Randomized Quantization and Optimal Design with a Marginal Constraint

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***Abstract*—We consider the problem of optimal randomized vector quantization under a constraint on the output's distribution. The problem is formalized by introducing a general representation of randomized quantization via probability measures over the space of joint distributions on the source and reproduction alphabets. Using this representation and results from optimal transport theory, we show the existence of an optimal (minimum distortion) randomized quantizer having a fixed output distribution under various conditions. For sources with densities and the mean square distortion measure, we show that this optimum can be attained by randomizing quantizers having convex code cells. We also consider a relaxed version of the problem where the output marginal must belong to some neighborhood (in the weak topology) of a fixed probability measure. We demonstrate that finitely randomized quantizers form an optimal class for the relaxed problem.**

I. Introduction

A quantizer maps each value of a source (input) alphabet into one in a finite collection of points (output levels) from a reproduction alphabet. A quantizer's performance is usually characterized by its rate, defined as the logarithm of the number of output levels, and its expected distortion when the input is a random variable. Although the introduction of randomization in the quantization procedure does not affect the optimal rate-distortion tradeoff, other measures of quantizer performance may be improved by using randomized quantizers, where the input is encoded by a quantizer randomly chosen from a given family of quantizers.

In what appears to be the first work on randomized quantization, Roberts [1] found that adding random noise to an image before quantization and subtracting the noise before reconstruction may result in a perceptually more pleasing image. Schuchman [2] and Gray and Stockham [3] analyzed versions of such so called *dithered* scalar quantizers where random noise (dither) is added to the input signal prior to uniform quantization. If the dither is subtracted after the quantization operation, the procedure is called subtractive dithering; otherwise it is called non-subtractive dithering. Under certain conditions, dithering results in uniformly distributed quantization noise that is independent of the input [2], [3], which allows a simple modeling of the quantization process by an additive noise channel. In the information theoretic literature the properties of entropy coded dithered lattice quantizers have been extensively studied. For example, such quantizers have been used to provide achievable bounds on the performance of universal lossy compression systems by Ziv [4] and Zamir and Feder [5], [6]. Recently Akyol and Rose [7] introduced a class of randomized *nonuniform* scalar quantizers obtained via applying companding to a dithered uniform quantizer and investigated optimality conditions for the design of such quantizers.

Dithered uniform/lattice and companding quantizers pick a random quantizer from a "small" structured subset of all possible quantizers. Such special randomized quantizers may be suboptimal for certain tasks and one would like to be able to work with more general (or completely general) classes of randomized quantizers. For example, Li *et al*. [8] considered *distribution preserving* dithered quantization to improve the perceptual quality of mean square optimal quantizers in audio and video coding. In this model, the output of the (random) scalar quantizer is restricted to have the same distribution as the source. One can slightly generalize this problem by requiring that the output of the quantizer have a prescribed distribution which is not necessarily equal to the input distribution. Then the goal is to find a randomized quantizer with a given rate minimizing the distortion under this distribution constraint. Clearly, the class of dithered quantizers may be strictly suboptimal in this problem.

In this paper we propose a general representation of randomized vector quantization which formalizes the notion of randomly picking a quantizer from the set of *all* M-level vector quantizers. This model is then used to rigorously formulate the minimum-distortion quantization problem under an output distribution constraint and to prove the existence of an optimal solution.

The paper is organized as follows. In Section II we introduce a representation of general randomized quantization by a certain set of probability measures on the space of all probability measures of the Borel subsets of $\mathbb{R}^n \times \mathbb{R}^n$. The equivalence of this model to other models more common in the information theoretic literature is also demonstrated. In Section III the randomized quantization problem with a fixed output distribution constraint is formulated and the existence of an optimal solution is shown using optimal transport theory. For the special but important special case of sources with densities and the mean square distortion measure, we show that this optimum can be attained by randomizing quantizers

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having convex code cells. In Section IV a relaxed version of the problem is considered where the output distribution belongs to some neighborhood (in the weak topology) of a fixed probability measure and the optimality of finitely randomized quantizers is demonstrated.

Since most of the proofs are quite technical, the results will be presented with only proof sketches or without proofs. Detailed proofs will be given in the full paper version of this work.

II. Defining Randomized Quantization

In this paper X denotes the input alphabet and Y is the reconstruction alphabet. Throughout the paper we set $\mathsf{X} = \mathsf{Y} = \mathbb{R}^n$ for some $n \geq 1$, although most of the results hold in more general settings. If M is a positive integer, an M-level quantizer from X to Y is a Borel measurable function[1] $q : \mathsf{X} \to \mathsf{Y}$ such that its range $q(\mathsf{Y})$ contains *at most* M points of Y. If $\mathcal{Q}_M$ denotes the set of all M-level quantizers, note that our definition implies $\mathcal{Q}_M \subset \mathcal{Q}_{M+1}$ for all M.

Model 1

One general model of rate $R = \log M$ randomized quantization (lossy coding), often used in the information theoretic literature, is depicted in Fig. 1.

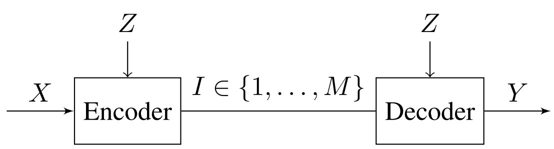

Figure 1. General randomized quantizer

Here X and Y are the input and output random variables taking values in X and Y, respectively. The index I takes values in $\{1, \ldots, M\}$, and Z is a $\mathsf{Z} = \mathbb{R}^m$-valued random variable which is independent of X and which is assumed to be available at both the encoder and the decoder. The encoder is a measurable function from $\mathsf{X} \times \mathsf{Z}$ into $\{1, \ldots, M\}$ which maps (X, Z) to I, and the decoder is a measurable mapping from $\{1, \ldots, M\} \times \mathsf{Z}$ into Y which maps (I, Z) to Y.

Model 2

Model 1 can be collapsed into a more tractable equivalent model. In this model a randomized quantizer is a pair (q, ν), where $\mathsf{q} : \mathsf{X} \times \mathsf{Z} \to \mathsf{Y}$ is a measurable mapping such that $\mathsf{q}(\cdot, z)$ is an M-level quantizer for all $z \in \mathsf{Z}$ and ν is a probability measure on Z (recall that $\mathsf{X} = \mathsf{Y} = \mathbb{R}^n$ and $\mathsf{Z} = \mathbb{R}^m$). In this model ν is the distribution of the randomizing random variable Z, and q is the composition of the encoder and the decoder in Model 1.

Let $\rho : \mathsf{X} \times \mathsf{Y} \to \mathbb{R}$ be a nonnegative measurable function, called the distortion measure. From now on we assume that the source X has distribution μ (denoted as $X \sim \mu$). The cost (or distortion) associated with (q, ν) is the expectation

$$L(\mathsf{q}, \nu) = \int_{\mathsf{Z}} \int_{\mathsf{X}} \rho(x, \mathsf{q}(x, z))\, \mu(dx) \nu(dz) = E\big[\rho(X, \mathsf{q}(X, Z))\big] \tag{1}$$

where $Z \sim \nu$ is independent of X.

The above model includes, as special cases, subtractive and non-subtractive dithering of uniform M-level quantizers, as well as the dithering of non-uniform quantizers considered in [8] and [7].

Model 3

This model is motivated by the generalization of the problem of distribution preserving quantization posed in [8]: Find a randomized quantizer that minimizes the distortion (1) under the constraint that the reconstruction $Y = \mathsf{q}(X, Z)$ has a prescribed distribution (say $Y \sim \psi$). In Model 2 the design parameters are q and ν, and it seems quite difficult to identify the set of randomized quantizers as a tractable subspace of the space of all measurable functions from $\mathsf{X} \times \mathsf{Z}$ to Y. For this reason, instead of considering quantizers as functions that map X into a finite subset of Y, first we represent them as the probability measures on $\mathsf{X} \times \mathsf{Y}$ (see, e.g, [9]). This leads to an alternative representation of randomized quantizers which enables us to easily identify the set which satisfies the constraints on the output distribution.

A quantizer from q into Y can be represented as stochastic kernel (regular conditional probability [10]) Q on Y given X by letting $Q(dy|x) = \delta_{q(x)}(dy)$, where δ_u denotes the point mass at u: $\delta_u(A) = 1$ if $u \in A$ and $\delta_u(A) = 0$ if $u \notin A$ for any Borel set $A \subset \mathsf{X}$. If $X \sim \mu$, then we can also represent q by a probability measure v on $\mathsf{X} \times \mathsf{Y}$ defined by $v(dx, dy) = \mu(dx)\delta_{q(x)}(dy)$. Let $\mathcal{P}(\mathsf{X} \times \mathsf{Y})$ denote the set of all probability measures on $\mathsf{X} \times \mathsf{Y}$ and recall that $\mathcal{Q}_M$ denotes the collection of all M-level quantizers from X to Y. Then we can identify $\mathcal{Q}_M$ with the following subset of $\mathcal{P}(\mathsf{X} \times \mathsf{Y})$:

$$\Gamma_\mu(M) = \big\{v \in \mathcal{P}(\mathsf{X} \times \mathsf{Y}) : v(dx, dy) = \mu(dx)\delta_{q(x)}(dy), \; q \in \mathcal{Q}_M\big\}. \tag{2}$$

Note that the mapping $q \mapsto \mu(dx)\delta_{q(x)}(dy)$ maps $\mathcal{Q}_M$ onto $\Gamma_\mu(M)$, but this mapping is one-to-one only if we consider equivalence classes of quantizers in $\mathcal{Q}_M$ that are equal μ almost everywhere (a.e.).

We equip $\mathcal{P}(\mathsf{X} \times \mathsf{Y})$ with the topology of weak convergence (weak topology) which is metrizable with the Prokhorov metric. Since $\mathsf{X} \times \mathsf{Y} = \mathbb{R}^n \times \mathbb{R}^n$ is a Polish (complete and separable metric) space, this makes $\mathcal{P}(\mathsf{X} \times \mathsf{Y})$ into a Polish space [11]. Then the following can be proved:

Lemma 1. *$\Gamma_\mu(M)$ is a Borel subset of $\mathcal{P}(\mathsf{X} \times \mathsf{Y})$.*

Now we are ready to introduce Model 3 for randomized quantization. Let P be a probability measure on $\mathcal{P}(\mathsf{X} \times \mathsf{Y})$ which is supported on $\Gamma_\mu(M)$, i.e., $P(\Gamma_\mu(M)) = 1$. Then P

[1] Unless otherwise specified, "measurable" will refer to Borel measurability. We equip $\mathbb{R}^k$, or more generally any metric space, with its Borel σ-algebra and all probability measures on such spaces will be Borel measures. The product of metric spaces will be equipped will the product Borel σ-algebra.

induces a "randomized quantizer" $v_P \in \mathcal{P}(\mathsf{X}\times\mathsf{Y})$ via

$$v_P(A\times B) = \int_{\Gamma_\mu(M)} v(A\times B)\, P(dv)$$

for Borel sets $A\subset\mathsf{X}$ and $B\subset\mathsf{Y}$, which we abbreviate to

$$v_P = \int_{\Gamma_\mu(M)} v\, P(dv). \tag{3}$$

Since each v in $\Gamma_\mu(M)$ corresponds to a quantizer with input distribution μ, P can be thought as a probability measure on the set of all M-level quantizers $\mathcal{Q}_M$.

Let $\mathcal{P}_0(\Gamma_\mu(M))$ denote the set of probability measures on $\mathcal{P}(\mathsf{X}\times\mathsf{Y})$ supported on $\Gamma_\mu(M)$. We define the set of M-level Model 3 randomized quantizers as

$$\Gamma_\mu^R(M) = \Big\{ v_P \in \mathcal{P}(\mathsf{X}\times\mathsf{Y}) : v_P = \int_{\Gamma_\mu(M)} vP(dv), \; P\in\mathcal{P}_0(\Gamma_\mu(M))\Big\} \tag{4}$$

Equivalence of models

The following theorem states that Models 2 and 3 of randomized quantization are essentially equivalent. The equivalence (in the sense of the theorem) of Models 1 and 2 is immediate.

Theorem 1. *Let $X\sim\mu$. For each Model 2 randomized quantizer (q,ν) there exists a Model 3 randomized quantizer $v_P\in\Gamma_\mu^R(M)$ such that $(X,Y)=(X,\mathsf{q}(X,Z))$ has distribution v_P. Conversely, for any $v_P\in\Gamma_\mu^R(M)$ there exists a Model 2 randomized quantizer (q,ν) such that $(X,Y)\sim v_P$.*

Sketch of proof. For any (q,ν) define $f:\mathbb{R}^m\to\Gamma_\mu(M)$ by $f(z)=\delta_{\mathsf{q}(x,z)}(dy)\mu(dx)$. It can be shown that the mapping f is measurable and thus we can define the probability measure P supported on $\Gamma_\mu(M)$ by $P=\nu\circ f^{-1}$ (i.e., $P(B)=\nu(f^{-1}(B))$ for any Borel set $B\subset\Gamma_\mu(M)$). It is straightforward to show that for the corresponding v_p we have $(X,Y)\sim v_P$.

Conversely, let v_P be defined as in (3) with P supported on $\Gamma_\mu(M)$. Define the mapping $\Gamma_\mu(M)\ni v\mapsto q_v$ where q_v is the μ-a.e. defined quantizer giving $v(dx,dy)=\mu(dx)\delta_{q_v(x)}(dy)$. Since $\Gamma_\mu(M)$ is an uncountable Borel space, there is a measurable bijection (Borel isomorphism) $g:\mathbb{R}^m\to\Gamma_\mu(M)$ between $\mathbb{R}^m$ and $\Gamma_\mu(M)$ [10]. Now define q by $\mathsf{q}(x,z)=q_{g(z)}$ and let $\nu=P\circ g$. Then for all z, $\mathsf{q}(\cdot,z)$ is a μ-a.e. defined M-level quantizer. It can be proved that the domain of q can enlarged so that $\mathsf{q}(\cdot,z)$ is an M-level quantizer for all z and q is measurable. It is straightforward to check that if $Z\sim\nu$ is independent of X and $Y=\mathsf{q}(X,Z)$, then $(X,Y)\sim v_P$. □

Note that the randomized quantizer (q,ν) and that represented by v_P in Theorem 1 have the same distortion:

$$\begin{aligned} L(\mathsf{q},\nu) &= E\big[\rho(X,\mathsf{q}(X,Z))\big] \\ &= \int_{\Gamma_\mu(M)}\int_{\mathsf{X}\times Y}\rho(x,y)v(dx,dy)P(dv) \\ &= \int_{\mathsf{X}\times Y}\rho(x,y)v_P(dx,dy) := L(v_P)\end{aligned}$$

In the rest of paper Model 3 will be used to represent randomized quantizers in treating the optimal randomized quantization problem under an output distribution constraint.

Remark 1. (a) Since the dimension m of the randomizing random vector Z was arbitrary, we can take $m=1$ in Theorem 1. In fact, any Model 2 or 3 randomized quantizer is equivalent (in the sense of Theorem 1) to some (q,ν), where $\mathsf{q}:\mathsf{X}\times[0,1]\to\mathsf{Y}$ and ν is the uniform distribution on $[0,1]$.

(b) Assume that $(\mathsf{Z},\mathcal{A},\nu)$ is an *arbitrary* probability space. It can also be proved that any randomized quantizer $\mathsf{q}:\mathsf{X}\times\mathsf{Z}\to\mathsf{Y}$ in the form $q(X,Z)$, where $Z\sim\nu$ is independent of X, is still equivalent to a Model 3 quantizer. In view of the previous remark and Theorem 1, this means that uniform randomization over the unit interval $[0,1]$ suffices under the most general circumstances.

(c) All results in this section remain valid if the input and reproduction alphabets X and Y are arbitrary uncountable Polish spaces. In this case, uniform randomization over the interval still provides the most general model possible.

III. Optimal Randomized Quantization with Fixed Output Marginal

Let ψ be a probability measure on Y and let $\Lambda(M,\psi)$ denote the set of all M-level Model 2 randomized quantizers (q,ν) such that the output $\mathsf{q}(X,Z)$ has distribution ψ. As before, we assume that $X\sim\mu$, $Z\sim\nu$, and Z and X are independent. We want to show the existence of a minimum distortion randomized quantizer having output distribution ψ, i.e, the existence of $(\mathsf{q}^*,\nu^*)\in\Lambda(M,\psi)$ such that

$$L(\mathsf{q}^*,\nu^*) = \inf_{(\mathsf{q},\nu)\in\Lambda(M,\psi)} L(\mathsf{q},\nu).$$

The set of M-level randomized quantizers is a fairly general (nonparametric) set of functions and it seems difficult to investigate the existence of an optimum directly. On the other hand, Model 3 provides a tractable framework for establishing the existence of an optimal randomized quantizer under quite general conditions.

Let $\Gamma_{\mu,\psi}$ be the set of all joint distributions $v\in P(\mathsf{X}\times\mathsf{Y})$ having X marginal μ and Y marginal ψ. Then

$$\Gamma_{\mu,\psi}^R(M) = \Gamma_\mu^R(M)\cap\Gamma_{\mu,\psi}$$

is the subset of Model 3 randomized quantizers which corresponds to class of output distribution constrained Model 2 randomized quantizers $\Lambda(M,\psi)$.

For any $v\in\mathcal{P}(\mathsf{X}\times\mathsf{Y})$ let

$$L(v) = \int_{\mathsf{X}\times\mathsf{Y}}\rho(x,y)v(dx,dy).$$

Using these definitions, finding optimal randomized quantizers with a given output distribution can be posed as finding v in $\Gamma_{\mu,\psi}^R(M)$ which minimizes $L(v)$, i.e.

(P1) minimize $L(v)$

subject to $v\in\Gamma_{\mu,\psi}^R(M)$.

We can prove the existence of the minimizer for **(P1)** under the following assumptions. Here $\|x\|$ denotes the Euclidean norm on $\mathbb{R}^n$.

ASSUMPTION 1: $\rho(x,y)$ is continuous and $\psi(B)=1$ for some compact subset B of Y.

ASSUMPTION 2: $\rho(x,y)=\|x-y\|^2$.

Theorem 2. *Suppose* $\inf_{v\in\Gamma^R_{\mu,\psi}(M)} L(v) < \infty$. *There exists a minimizer with finite cost for problem* **(P1)** *under either Assumption 1 or Assumption 2.*

Remark 2. Note that the product distribution $\mu\otimes\psi$ corresponds to a 1-level randomized quantizer and thus $\mu\otimes\psi\in\Gamma^R_{\mu,\psi}(M)$ for all $M\geq 1$. Hence if $L(\mu\otimes\psi)<\infty$, then the condition $\inf_{v\in\Gamma^R_{\mu,\psi}(M)} L(v)<\infty$ holds. In particular, if under Assumption 2 both μ and ψ have finite second moments, then $\inf_{v\in\Gamma^R_{\mu,\psi}(M)} L(v)<\infty$.

Sketch of proof. We only outline the proof under Assumption 1; the proof under Assumption 2 then follows by a one-point compactification argument. It follows from the continuity of ρ that L is a lower semicontinuous on $\mathcal{P}(\mathsf{X}\times\mathsf{Y})$ for the weak topology (see, e.g., [12]). Hence, to show the existence of a minimizer for problem **(P1)** it would suffice to prove that $\Gamma^R_{\mu,\psi}(M)=\Gamma^R_{\mu}(M)\cap\Gamma_{\mu,\psi}$ is compact. It is known that $\Gamma_{\mu,\psi}$ is compact [12, Chapter 4], but unfortunately $\Gamma_\mu(M)$ is not closed [13] and it seems doubtful that $\Gamma^R_\mu(M)$ is compact. Hence, we will develop a different argument which is based on optimal transport theory.

Recall that $\psi(B)=1$ for a compact set B in Y. Let $\mathcal{P}_M(B)=\{\psi_0\in\mathcal{P}(\mathsf{Y}):\psi_0(B)=1, |\operatorname{supp}(\psi_0)|\leq M\}$ be the set of discrete distributions on Y supported on B and having at most M atoms. Define

$$\hat{\Gamma}_\mu(M)=\big\{\hat{v}\in\mathcal{P}(\mathsf{X}\times\mathsf{Y}):\hat{v}=\operatorname*{arg\,min}_{v\in\Gamma_{\mu,\psi_0}} L(v), \psi_0\in\mathcal{P}_M(B)\big\}.$$

The elements of $\hat{\Gamma}_\mu(M)$ are the probability measures which solve the optimal mass transport problem (see, e.g., [12]) for fixed input marginal μ and some output marginal ψ_0 in $\mathcal{P}_M(B)$. One can show that $\hat{\Gamma}_\mu(M)$ is a Borel set. Let $\hat{\Gamma}^R_\mu(M)$ be the randomization of $\hat{\Gamma}_\mu(M)$, obtained by replacing $\Gamma_\mu(M)$ with $\hat{\Gamma}_\mu(M)$ in (4). Define the optimization problem **(P2)** as

$$\textbf{(P2)}\ \text{minimize } L(v) \quad \text{subject to } v\in\hat{\Gamma}^R_{\mu,\psi}(M)$$

where $\hat{\Gamma}^R_{\mu,\psi}(M)=\hat{\Gamma}^R_\mu(M)\cap\Gamma_{\mu,\psi}$.

Clearly, the distortion of any minimizer in **(P2)** is less than or equal to the distortion of a minimizer in **(P1)**. Furthermore, we can prove that if **(P2)** has a minimizer $\hat{v}$, then we can find $v\in\Gamma^R_{\mu,\psi}(M)$ such that $L(v)=L(\hat{v})$, implying that v is a minimizer for **(P1)**. Hence, to prove the existence of a minimizer for **(P1)** it is enough prove the existence of a minimizer for **(P2)**.

Using a slight modification of [12, Corollary 5.21] we can show that $\hat{\Gamma}^{\mathrm{f}}_\mu(M):=\{v\in\mathcal{P}(\mathsf{X}\times\mathsf{Y}):L(v)<\infty\}\cap\hat{\Gamma}_\mu(M)$ is compact. Thus it follows that $\mathcal{P}(\hat{\Gamma}^{\mathrm{f}}_\mu(M))$ is also compact. It is clear that the randomization can be restricted to $\hat{\Gamma}^{\mathrm{f}}_\mu(M)$ when defining $\hat{\Gamma}^R_\mu(M)$ for **(P2)**. Let $\hat{\Gamma}^{\mathrm{f},R}_\mu(M)$ be the randomization of $\hat{\Gamma}^{\mathrm{f}}_\mu(M)$ obtained by replacing $\Gamma_\mu(M)$ with $\hat{\Gamma}^{\mathrm{f}}_\mu(M)$ in (4). One can show that the mapping $\mathcal{P}(\hat{\Gamma}^{\mathrm{f}}_\mu(M))\ni P\mapsto v_P\in\hat{\Gamma}^{\mathrm{f},R}_\mu(M)$ is continuous. Thus $\hat{\Gamma}^{\mathrm{f},R}_\mu(M)$ is the continuous image of a compact set, and as such it is also compact. This, together with the compactness of $\Gamma_{\mu,\psi}$ and the lower semicontinuity of L, implies the existence of the minimizer for **(P2)** □

Optimal transport theory can also be used to show that, under some regularity conditions on the input distribution and the distortion measure, the randomization can be restricted to quantizers having a certain structure. In what follows we consider sources with densities and the mean square distortion. An M-level quantizer q with range (code points) $q(\mathsf{X})=\{y_1,\ldots,y_k\}\subset\mathsf{Y}$ ($k\leq M$) is said to have *convex code cells* if $q^{-1}(y_i)=\{x:q(x)=y_i\}$ is a convex subset of $\mathsf{X}=\mathbb{R}^n$ for all $i=1,\ldots,k$. Let $\mathcal{Q}_{M,C}$ denote the set of all M-level quantizers having convex code cells.

Theorem 3. *Suppose* $\rho(x,y)=\|x-y\|^2$ *and* μ *admits a probability density function. Then an optimal randomized quantizer in Theorem 2 can be obtained by randomizing over quantizers with convex cells. That is*

$$\min_{v\in\Gamma^R_{\mu,\psi}(M)} L(v)=\min_{v\in\Gamma^{R,C}_{\mu,\psi}(M)} L(v)$$

where $\Gamma^{R,C}_{\mu,\psi}(M)$ *represents the Model 3 quantizers with output distribution* ψ *that are obtained by replacing* $\mathcal{Q}_M$ *with* $\mathcal{Q}_{M,C}$ *in* (2).

Sketch of proof. From the proof of Theorem 2 recall the set $\hat{\Gamma}_\mu(M)$ of probability measures which solve the optimal mass transport problem for fixed input marginal μ and some output marginal ψ_0 in $\mathcal{P}_M(B)$, where we now take $B=\mathsf{Y}$. It is known that if μ admits a density and $\rho(x,y)=\|x-y\|^2$, then each $v\in\hat{\Gamma}_\mu(M)$ is in the form $v(dx,dy)=\mu(dx)\delta_{q(x)}(dy)$ for some $q\in\mathcal{Q}_{M,C}$ (see, e.g. [14]). Thus in this case $\hat{\Gamma}_\mu(M)\subset\Gamma_\mu(M)$, which implies that $\hat{\Gamma}^R_{\mu,\psi}(M)\subset\Gamma^{R,C}_{\mu,\psi}(M)$. Since $\Gamma^{R,C}_{\mu,\psi}(M)\subset\Gamma^R_{\mu,\psi}(M)$, any solution to **(P2)** is a solution to **(P1)**, which in addition, is obtained by randomizing over the set of quantizers having convex code cells represented by $\hat{\Gamma}_\mu(M)$. □

Remark 3. It is easy to show that under the conditions of Theorem 3, **(P1)** is solved by some nonrandomized quantizer $v\in\Gamma_\mu(M)$ if and only if $|\operatorname{supp}(\psi)|\leq M$. (Of course, in this case v also has convex cells.)

IV. APPROXIMATION WITH FINITE RANDOMIZATION

Since randomized quantizers require common randomness that must be shared between the encoder and the decoder, it is of interest to see how one can approximate the optimal cost by using only finite randomization (i.e., a P that is supported on a finite subset of $\Gamma_\mu(M)$). Clearly, if the target probability measure ψ on Y is not finitely supported, then no optimal finite randomization exists. In the next section we relax the fixed output distribution constraint and consider the problem where the output distribution belongs to some neighborhood (in the weak topology) of ψ. We show that one can always find a finitely randomized quantizer which is optimal (resp., ε-optimal) for this relaxed problem if the distortion measure is continuous and bounded (resp., arbitrary).

Let $B(\psi, \delta)$ denote the open ball in $\mathcal{P}(\mathsf{Y})$ (with respect to the Prokhorov metric) having radius $\delta > 0$ and centered at the target input distribution ψ. Also, let $M_{\mu,\psi}^{\delta}$ denote the set of all $v \in \Gamma_{\mu}^{R}(M)$ whose Y marginal belongs to $B(\psi, \delta)$. That is, $M_{\mu,\psi}^{\delta}$ represents all randomized quantizers in $\Gamma_{\mu}^{R}(M)$ whose output distribution is within distance δ of the target distribution ψ. We consider the following relaxed version of the minimization problem **(P1)**:

$$\textbf{(P3)} \text{ minimize } L(v) \quad \text{subject to } v \in M_{\mu,\psi}^{\delta}.$$

The set of *finitely randomized* quantizers in $\Gamma_{\mu}^{R}(M)$ is obtained by taking finite mixtures of quantizers in $\Gamma_{\mu}(M)$, i.e.,

$$\Gamma_{\mu}^{FR}(M) = \Big\{ v_P \in \Gamma_{\mu}^{R}(M) : v_P = \int_{\Gamma_{\mu}(M)} v P(dv), \; |\operatorname{supp}(P)| < \infty \Big\}$$

Proposition 1. *Let $v \in M_{\mu,\psi}^{\delta}$ and assume the distortion measure ρ is continuous and bounded. Then we can find v_F in $M_{\mu,\psi}^{\delta} \cap \Gamma_{\mu}^{FR}(M)$ such that $L(v_F) \le L(v)$.*

Sketch of proof. First we treat the case $L(v) > \inf_{v' \in \Gamma_{\mu}(M)} L(v')$. If ρ is continuous and bounded, then L is continuous. Hence, the set $\{v' \in \Gamma_{\mu}^{R}(M) : L(v') < L(v)\}$ is relatively open in $\Gamma_{\mu}^{R}(M)$ and its intersection with $M_{\mu,\psi}^{\delta}$ can be shown to be *nonempty* and relatively open in $\Gamma_{\mu}^{R}(M) \cap M_{\mu,\psi}^{\delta}$. We can show that $\Gamma_{\mu}^{FR}(M)$ is dense in $\Gamma_{\mu}^{R}(M)$. The proposition follows from these facts.

The case $L(v) = \inf_{v' \in \Gamma_{\mu}(M)} L(v') := L^*$ is handled similarly; here one can show that the set $\Gamma_{\mu,opt}^{FR}(M)$ obtained by the finite randomization of optimal quantizers (i.e., quantizers $v' \in \Gamma_{\mu}(M)$ such that $L(v') = L^*$) is dense in the set $\Gamma_{\mu,opt}^{R}(M)$ of (arbitrarily) randomized optimal quantizers (note that $v \in \Gamma_{\mu,opt}^{R}(M)$). □

Although the minimum in **(P3)** may not be achieved by any $v \in M_{\mu,\psi}^{\delta}$, the proposition implies that if the problem has a solution, it also has a solution in the set of finitely randomized quantizers.

Theorem 4. *Assume ρ is continuous and bounded and suppose there exists $v^* \in M_{\mu,\psi}^{\delta}$ with $L(v^*) = \inf_{v \in M_{\mu,\psi}^{\delta}} L(v)$. Then there exists $v_F \in M_{\mu,\psi}^{\delta} \cap \Gamma_{\mu}^{FR}(M)$ such that $L(v_F) = L(v^*)$.*

The continuity of L, implied by the boundedness of ρ is crucial in the proof of Theorem 4. However, for an arbitrary ρ we can still show that for any $\varepsilon > 0$ and $v \in M_{\mu,\psi}^{\delta}$ there exists v_F in $M_{\mu,\psi}^{\delta} \cap \Gamma_{\mu}^{FR}(M)$ such that $L(v_F) \le L(v) + \varepsilon$. That is, for any $\varepsilon > 0$ there exists an ε-optimal finitely randomized quantizer for **(P3)**.

Theorem 5. *Let ρ be an arbitrary distortion measure and assume $\inf_{v \in M_{\mu,\psi}^{\delta}} L(v) < \infty$. Then,*

$$\inf_{v \in M_{\mu,\psi}^{\delta} \cap \Gamma_{\mu}^{FR}(M)} L(v) = \inf_{v \in M_{\mu,\psi}^{\delta}} L(v).$$

The theorem has a simple probabilistic proof that is based on the strong law of large numbers and the almost sure convergence of empirical measures [10, Theorem 11.4.1].

V. Discussion

We investigated a general abstract model for randomized quantization that provides a more suitable framework for certain optimality problems than the ones usually considered in the source coding literature. Using this model, we demonstrated the existence of an optimal randomized vector quantizer under the constraint that the quantizer output has a given distribution. Our results are not constructive and it is an open problem how to find (or well approximate) such optimal quantizers. A special case where a scalar source has a density and the output distribution is constrained to be equal to the source distribution was considered in [8] and construction based on dithered uniform quantization followed by a nonlinear mapping was given. Although this construction is asymptotically (as $M \to \infty$) optimal, it is very likely suboptimal for any finite M. In general, it would be interesting to better characterize optimal randomized quantizers in Theorem 2, for example, by finding useful necessary conditions for optimality.

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