delta) = cE_{x,\Delta}\left[\tau_\Lambda - 1\right] + c \log_\rho \alpha. \tag{A.32}$$

First, suppose $\Delta > D$ (so that $(x, \Delta) \notin C$), then by construction of D and c, (A.32) becomes

$$\begin{aligned} cE_{x,\Delta}\left[\tau_\Lambda - 1\right] + c \log_\rho \alpha &\leq \tfrac{1}{2} c \log_\rho(\tfrac{1}{\alpha}) + c \log_\rho \alpha = -\tfrac{1}{2} c \log_\rho(\tfrac{1}{\alpha}) \\ &\leq -\frac{1}{2}\left(\frac{2d}{\log_\rho(\frac{1}{\alpha})}\right) \log_\rho(\tfrac{1}{\alpha}) \\ &= -d = -d + b\mathbb{1}_{\{(x,\Delta) \in C\}} \end{aligned}$$

which satisfies (A.31). Next, suppose that $\Delta \leq D$, then by construction of b and d, (A.32) becomes

$$\begin{aligned} cE_{x,\Delta}\left[\tau_\Lambda - 1\right] + c \log_\rho \alpha &\leq c(d - 1) + c \log_\rho \alpha \\ &= -d + b = -d + b\mathbb{1}_{\{(x,\Delta) \in C\}} \end{aligned}$$

and so (A.31) is satisfied over Λ. It follows then by Lemma 1.7.3 that $\{(x_t, \Delta_t)\}_{t=0}^{\infty}$ is positive Harris recurrent, provided that we can show that $P_{x,\Delta}\left(\tau_C < \infty\right) = 1$ for all

$(x, \Delta) \in \mathbb{R}^n \times \Omega_\Delta$. We do this to complete the proof.

Since we have proven the drift condition (1.8) holds at all $(x, \Delta) \in \Lambda$, it follows from the proof of Lemma 1.7.3 until (A.3) that for all $(x, \Delta) \in \Lambda$ we have

$$E_{x,\Delta}\left[\tau_C\right] \leq V(x, \Delta) + b < \infty$$

so we must have that $P_{x,\Delta}\left(\tau_C < \infty\right) = 1$ for all $(x, \Delta) \in \Lambda$. From here, it suffices to verify that all states return to Λ in finite time. This is automatic for initial $(x, \Delta) \in \Lambda$ by Corollary A.6. Now suppose that $(x, \Delta) \in \Lambda^C$. By nearly identical arguments to those of Lemma A.5, we can show that

$$P_{x,\Delta}\left(\tau_\Lambda \geq k+1\right) \leq P_{x,\Delta}\left(\sum_{t=0}^{k-1}\left(\left\|w_t\right\|_\infty + \tfrac{\Delta_{(N)}}{2}\right) > \tfrac{\Delta}{2} K\xi^k - \left\|x\right\|_\infty\right).$$

For k sufficiently large we have $\frac{\Delta}{2}K\xi^k > \left\|x\right\|_\infty$, so we may apply Proposition A.3 which yields that for k sufficiently large,

$$P_{x,\Delta}\left(\tau_\Lambda \geq k+1\right) \leq kT_{w_0}\left(\frac{\frac{\Delta}{2}K\xi^k - \left\|x\right\|_\infty}{k} - \frac{\Delta_{(N)}}{2}\right)$$

which converges to zero as k grows large ($T_{w_0}(u) = \mathcal{O}_u\left(u^{-2}\right)$, as in the proof of Corollary A.6). Therefore, we must have $P_{x,\Delta}\left(\tau_\Lambda = \infty\right) = 0$, which completes the proof. $\square$

Proof of Proposition 4.2.3. First, we note that if λ^* is the maximum eigenvalue of Q then the following holds for arbitrary $x \in \mathbb{R}^n$,

$$x^T Q x \leq \lambda^* \left\|x\right\|_2^2 \leq \lambda^* \left(\sqrt{n}\left\|x\right\|_\infty\right)^2 = n\lambda^* \left\|x\right\|_\infty^2 \tag{A.33}$$

where the first bound follows by properties of positive semidefinite matrices and the second follows by (1.4). Since $\varepsilon < \beta - 2$ it follows that $x^T Q x$ is bounded by $\|x\|_\infty^{\beta-\varepsilon}$ in the sense of Lemma 1.7.4.

We will show in the proof of Proposition A.8 that Condition 1.7.1 holds in the form required by Lemma 1.7.4 with the function $f(x, \Delta) = C\,\|x\|_\infty^{\beta-\varepsilon}$, for a constant $C > 0$. Since $x^T Q x$ is bounded by $\|x\|_\infty^{\beta-\varepsilon}$, the proof completes in view of the ergodicity result (1.9). □

To complete the proof program, we must show the moment condition of Lemma 4.2.5 holds. The rest of the appendix is dedicated to this proof. We first have some intermediate results.

Proposition A.7. *Under Scheme P(β, ε), for $(x, \Delta) \in \Lambda$ the return time τ_Λ satisfies the following independently of N.*

$$\sup_{(x,\Delta)\in\Lambda} E\left[\left(\rho^{\beta-\varepsilon}\right)^{\tau_\Lambda - 1}\right] < \infty$$

$$\lim_{\Delta\to\infty} E\left[\left(\rho^{\beta-\varepsilon}\right)^{\tau_\Lambda - 1}\right] = 1.$$

Proof. Note that by Condition 3.0.1 with $\beta > 2$ and Markov's inequality we have for $u > 0$ that

$$T_{w_0}(u) \le E\left[\|w_0\|_\infty^\beta\right] u^{-\beta}.$$

Let $r := \rho^{\beta-\varepsilon}$ for brevity. Then we have by Lemma A.5 that

$$\begin{aligned}
E_{x,\Delta}\left[r^{\tau_\Lambda - 1}\right] &= \sum_{k=0}^{\infty} P_{x,\Delta}\left(\tau_\Lambda = k+1\right) r^k \leq 1 + \sum_{k=1}^{\infty} P_{x,\Delta}\left(\tau_\Lambda \geq k+1\right) r^k \\
&\leq 1 + \sum_{k=1}^{\infty} k T_{w_0}\left(\frac{\Delta}{2}\left(\frac{h\xi^k - 1}{k} - \frac{\Delta_{(N)}}{\alpha L}\right)\right) r^k \\
&\leq 1 + \sum_{k=1}^{\infty} k r^k E\left[\|w_0\|_\infty^\beta\right]\left(\frac{\Delta}{2}\left(\frac{h\xi^k - 1}{k} - \frac{\Delta_{(N)}}{\alpha L}\right)\right)^{-\beta} \\
&= 1 + \Delta^{-\beta} \cdot \sum_{k=1}^{\infty} 2^\beta E\left[\|w_0\|_\infty^\beta\right] k^{\beta+1} r^k \left(h\xi^k - 1 - \frac{\Delta_{(N)}}{\alpha L} k\right)^{-\beta}.
\end{aligned}$$

Above, the series converges since the summand is $\mathcal{O}_k\left(k^{\beta+1}\left(r\xi^{-\beta}\right)^k\right)$ where $r\xi^{-\beta} < 1$ (this is ensured by the assumption that $\rho > (\|A\|_\infty)^{\frac{\beta}{\varepsilon}}$). Then, we have shown that

$$E_{x,\Delta}\left[r^{\tau_\Lambda - 1}\right] = 1 + \mathcal{O}_\Delta\left(\Delta^{-\beta}\right)$$

which (in light of the fact that $\Delta \geq \alpha L$) completes the proof. □

Proposition A.8. *Under Scheme* $P(\beta, \varepsilon)$*, the invariant state* $x_{*,N}$ *has finite* $(\beta-\varepsilon)$*-th moment uniformly in* $N \geq 2$*. That is,*

$$\sup_{N \geq 2} E\left[\|x_{*,N}\|_\infty^{\beta-\varepsilon}\right] < \infty.$$

Proposition A.9. *Under Scheme* $P(\beta, \varepsilon)$*, the invariant adaptive bin size* $\Delta_{*,N}$ *has finite* $(\beta - \varepsilon)$*-th moment uniformly in* $N \geq 2$*. That is,*

$$\sup_{N \geq 2} E\left[(\Delta_{*,N})^{\beta-\varepsilon}\right] < \infty.$$

We will prove these two results shortly via Lyapunov drift arguments, but using them we will first provide a short proof of Lemma 4.2.5.

Proof of Lemma 4.2.5. Let $(x_{*,N}, \Delta_{*,N}) \sim \pi_N$ and let $e_{*,N} = x_{*,N} - Q_K^{\Delta_{*,N}}(x_{*,N})$. It follows that $e_{*,N}$ is distributed as the invariant system adaptive error. Note that for any (x, Δ) we have the inequality $\left\|x - Q_K^{\Delta}(x)\right\|_\infty \leq \|x\|_\infty + \frac{\Delta}{2}$, where the extra $\frac{\Delta}{2}$ term accounts for the possibility that $\|x\|_\infty < \frac{\Delta}{2}$. Therefore, we have that

$$\|e_{*,N}\|_\infty = \left\|x_{*,N} - Q_K^{\Delta_{*,N}}(x_{*,N})\right\|_\infty \leq \|x_{*,N}\|_\infty + \frac{\Delta_{*,N}}{2}$$

and so,

$$\begin{aligned} E\left[\|e_{*,N}\|_\infty^{\beta-\varepsilon}\right] &\leq E\left[\left(\|x_{*,N}\|_\infty + \frac{\Delta_{*,N}}{2}\right)^{\beta-\varepsilon}\right] \\ &\leq 2^{\beta-\varepsilon} E\left[\|x_{*,N}\|_\infty^{\beta-\varepsilon}\right] + E\left[(\Delta_{*,N})^{\beta-\varepsilon}\right]. \end{aligned} \tag{A.34}$$

Above, the second bound follows from the general fact that for any $\theta > 0$ and non-negative random variables X, Y we have

$$E\left[(X+Y)^\theta\right] \leq 2^\theta \left(E\left[X^\theta\right] + E\left[Y^\theta\right]\right) \tag{A.35}$$

which follows by convexity/concavity of $u \mapsto u^\theta$ for $\theta \geq 1$ when $\theta \geq 1$ and $\theta < 1$, respectively.

Therefore we have by (A.34) that

$$\sup_{N\geq 2} E\left[\|e_{*,N}\|_\infty^{\beta-\varepsilon}\right] \leq 2^{\beta-\varepsilon} \sup_{N\geq 2} E\left[\|x_{*,N}\|_\infty^{\beta-\varepsilon}\right] + \sup_{N\geq 2} E\left[(\Delta_{*,N})^{\beta-\varepsilon}\right] < \infty$$

which is finite by Propositions A.8 and A.9. □

All that remains is to prove Propositions A.8 and A.9, which we do here via Lyapunov drift arguments.

Proof of Proposition A.8. Recall that we must show that

$$\sup_{N\geq 2} E\left[\|x_{*,N}\|_\infty^{\beta-\varepsilon}\right] < \infty.$$

We will do this via random-time Lyapunov drift arguments, in particular Lemma 1.7.4 using the drift condition (1.8). Let $r \coloneqq \rho^{\beta-\varepsilon}$ as in Proposition A.7. We set the following:

$$\begin{aligned}
&V(x,\Delta) = \Delta^{\beta-\varepsilon},\\
&d(x,\Delta) = s\Delta^{\beta-\varepsilon},\\
&f(x,\Delta) = cs\left(\tfrac{2}{K}\|x\|_\infty\right)^{\beta-\varepsilon},\\
&C = \Lambda \cap \left\{(x,\Delta)\in\mathbb{R}^n\times\Omega_\Delta : \Delta\leq D\right\},\\
&b = D^{\beta-\varepsilon}\left(\alpha^{\beta-\varepsilon}\sup_{x,\Delta\in\Lambda}E_{x,\Delta}\left[r^{\tau_\Lambda-1}\right]-1+s\right),
\end{aligned}$$

where $s\in(0,1-\alpha^{\beta-\varepsilon})$ is arbitrary, $D>0$ is such that

$$(x,\Delta)\in\Lambda \text{ and } \Delta > D \implies \alpha^{\beta-\varepsilon}E_{x,\Delta}\left[r^{\tau_\Lambda-1}\right]-1+s\leq 0$$

(such a D exists by Proposition A.7), b is finite by Proposition A.7, and $c>0$ is a sufficiently small constant (which will be specified shortly). We will show that the drift condition (1.8) holds for the choices above.

First, note that for $(x, \Delta) \in \Lambda$ we have that $\Delta_{\tau_\Lambda} = \rho^{\tau_\Lambda - 1}\alpha\Delta$, so

$$\begin{aligned}
&E_{x,\Delta}\left[V(x_{\tau_\Lambda}, \Delta_{\tau_\Lambda})\right] - V(x,\Delta) + d(x,\Delta) \\
&\qquad = E_{x,\Delta}\left[\left(\rho^{\tau_\Lambda - 1}\alpha\Delta\right)^{\beta-\varepsilon}\right] - \Delta^{\beta-\varepsilon} + s\Delta^{\beta-\varepsilon} \\
&\qquad = \Delta^{\beta-\varepsilon}\left(\alpha^{\beta-\varepsilon}E_{x,\Delta}\left[r^{\tau_\Lambda - 1}\right] - 1 + s\right). \qquad \text{(A.36)}
\end{aligned}$$

For $\Delta \leq D$, by construction of b we have that (A.36) is bounded by b. For $\Delta > D$, by construction of D we have that (A.36) is nonpositive. In either case, the first drift inequality is satisfied.

Next we show that the second inequality of (1.8) holds, when c is sufficiently small. To start, we have that for $(x, \Delta) \in \Lambda$ that

$$\begin{aligned}
E_{x,\Delta}\left[\sum_{t=0}^{\tau_\Lambda - 1} f(x_t, \Delta_t)\right] &= f(x,\Delta) + E_{x,\Delta}\left[\sum_{t=1}^{\tau_\Lambda - 1} f(x_t, \Delta_t)\right] \\
&= cs\left(\tfrac{2}{K}\|x\|_\infty\right)^{\beta-\varepsilon} + cs\left(\tfrac{2}{K}\right)^{\beta-\varepsilon} E_{x,\Delta}\left[\sum_{t=1}^{\tau_\Lambda - 1}\|x_t\|_\infty^{\beta-\varepsilon}\right] \\
&\leq cs\Delta^{\beta-\varepsilon} + cs\left(\tfrac{2}{K}\right)^{\beta-\varepsilon} E_{x,\Delta}\left[\sum_{t=1}^{\tau_\Lambda - 1}\|x_t\|_\infty^{\beta-\varepsilon}\right] \\
&= cd(x,\Delta) + cs\left(\tfrac{2}{K}\right)^{\beta-\varepsilon} E_{x,\Delta}\left[\sum_{t=1}^{\tau_\Lambda - 1}\|x_t\|_\infty^{\beta-\varepsilon}\right]. \qquad \text{(A.37)}
\end{aligned}$$

The remainder of the proof will be dedicated to showing that there exists a constant $M > 0$ so that uniformly over $(x, \Delta) \in \Lambda$ we have,

$$E_{x,\Delta}\left[\sum_{t=1}^{\tau_\Lambda - 1}\|x_t\|_\infty^{\beta-\varepsilon}\right] \leq M\Delta^{\beta-\varepsilon} \qquad \text{(A.38)}$$

so that we may bound (A.37) as

$$\text{(A.37)} \leq cd(x,\Delta) + cs\left(\tfrac{2}{K}\right)^{\beta-\varepsilon} M\Delta^{\beta-\varepsilon} = c\left(1 + \left(\tfrac{2}{K}\right)^{\beta-\varepsilon} M\right) d(x,\Delta).$$

It then suffices to take $c = (1 + (\frac{2}{K})^{\beta-\varepsilon}M)^{-1}$, and the second drift inequality holds. Therefore, we dedicate the remainder of this proof to establishing (A.38).

As in the proof of Lemma A.5, the "zoom-out" process y_t defined by $y_0 = e_0$, $y_{t+1} = AS(y_t) + w_t$ agrees with x_t for $1 \leq t \leq \tau_\Lambda$. Therefore, we have that

$$\begin{aligned} E_{x,\Delta}\left[\sum_{t=1}^{\tau_\Lambda - 1} \|x_t\|_\infty^{\beta-\varepsilon}\right] &= E_{x,\Delta}\left[\sum_{t=1}^{\tau_\Lambda - 1} \|y_t\|_\infty^{\beta-\varepsilon}\right] \\ &= E_{x,\Delta}\left[\sum_{t=1}^{\infty} \mathbb{1}_{\{\tau_\Lambda \geq t+1\}} \|y_t\|_\infty^{\beta-\varepsilon}\right] \\ &= \sum_{t=1}^{\infty} E_{x,\Delta}\left[\mathbb{1}_{\{\tau_\Lambda \geq t+1\}} \|y_t\|_\infty^{\beta-\varepsilon}\right] \end{aligned} \tag{A.39}$$

where the exchange of summation and expectation is justified by the monotone convergence theorem [40, Theorem 1.26].

Let $a \coloneqq \|A\|_\infty$ for brevity. Now, choose $q \in \left(\frac{\beta}{\varepsilon},\ \frac{\beta}{\beta-\varepsilon}\log_a \xi\right)$ arbitrarily (this is a valid interval by the condition that $\rho > a^{\frac{\beta}{\varepsilon}}$). We let p be the Hölder conjugate of q (i.e., $\frac{1}{p} + \frac{1}{q} = 1$) and apply Hölder's inequality to (A.39).

$$\text{(A.39)} \leq \sum_{t=1}^{\infty} P_{x,\Delta}\left(\tau_\Lambda \geq t+1\right)^{\frac{1}{q}} E_{x,\Delta}\left[\|y_t\|_\infty^{p(\beta-\varepsilon)}\right]^{\frac{1}{p}}. \tag{A.40}$$

We will consider each factor of the summand separately. First, by Lemma A.5 and familiar arguments we have

$$\begin{aligned}P_{x,\Delta}\left(\tau_\Lambda \geq t+1\right) &\leq tT_{w_0}\left(\frac{\Delta}{2}\left(\frac{h\xi^t-1}{t}-\frac{\Delta_{(N)}}{\alpha L}\right)\right)\\ &\leq tE\left[\left\|w_0\right\|_\infty^\beta\right]\left(\frac{\Delta}{2}\left(\frac{h\xi^t-1}{t}-\frac{\Delta_{(N)}}{\alpha L}\right)\right)^{-\beta}\\ &\leq E\left[\left\|w_0\right\|_\infty^\beta\right]2^\beta(\alpha L)^{-\beta}\cdot t^{\beta+1}\left(h\xi^t-1-\frac{\Delta_{(N)}}{\alpha L}t\right)^{-\beta}\end{aligned}$$

so that for series convergence,

$$P_{x,\Delta}\left(\tau_\Lambda \geq t+1\right)^{\frac{1}{q}} = \mathcal{O}_t\left(t^{\frac{\beta+1}{q}}\left(\xi^{-\frac{\beta}{q}}\right)^t\right) \tag{A.41}$$

and this term is $\mathcal{O}_\Delta(1)$. Next, we consider the second summand factor of (A.40). For brevity, let $m := p(\beta-\varepsilon)$. Since $\left\|Av\right\|_\infty \leq a\left\|v\right\|_\infty$ and $\left\|S(v)\right\|_\infty \leq \left\|v\right\|_\infty + \frac{\Delta_{(N)}}{2}$, we have for $t \geq 1$ that

$$\left\|y_t\right\|_\infty = \left\|AS(y_{t-1})+w_{t-1}\right\|_\infty \leq a\left\|y_{t-1}\right\|_\infty + \left\|w_{t-1}\right\|_\infty + a\cdot\frac{\Delta_{(N)}}{2}.$$

Repeating these arguments $t-1$ times yields that

$$\left\|y_t\right\|_\infty \leq a^t\left(\left\|y_0\right\|_\infty + \sum_{i=0}^{t-1}a^{-i}\left(\frac{\left\|w_i\right\|_\infty}{a}+\frac{\Delta_{(N)}}{2}\right)\right)$$

and since $a>1$ and $\left\|y_0\right\|_\infty = \left\|e_0\right\|_\infty \leq \frac{\Delta}{2}$ we find that

$$\left\|y_t\right\|_\infty \leq a^t\left(\frac{\Delta}{2}+\sum_{i=0}^{t-1}\left(\left\|w_i\right\|_\infty+\frac{\Delta_{(N)}}{2}\right)\right).$$

Therefore, we have by Corollary A.4 and repeated application of (A.35) that

$$
\begin{aligned}
E_{x,\Delta}\left[\|y_t\|_\infty^m\right] &\leq a^{mt} E_{x,\Delta}\left[\left(\frac{\Delta}{2}+\sum_{i=0}^{t-1}\left(\|w_i\|_\infty+\tfrac{\Delta_{(N)}}{2}\right)\right)^m\right] \\
&\leq a^{mt}\left(\Delta^m+2^m E\left[\left(\sum_{i=0}^{t-1}\left(\|w_i\|_\infty+\tfrac{\Delta_{(N)}}{2}\right)\right)^m\right]\right) \\
&\leq a^{mt}\left(\Delta^m+2^m t^{m+1} E\left[\left(\|w_0\|_\infty+\tfrac{\Delta_{(N)}}{2}\right)^m\right]\right) \\
&\leq a^{mt}\left(\Delta^m+t^{m+1}4^m\left(E\left[\|w_0\|_\infty^m\right]+\left(\tfrac{\Delta_{(N)}}{2}\right)^m\right)\right) \\
&\leq \Delta^m a^{mt}\left(1+t^{m+1}\left(\tfrac{4}{\alpha L}\right)^m\left(E\left[\|w_0\|_\infty^m\right]+\left(\tfrac{\Delta_{(N)}}{2}\right)^m\right)\right) \\
&= \Delta^m a^{mt}\mathcal{O}_t\left(t^{m+1}\right) \\
&= \Delta^m a^{mt}\mathcal{O}_t\left(t^{\beta+1}\right).
\end{aligned}
$$

where $q > \frac{\beta}{\varepsilon}$ implies that $\beta > m = p(\beta-\varepsilon)$. Note that the term $\mathcal{O}_t\left(t^{\beta+1}\right)$ is $\mathcal{O}_\Delta(1)$. Then finally we have,

$$
E_{x,\Delta}\left[\|y_t\|_\infty^{p(\beta-\varepsilon)}\right]^{\frac{1}{p}} \leq \Delta^{\beta-\varepsilon}\mathcal{O}_t\left(t^{\frac{\beta+1}{p}} a^{(\beta-\varepsilon)t}\right) \tag{A.42}
$$

which in combination with (A.41) yields that (A.40) is bounded by

$$
\text{(A.40)} \leq \Delta^{\beta-\varepsilon}\sum_{t=1}^{\infty}\mathcal{O}_t\left(t^{\beta+1}\left(a^{\beta-\varepsilon}\xi^{-\frac{\beta}{q}}\right)^t\right)
$$

where the series above converges to a constant $M > 0$ when $a^{\beta-\varepsilon}\xi^{-\frac{\beta}{q}} < 1$. This is ensured by the condition $q < \frac{\beta}{\beta-\varepsilon}\log_a \xi$, since then we have

$$
a^{\beta-\varepsilon}\xi^{-\frac{\beta}{q}} < a^{\beta-\varepsilon}\xi^{-\beta\left(\frac{\beta-\varepsilon}{\beta}\right)\frac{1}{\log_a \xi}} = a^{\beta-\varepsilon}\xi^{-(\beta-\varepsilon)\log_\xi a} = a^{\beta-\varepsilon}a^{-(\beta-\varepsilon)} = 1.
$$

This establishes (A.38) and completes the proof. □

Proof of Proposition A.9. Recall that we must show that

$$\sup_{N\geq 2} E\left[(\Delta_{*,N})^{\beta-\varepsilon}\right] < \infty.$$

We will do this again via random-time Lyapunov drift arguments. In particular, we will use most of the same Lyapunov parameters as in the previous proof. To be precise, with $r \coloneqq \rho^{\beta-\varepsilon}$ we once again set:

$$\begin{aligned}
&V(x,\Delta) = \Delta^{\beta-\varepsilon},\\
&d(x,\Delta) = s\Delta^{\beta-\varepsilon},\\
&C = \Lambda \cap \{(x,\Delta) \in \mathbb{R}^n \times \Omega_\Delta : \Delta \leq D\},\\
&b = D^{\beta-\varepsilon}\left(\alpha^{\beta-\varepsilon} \sup_{x,\Delta\in\Lambda} E_{x,\Delta}\left[r^{\tau_\Lambda - 1}\right] - 1 + s\right),
\end{aligned}$$

where $s \in (0, 1-\alpha^{\beta-\varepsilon})$ is arbitrary, $D > 0$ is such that

$$(x,\Delta) \in \Lambda \text{ and } \Delta > D \implies \alpha^{\beta-\varepsilon} E_{x,\Delta}\left[r^{\tau_\Lambda - 1}\right] - 1 + s \leq 0$$

(such a D exists by Proposition A.7), and b is finite by Proposition A.7.

Instead of f proportional to $\|x\|_\infty^{\beta-\varepsilon}$, we set f proportional to $\Delta^{\beta-\varepsilon}$. That is, we set:

$$f(x,\Delta) = cs\Delta^{\beta-\varepsilon}$$

where

$$c = \left(1 + \frac{\alpha^{\beta-\varepsilon}}{r-1}\left(\sup_{(x,\Delta)\in\Lambda} E\left[r^{\tau_\Lambda - 1}\right] - 1\right)\right)^{-1}$$

is well-defined by Proposition A.7. We will show that these Lyapunov parameters satisfy the drift condition (1.8) and invoke Lemma 1.7.4 to complete the proof.

To begin, note that the first drift inequality of (1.8) does not involve f, and with otherwise identical Lyapunov parameters we showed in the proof of Proposition A.8 that this drift inequality holds. Therefore, what remains is to show only that the second inequality in (1.8) holds. Explicitly, we must show that

$$csE_{x,\Delta}\left[\sum_{t=0}^{\tau_\Lambda-1}\Delta_t^{\beta-\varepsilon}\right]\leq s\Delta^{\beta-\varepsilon}$$

which is equivalent to,

$$E_{x,\Delta}\left[\sum_{t=0}^{\tau_\Lambda-1}\left(\frac{\Delta_t}{\Delta}\right)^{\beta-\varepsilon}\right]\leq c^{-1}=1+\frac{\alpha^{\beta-\varepsilon}}{r-1}\left(\sup_{(x,\Delta)\in\Lambda}E\left[r^{\tau_\Lambda-1}\right]-1\right). \tag{A.43}$$

We establish this now. Note that $\Delta_0=\Delta$ and for $1\leq t\leq\tau_\Lambda$ we have $\Delta_t=\alpha\Delta\rho^{t-1}$ so,

$$\begin{aligned}
E_{x,\Delta}\left[\sum_{t=0}^{\tau_\Lambda-1}\left(\frac{\Delta_t}{\Delta}\right)^{\beta-\varepsilon}\right]&=1+\alpha^{\beta-\varepsilon}E_{x,\Delta}\left[\sum_{t=1}^{\tau_\Lambda-1}\left(\rho^{t-1}\right)^{\beta-\varepsilon}\right]\\
&=1+\alpha^{\beta-\varepsilon}E_{x,\Delta}\left[\sum_{t=1}^{\tau_\Lambda-1}r^{t-1}\right]\\
&=1+\alpha^{\beta-\varepsilon}\cdot\frac{E_{x,\Delta}\left[r^{\tau_\Lambda-1}\right]-1}{r-1}\\
&=1+\frac{\alpha^{\beta-\varepsilon}}{r-1}\left(E_{x,\Delta}\left[r^{\tau_\Lambda}\right]-1\right)\leq c^{-1}
\end{aligned}$$

by construction of c. Therefore the drift condition (1.8) is satisfied for choice of b and f independent of N, which by Lemma 1.7.4 completes the proof. □

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