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# Jointly Optimal LQG Quantization and Control Policies for Multi-Dimensional Linear Gaussian Sources

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***Abstract*— For controlled $\mathbb{R}^n$-valued linear systems driven by Gaussian noise under quadratic cost criteria, we investigate the existence and the structure of optimal quantization and control policies. For a fully observed system, we show that an optimal quantization policy exists, provided that the quantizers allowed are ones which have convex codecells. Furthermore, optimal control policies are linear in the conditional estimate of the state. A form of separation and estimation applies. As a minor side result, towards obtaining the main results of the paper, structural results in the literature for optimal causal (zero-delay) quantization of Markov sources is extended to systems driven by control. For the partially observed case, structure of optimal coding and control policies is presented.**

## I. Jointly Optimal Encoding and Control Policies

Consider a Linear Quadratic Gaussian setup, where a sensor encodes its noisy information to a controller. Let $x_t \in \mathbb{R}^n$ and the evolution of the system be given by the following:

$$\begin{aligned} x_{t+1} &= Ax_t + Bu_t + w_t, \\ y_t &= Cx_t + v_t, \end{aligned} \tag{1}$$

Here, $\{w_t, v_t\}$ is a mutually independent, zero-mean i.i.d. Gaussian noise sequence, $\{u_t\}$ is an $\mathbb{R}^m-$valued control action, $y_t \in \mathbb{R}^p$ is the observation variable, and $A, B, C$ are matrices of appropriate dimensions. We assume that $x_0$ is a zero-mean Gaussian random variable.

Let there be an encoder who has access to the observation variable $y_t$, and who transmits his information to a receiver/controller, over a discrete noiseless channel with finite capacity; that is, he quantizes his information.

**Definition 1.1:** Let $\mathcal{M} = \{1, 2, \ldots, M\}$ with $M = |\mathcal{M}|$. Let $\mathbb{A}$ be a topological space. A quantizer $Q(\mathbb{A}; \mathcal{M})$ is a Borel measurable map from $\mathbb{A}$ to $\mathcal{M}$. ⋄

When the spaces $\mathbb{A}$ and $\mathcal{M}$ are clear from context, we will drop the notation and denote the quantizer simply by $Q$.

Following [30], we refer by a *Composite Quantization (Coding) Policy* $\Pi^{comp}$, a sequence of functions $\{Q_t^{comp}, t \geq 0\}$ which are causal such that the quantization output at time $t$, $q_t$, under $\Pi^{comp}$ is generated by a causally measurable function of its local information, that is, a mapping measurable on the sigma-algebra generated by

$$\mathcal{I}_t^e = \{y_{[0,t]}\}$$

to a finite set $\mathcal{M}$, the quantization output alphabet given by

$$\mathcal{M} := \{1, 2, \ldots, M\},$$

for $0 \leq t \leq T-1$ and $i = 1, 2$. Here, we have the notation for $t \geq 1$:

$$y_{[0,t-1]} = \{y_s, 0 \leq s \leq t-1\}.$$

Let $\mathbb{I}_t = (\mathbb{R}^p)^{t+1}$, be information spaces such that for all $t \geq 0$, $\mathcal{I}_t^e \in \mathbb{I}_t$. Thus,

$$Q_t^{comp} : \mathbb{I}_t \to \mathcal{M}.$$

As elaborated on in [30], we may express the policy $\Pi^{comp}$ as a composition of a *Quantization Policy* $\Pi^i$ and a *Quantizer*. A quantization policy $\mathcal{T}^i$ is a sequence of functions $\{T_t\}$, such that for each $t \geq 0$, $T_t$ is a mapping from the information space $\mathbb{I}_t$ to a space of quantizers $\mathbb{Q}_t$. A quantizer, subsequently is used to generate the quantizer output. A quantizer will be generated based on the common information at the encoder and the controller/receiver, and the quantizer will map the relevant private information at the encoder to the quantization output. Such a separation in the design will also allow us to use the machinery of Markov Decision Processes to obtain a structural method to design optimal quantizers, to be clarified further, without any loss in optimality. Thus, with the information at the controller at time $t$ being

$$\mathcal{I}_t^c = \{q_{[0,t]}\}, \quad t \geq 0,$$

we can express the composite quantization policy as:

$$Q_t^{comp}(\mathcal{I}_t^e) = (T_t(\mathcal{I}_t^c))(\mathcal{I}_t^e \setminus \mathcal{I}_t^c), \tag{2}$$

We note that, any composite quantization policy $Q_t^{comp}$ can be expressed in the form above; that is there is no loss in the space of possible such policies, since for any $Q_t^{comp}$, one could define

$$T_t(\mathcal{I}_t^c)(\cdot) := Q_t^{comp}(\mathcal{I}_t^c, \cdot).$$

Thus, we let the encoder have policy $\mathcal{T}$ and under this policy generate quantizer actions $\{Q_t, t \geq 0\}$, $Q_t \in \mathbb{Q}_t$ ($Q_t$ is the quantizer used at time $t$). Under action $Q_t$, and given the local information, the encoder generates $q_t^i$, as the *quantization output* at time $t$. See [30] for further discussion on such a construction.

The controller, upon receiving the information from the encoders, generates its decision at time $t$, also causally: An admissible causal controller policy is a sequence of measurable functions $\gamma = \{\gamma_t\}$ such that

$$\gamma_t : \mathcal{M}^{t+1} \to \mathbb{R}^m, \qquad t \geq 0.$$

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We call such encoding and control policies, *causal* or *admissible*. Now, suppose that the goal is the computation of

$$\inf_{\Pi^{comp}} \inf_{\gamma} J(\Pi^{comp}, \gamma, T) \tag{3}$$

where

$$J(\Pi^{comp}, \gamma, T) := \frac{1}{T} E_{\nu_0}^{\Pi^{comp},\gamma}[\sum_{t=0}^{T-1} x_t' Q x_t + u_t' R u_t].$$

Here, $Q \geq 0$ a positive semi-definite matrix, and $R > 0$ a positive definite matrix.

Finally, we assume that the quantizer and the controller have an agreement on the zero-mean Gaussian probability measure $\nu_0$ on the initial state.

### *A. Relevant Literature*

In this subsection, we provide a literature review on structural results, existence results, and results on jointly optimal coding and control policies.

Regarding structural results on optimal causal (zero-delay or real-time) coding, all the papers to our knowledge consider a control-free setup: Related papers on causal coding include the following. If the source is $k$th-order Markov, then an optimal causal fixed-rate coder minimizing any measurable distortion uses only the last $k$ source symbols, together with the current state at the receiver's memory [28]. Reference [27] considers optimal causal coding problem of finite-state Markov sources over noisy channels with feedback. [30] has considered optimal causal coding for sources taking values in Polish spaces, partially observed Markov sources and also for a class of multi-encoder systems. References [25] and [17] have considered optimal causal coding of Markov sources over noisy channels without feedback. [16] has considered the optimal causal coding over a noisy channel with noisy feedback. [20] has considered, within a multi-terminal setup, decentralized coding of correlated sources when the encoders observe conditionally independent messages given a finitely valued random variable, and obtained separation results for optimal encoders. Reference [15] has considered the causal coding of more general sources, stationary sources, under a high-rate assumption. An earlier reference on quantizer design is [7]. Relevant discussions on optimal quantization, randomized decisions, and optimal quantizer design can be found in [11] and [32]. A more relaxed version of causality (allowing delay at the decoder, but not at the encoder) has been considered in [21], which has established that the optimal optimal causal encoder minimizing the data rate subject to a distortion for an i.i.d sequence is memoryless. A parallel line of consideration which is of a rate-distortion theoretic nature is the *sequential-rate distortion* proposed in [24], and the *feedforward* setup, which has been investigated in [26] and [8]. For a further literature review, the reader is referred to [30].

Regarding existence results, there have been few studies: The existence of optimal quantizers for a one-stage cost problem has been investigated in [1] and [23] (which have considered nearest neighbor encoding/decoding rules), and [32] which has considered quantizers with convex codecells. For multi-stage settings, reference [31] has considered the existence problem for the optimal quantization of control-free Markov sources for a class of Markov sources driven by an additive Gaussian noise under the restriction that the quantizers have convex codecells. This class of systems also includes the setup contained here except that control is not available in the systems considered in [31]. Also for optimal multi-stage vector quantizers, [5] has obtained existence results for an infinite horizon setup with discounted costs under a uniform boundedness assumption on the reconstruction levels.

There is a large literature on jointly optimal quantization for the LQG problem. In this literature, references [9], [7], [18], [24], [19], [4] and [10] have considered the optimal LQG quantization and control with various results on the optimality or the lack of optimality of the separation principle with different assumptions in the setups and various conclusions on the structural properties of optimal policies. Among these, we refer the reader to [9] and [10] for a detailed account of further, earlier, contributions in the literature. This literature has considered either the quantization of control signals (as in [9] and [18]) or the quantization of sensor information as in the current paper (see [24], [19], [10]). We also note that [30] provides a discussion for optimal quantization of control-free linear Gaussian systems.

On a related setting, for scalar Gaussian sources controlled over scalar Gaussian channels, the jointly optimal policies for costs of the form (3) have been established in [2] and [3], where both encoding and control policies are shown to admit linear forms. This result does not extend to general multi-dimensional sources and channels, where further conditions are required in view of what is known as the *matching between the source and channel pairs*, see [24] and [12] and for a counterexample on suboptimality of linear policies, see [33].

### *B. Contributions of the paper and summary*

Our contribution in this paper, in view of the literature reviewed above, is that (i) we provide a structural result for optimal encoders for systems driven by control, and building on this structural result, (ii) establish a separation theorem between coding and control which is new to our knowledge in its generality, as well as (iii) an existence theorem for optimal quantizers (building on [32] and [31]), and also (iv) establish the structure of optimal control policies.

Here is a summary of the rest of the paper. In Section II, we establish the structure of optimal causal (zero-delay) coding policies for fully observed controlled Markov sources. In Section III, we consider the fully observed setting in (1) (that is with $y_t = x_t$) and obtain the structure of optimal control policies. In Section IV, we establish the existence of optimal quantization policies. The partially observed setting is discussed in Section V.

## II. STRUCTURAL RESULTS FOR OPTIMAL ZERO-DELAY CODES FOR CONTROLLED MARKOV SOURCES

In this section, toward obtaining a solution to (3), we develop structural results for optimal causal composite quantization policies. Consider a fully observed system described by the following equations

$$x_{t+1} = f(x_t, u_t, w_t),$$
$$y_t = x_t, \quad (4)$$

where the realizations satisfy $x_t \in \mathbb{X}, u_t \in \mathbb{U}$, with $\mathbb{X}, \mathbb{U}$ being complete, separable, metric (that is Polish) spaces (thus including spaces such as $\mathbb{R}^n$ or a finite set). Suppose that the goal is the minimization,

$$\inf_{\Pi^{comp}} \inf_{\gamma} E_{\nu_0}^{\Pi^{comp},\gamma}[\sum_{t=0}^{T-1} c(x_t, u_t)], \quad (5)$$

over all policies $\Pi^{comp}$, $\gamma$ with the random initial condition $x_0$ having probability measure $\nu_0$. Here $c(\cdot,\cdot)$, is a measurable function and $u_t = \gamma_t(q_{[0,t]})$ for $t \geq 0$. Here, the information and quantization restrictions are as stated in Section I.

The proofs of the results below essentially follow from Theorems 2.4 and 2.5 in [30] with additional technical intricacies due to the presence of control actions. The first one can be regarded as an extension of Witsenhausen's structural theorem [28], and the second one can be regarded as an extension of the results of Walrand and Varaiya [27] (see also [25]).

For the proofs of the results below, see [29].

**Theorem 2.1:** For system (4), under the information structure described in the previous section and the objective given in (5), any composite quantization policy (with a given control policy) can be replaced, without any loss in performance, by one which only uses $x_t$ and $q_{[0,t-1]}$ at time $t \geq 1$ while keeping the control policy unaltered. This can be expressed as a quantization policy which only uses $q_{[0,t-1]}$ to generate a quantizer, where the quantizer uses $x_t$ to generate the quantization output at time $t$.

Let $\mathcal{P}(\mathbb{X})$ denote the set of probability measures on $\mathcal{B}(\mathbb{X})$ (where $\mathcal{B}(\mathbb{X})$ denotes the Borel $\sigma$-field on $\mathbb{X}$) under the topology of weak convergence (please see Section II.B in [30] for the use of such a topology) and define $\pi_t \in \mathcal{P}(\mathbb{X})$ to be the regular conditional probability measure given by $\pi_t(\cdot) = P(x_t \in \cdot | q_{[0,t-1]}, u_{0,t-1})$ or since the control actions are determined by the quantizer outputs given a control policy, $\pi_t(\cdot) = P(x_t \in \cdot | q_{[0,t-1]})$, that is

$$\pi_t(A) = P(x_t \in A | q_{[0,t-1]}), \quad A \in \mathcal{B}(\mathbb{X}).$$

**Theorem 2.2:** For system (4), under the information structure described in the previous section and the objective given in (5), any *causal composite quantization policy* can be replaced, without any loss in performance, by one which only uses the conditional probability measure $\pi_t(\cdot) = P(x_t \in \cdot | q_{[0,t-1]})$, the state $x_t$, and the time information $t$, at time $t$. This can be expressed as a quantization policy which only uses $\{\pi_t, t\}$ to generate a quantizer, where the quantizer uses $x_t$ to generate the quantization output at time $t$.

**Definition 2.1:** *An $M$-cell quantizer $Q$ on $\mathbb{R}^n$ is a (Borel) measurable mapping $Q : \mathbb{R}^n \to \mathcal{M}$, and $\mathcal{Q}$ denotes the collection of all $M$-cell quantizers on $\mathbb{R}^n$.*

Note that each $Q \in \mathcal{Q}$ is uniquely characterized by its *quantization cells* (or bins) $B_i = \{x : Q(x) = i\}$, $i = 1, \ldots, M$ which form a measurable partition of $\mathbb{R}^n$.

**Remark 2.1:**

(a) As in [32], we allow for the possibility that some of the cells of the quantizer are empty.

(b) In source coding theory, a quantizer is a mapping $Q : \mathbb{R}^n \to \mathbb{R}^n$ with a finite range. Thus $Q$ is defined by a partition and a reconstruction value in $\mathbb{R}^n$ for each cell in the partition. That is, for given cells $\{B_1, \ldots, B_M\}$ and reconstruction values $\{c_1, \ldots, c_M\} \subset \mathbb{R}^n$, we have $Q(x) = c_i$ if and only if $x \in B_i$. In our definition, we do not include the reconstruction values (and the controller/receiver policy computes the decision outputs).

As discussed in [32], a quantizer $Q$ with cells

$$\{B_1, \ldots, B_M\}$$

can also be characterized as a stochastic kernel $Q$ from $\mathbb{R}^n$ to $\{1, \ldots, M\}$ defined by

$$Q(i|x) = 1_{\{x \in B_i\}}, \quad i = 1, \ldots, M$$

We will endow the quantizers with a topology induced by such a stochastic kernel interpretation. If $P$ is a probability measure on $\mathbb{R}^n$ and $Q$ is a stochastic kernel from $\mathbb{R}^n$ to $\mathcal{M}$, then $PQ$ denotes the resulting joint probability measure on $\mathbb{R}^n \times \mathcal{M}$.

Let $\mathcal{P}(\mathbb{R}^{\mathbb{N}})$ denote the family of all probability measures on $(\mathbb{R}^N, \mathcal{B}(\mathbb{R}^N))$ for some $N \in \mathbb{N}$. Let $\{\mu_n, n \in \mathbb{N}\}$ be a sequence in $\mathcal{P}(\mathbb{R}^N)$. It is said to converge to $\mu \in \mathcal{P}(\mathbb{R}^N)$ *weakly* if

$$\int_{\mathbb{R}^N} c(x)\mu_n(dx) \to \int_{\mathbb{R}^N} c(x)\mu(dx)$$

for every continuous and bounded $c : \mathbb{R}^N \to \mathbb{R}$.

The following sequential convergence notion is considered for quantizers in this paper.

**Definition 2.2 ([32]):** *A quantizer sequence $Q_n$ converges to $Q$ weakly at $P$ ($Q_n \to Q$ weakly at $P$) if $PQ^n \to PQ$ weakly.*

Consider the set of probability measures

$$\Theta := \{\zeta \in P(\mathbb{R}^n \times \mathcal{M}) : \zeta = PQ, Q \in \mathcal{Q}\},$$

on $\mathbb{R}^n \times \mathcal{M}$ having fixed input marginal $P$, equipped with weak topology. This set is the (Borel measurable) set of the extreme points on the set of probability measures on $\mathbb{R}^n \times \mathcal{M}$ with a fixed input marginal $P$ [6]. Borel measurability of $\Theta$ follows from [22] since set of probability measures on $\mathbb{R}^n \times \mathcal{M}$ with a fixed input marginal $P$ is a convex and compact set in a complete separable metric space, and therefore, the set of its extreme points is Borel measurable.

In view of this observation, we note that the class of quantization policies which admit the structure suggested in Theorem 2.2 is an important one. We henceforth define:

$$\Pi_W := \Big\{ \Pi^{comp} = \{Q_t^{comp}, t \geq 0\} \\ : \exists \Upsilon_t : \mathcal{P}(\mathbb{X}) \to \mathcal{Q} \\ Q_t^{comp}(I_t) = (\Upsilon_t(\pi_t))(x_t), \forall I_t, P a.s. \Big\}, \quad (6)$$

to represent this class of policies.

## III. Fully Observed Case: Separation of estimation error and control and dual effect

We now consider the original LQG problem. By Theorem 2.2, an optimal composite quantization policy will be within the class $\Pi_W$. Let us fix such a composite quantization policy. In the following, we adopt a dynamic programming approach and establish that the optimal controller is linear in its estimate. This fact applies naturally for the terminal time stage control. That this also applies for the previous time stages follows from dynamic programming as we observe in the following.

First consider the terminal time $t = T-1$. For this time stage, to minimize $E[x_t'Qx_t + u_t'Ru_t]$, the optimal control is $u_{T-1} = 0$ a.s..

To obtain a solution for $t = T-2$, we look for a solution to:

$$\min_{\gamma_t} E\Bigg[\bigg(x_t'Qx_t + u_t'Ru_t \\ + E[(Ax_t + Bu_t + w_t)'Q(Ax_t + Bu_t + w_t)|\mathcal{I}_t^c, u_t]\bigg) \\ \bigg|\mathcal{I}_t^c\Bigg]. \quad (7)$$

By completing the squares, and using the *Orthogonality Principle*, we obtain that the optimal control is linear and is given by

$$u_{T-2} = L_{T-2}E[x_{T-2}|q_{[0,T-2]}],$$

with

$$L_{T-2} = -R^{-1}B'QA.$$

For $t < T-2$, to obtain the solutions, we will first establish that the estimation errors are uncorrelated. Towards this end, define for $1 \leq t \leq T-1$ (recall that the control actions are determined by the quantizer outputs):

$$\mathcal{I}_t^c = \{q_{[0,t]}, u_{[0,t-1]}\},$$

and note that

$$\tilde{m}_{t+1} := E[x_{t+1}|\mathcal{I}_{t+1}^c] = E[Ax_t + Bu_t + w_t|\mathcal{I}_{t+1}^c].$$

It then follows that

$$\begin{aligned} \tilde{m}_{t+1} &= E[x_{t+1}|\mathcal{I}_{t+1}^c] \\ &= E[x_{t+1} - E[x_{t+1}|\mathcal{I}_t^c] + E[x_{t+1}|\mathcal{I}_t^c]|\mathcal{I}_{t+1}^c] \\ &= E[x_{t+1}|\mathcal{I}_t^c] + E[x_{t+1} - E[x_{t+1}|\mathcal{I}_t^c]|\mathcal{I}_{t+1}^c] \\ &= A\tilde{m}_t + Bu_t + \bar{w}_t, \end{aligned}$$

with

$$\bar{w}_t = (E[x_{t+1}|\mathcal{I}_{t+1}^c] - E[x_{t+1}|\mathcal{I}_t^c]). \quad (8)$$

Here, $\bar{w}_t$ is orthogonal to the control action variable $u_t$, as control actions are determined by the past quantizer outputs and iterated expectation leads to the result that conditioned on $\mathcal{I}_t^c$, $\bar{w}_t$ is zero mean, and is orthogonal to $\mathcal{I}_t^c$. Now, for going into earlier time stages, the dynamic programming recursion for linear systems driven by an uncorrelated noise process would normally apply, since the estimate process $\tilde{m}_t$ is driven an uncorrelated noise (though, not necessarily independent) process $E[x_{t+1}|\mathcal{I}_{t+1}^c] - E[x_{t+1}|\mathcal{I}_t^c]$. However, this lack of independence may be important, as elaborated on in [19]. Using the completion of the squares method, we can establish that the optimal controller at time $t$ will be linear, provided that the random variable $\bar{w}_t'Q\bar{w}_t$ does not depend on $u_k, k \leq t$ under any policy. A sufficient condition for this is that the encoder is a predictive one (see [4], [19] and [24]).

**Definition 3.1:** A predictive quantizer policy is one where for each time stage $t$, the quantization has the form that the quantizer at all time stages subtracts the effect of the past control terms, that is, at time $t$ it has the form

$$Q_t(x_t - \sum_{k=0}^{t-1} A^{t-k-1}Bu_k),$$

and the past control terms are added at the receiver. Hence, the encoder quantizes a control-free process, defined by:

$$\bar{x}_{t+1} = A\bar{x}_t + w_t,$$

the receiver generates the quantized estimate and adds

$$\sum_{k=0}^{t-1} A^{t-k-1}Bu_k,$$

to compute the estimate of the state at time $t$. ◇

A predictive encoder is depicted in Figure 1.

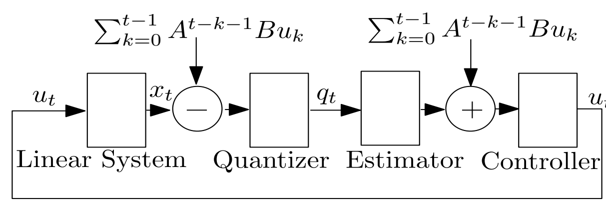


Fig. 1: For the LQG problem, a predictive encoder is without loss.

One question, which has not been explicitly addressed in [4], [19] and [24], is whether restriction to this class of quantization policies is without loss.

We have the following key lemma. See [29] for a proof.

**Lemma 3.1:** For problem (3), for any quantizer policy in class $\Pi_W$ (which is without any loss as a result of Theorem 2.2), there exists a quantizer which satisfies the form of a predictive quantizer (see Definition 3.1) and attains the same performance under an optimal control policy.

**Remark 3.1:** We note that the structure in Definition 3.1 separates the estimation from the control process in the sense that the estimation errors are independent of the control actions or policies. Hence, there is no *dual effect* of the control actions, in that the estimation error at any given time is independent of the past applied control actions. ◇

As a consequence of the lack of dual effect, the cost function becomes

$$J(\Pi^{comp}, \gamma, T) := \frac{1}{T} E_{\nu_0}^{\Pi^{comp},\gamma}[\sum_{t=0}^{T-1} \tilde{m}_t' Q \tilde{m}_t' Q x_t + u_t' R u_t + (x_t - \tilde{m}_t)' Q (x_t - \tilde{m}_t)].$$

We have thus established above that the optimal control is linear at $t = T-3$ as well, and as part of the proof of Lemma 3.1, we saw that this applies for all time stages. We have the following result (see also [19] in a related context which assumes the structure of Definition 3.1):

**Theorem 3.1:** For the minimization problem (3), an optimal control policy is given by $u_t = L_t E[x_t | q_{[0,t]}]$, where

$$L_t = -(R + B' P_{t+1} B)^{-1} B' K_{t+1} A,$$

and

$$P_t = A_t' K_{t+1} B (R + B' K_{t+1} B)^{-1} B' K_{t+1} A,$$
$$K_t = A_t' K_{t+1} A_t - P_t + Q,$$

with $K_T = P_{T-1} = 0$.

Therefore, we obtain for $t \geq 0$, the unnormalized value function for any time stage $t$ as

$$J_t(\mathcal{I}_t^c) = E[x_t' K_t x_t | \mathcal{I}_t^c] + \sum_{k=t}^{T-1} \Bigg( E[(x_t - E[x_t|\mathcal{I}_t^c])' Q (x_t - E[x_t|\mathcal{I}_t^c])] + E[\bar{w}_t' K_{t+1} \bar{w}_t] \Bigg),$$

with

$$J(\Pi^{comp}, \gamma, T) = \frac{1}{T} J_0(\mathcal{I}_0^c).$$

To obtain a more explicit expression for the value function $J_t$, we have the following analysis. Given a positive definite matrix $\Lambda$ define an inner-product as

$$\langle z_1, z_2 \rangle_\Lambda = z_1' \Lambda z_2.$$

and the norm generated by this inner-product as $||z||_\Lambda = \sqrt{z' \Lambda z}$. We now note the following:

$$E\Bigg[||E[x_{t+1}|\mathcal{I}_{t+1}^c] - E[x_{t+1}|\mathcal{I}_t^c]||_\Lambda^2\Bigg]$$
$$= E\Bigg[||(E[x_{t+1}|\mathcal{I}_{t+1}^c] - x_{t+1})||_\Lambda^2] + E[||(x_{t+1} - E[x_{t+1}|\mathcal{I}_t^c])||_\Lambda^2\Bigg]$$
$$+2E[\langle (E[x_{t+1}|\mathcal{I}_{t+1}^c] - x_{t+1}), (x_{t+1} - E[x_{t+1}|\mathcal{I}_t^c]) \rangle_\Lambda]$$

Note that

$$E\Bigg[\langle (E[x_{t+1}|\mathcal{I}_{t+1}^c] - x_{t+1}), (x_{t+1} - E[x_{t+1}|\mathcal{I}_t^c]) \rangle_\Lambda\Bigg]$$
$$= E\Bigg[ - \langle (E[x_{t+1}|\mathcal{I}_{t+1}^c] - x_{t+1}), (E[x_{t+1}|\mathcal{I}_t^c]) \rangle_\Lambda + \langle (E[x_{t+1}|\mathcal{I}_{t+1}^c] - x_{t+1}), (x_{t+1}) \rangle_\Lambda\Bigg]$$
$$= E\Bigg[\langle (E[x_{t+1}|\mathcal{I}_{t+1}^c] - x_{t+1}), (x_{t+1}) \rangle_\Lambda\Bigg] \quad (9)$$
$$= -E[||(E[x_{t+1}|\mathcal{I}_{t+1}^c] - x_{t+1})||_\Lambda^2] \quad (10)$$

where (9)-(10) follow from the orthogonality property of minimum mean-square estimation and that $E[x_{t+1}|\mathcal{I}_t^c]$ is measurable on $\sigma(\mathcal{I}_{t+1}^c)$, the sigma-field generated by $\mathcal{I}_{t+1}^c$.

Using these, with some further analysis [29], the cost function writes as:

$$J_t(\mathcal{I}_t^c) = E[x_t' K_t x_t | \mathcal{I}_t^c] + E[(x_t - E[x_t|\mathcal{I}_t^c])' (Q + A' K_{t+1} A)(x_t - E[x_t|\mathcal{I}_t^c])] + \sum_{k=t+1}^{T-1} E[(x_k - E[x_k|\mathcal{I}_k^c])' (Q + A' K_{k+1} A - K_k) \times (x_k - E[x_k|\mathcal{I}_k^c])] + \sum_{k=t}^{T-1} E[w_k' K_{k+1} w_k]. \quad (11)$$

Now that we have established the solution to the optimal control problem, we address the optimal quantization problem below.

## IV. Existence of optimal quantization policies

In (11) above, we have separated the costs due to control and quantization. Therefore, for the optimal quantization policy, we can effectively consider the setting where in (1), $u_t = 0$ and the quantizer is designed for this system. Hence, we consider below the system $x_{t+1} = A x_t + w_t$.

We first note that, for $K > 0$,

$$E\Bigg[(x_t - E[x_t|\mathcal{I}_t^c])' K (x_t - E[x_t|\mathcal{I}_t^c]) \Bigg| \mathcal{I}_{t-1}^c\Bigg] = \sum_{i \in \mathcal{M}} \inf_{\gamma_t(i)} \int_{\mathbb{R}^n} 1_{\{q_t = i\}} \pi_t(dx)(x_t - \gamma_t(i))' K \times (x_t - \gamma_t(i))$$

Thus, from (11), for $T \in \mathbb{N}$, we can define a cost to be minimized by a composite quantization policy as:

$$J(\Pi^{comp}, T) = E_{\nu_0}^{\Pi^{comp}}[\frac{1}{T}\Bigg(x_0' K_0 x_0 + \sum_{t=0}^{T-1} c_t(\pi_t, Q_t) + E[w_t' K_{t+1} w_t]\Bigg)],$$

where

$$\pi_t(\cdot) = P(x_t \in \cdot | q_{[0,t-1]}),$$

and $c_t(\pi_t, Q_t)$ is

$$\sum_{i\in\mathcal{M}} \inf_{\gamma_t(i)} \int_{\mathbb{R}^n} 1_{\{q_t=i\}} \pi_t(dx_t)(x_t - \gamma_t(i))' P_t (x_t - \gamma_t(i)),$$

with $\tilde{\gamma} = \{\gamma_t, t \geq 0\}$ now denoting the receiver policy and $P_t = (Q + A'K_{t+1}A - K_t)$ and $P_0 = Q + A'K_1A$. Note that, $E[1_{\{q_t=i\}}(\bar{x}_t - \gamma_t(i))'P_t(\bar{x}_t - \gamma_t(i))]$ is minimized by the conditional expectation given the bin information. As a consequence, an optimal receiver and hence control policy always exists.

In the analysis for an optimal quantization policy, as was also motivated in [32], we will restrict the quantizers to have convex codecells. As discussed in [13], by the separating hyperplane theorem, there exist pairs of complementary closed half spaces $\{(H_{i,j}, H_{j,i}) : 1 \leq i, j \leq M, i \neq j\}$ such that for all $i = 1, \ldots, M$,

$$B_i \subset \bigcap_{j\neq i} H_{i,j}.$$

Since $\bar{B}_i := \bigcap_{j\neq i} H_{i,j}$ is a closed convex polytope for each $i$, if the probability measure $P$ admits a density function, then one has $P(\bar{B}_i \setminus B_i) = 0$ for all $i = 1, \ldots, M$. One can thus obtain a ($P$–a.s) representation of $Q$ by the $M(M-1)/2$ hyperplanes $h_{i,j} = H_{i,j} \cap H_{j,i}$. One can represent a hyperplane in $\mathbb{R}^n$ by a vector of $n+1$ components $a_1, a_2, \ldots, b$ with $\sum_k |a_k|^2 = 1$, and $h = \{x \in \mathbb{R}^n : \sum a_i x_i = b\}$. A sequence of quantizers converges if each of the coefficients defining hyperplanes in the quantizer converges pointwise.

**Assumption 4.1:** The quantizers have convex codecells with at most a given number of cells; that is the quantizers live in $\mathcal{Q}_c(M)$, the collection of $k$-cell quantizers with convex cells where $1 \leq k \leq M$.

Let $\Pi_W^C$ denote the set of all policies in $\Pi_W$ (defined in (6)) which in addition satisfy Assumption 4.1 (i.e., $Q_t \in \mathcal{Q}_c(M)$ for all $t \geq 0$).

The properties of conditional probability lead to the filtering expression for $\pi_t(dx_t)$:

$$\frac{\int_{x_{t-1}} \pi_{t-1}(dx_{t-1}) P(q_{t-1}|\pi_{t-1}, x_{t-1}) P(dx_t|x_{t-1})}{\int_{x_{t-1}} \int_{x_t} \pi_{t-1}(dx_{t-1}) P(q_{t-1}|\pi_{t-1}, x_{t-1}) P(dx_t|x_{t-1})}.$$

Here, the term $P(q_{t-1}|\pi_{t-1}, x_{t-1})$ is determined by the quantizer action $Q_{t-1}$. In view of this observation, we have the following result.

**Lemma 4.1:** [30] With $\mathcal{P}(\mathbb{R}^n)$ denoting the set of probability measures on $\mathcal{B}(\mathbb{R}^n)$ under weak convergence topology, the conditional probability measure process and the quantization process $(\pi_t(x), Q_t)$ form a controlled Markov process in $\mathcal{P}(\mathbb{R}^n) \times \mathcal{Q}_c(M)$.

**Theorem 4.1:** There exists an optimal composite coding policy in $\Pi_W^C$ such that

$$\inf_{\Pi^{comp} \in \Pi_W^C} J(\Pi^{comp}, T),$$

is achieved. With,

$$J_0'(\pi) := \min_{\Pi^{comp} \in \Pi_W^C} J(\Pi^{comp}, T)$$

the following dynamic programming recursion holds for $0 \leq t \leq T-1$:

$$TJ_t'(\pi_t) = \min_{Q\in\mathbb{Q}} \left( c_t(\pi_t, Q_t) + TE[J_{t+1}'(\pi_{t+1})|\pi_t, Q_t] \right)$$

with $J_T'(\cdot) = 0$. Furthermore, the optimal control policy is linear in the conditional estimate and is given in Theorem 3.1. ⋄

The proof of Theorem 4.1 follows from the separation argument considered since one can consider a control-free Markov source which is to be quantized. Therefore, the existence result follows from Yüksel and Linder [31] which considers a control-free setting.

## V. Partially Observed Case: Structural Results

In this section, we consider the partially observed model (1) with $W = E[w_t w_t'], V = E[v_t v_t']$.

To obtain a solution, we again first separate the estimation and control terms, as in the fully observed case. The solution to the control terms then will follow from classical results in LQG theory. The solution for the quantization component will follow from the results earlier and Theorem 4.1 in [30].

Define $\bar{m}_t := E[x_t|y_{[0,t]}]$, which is computed through a Kalman Filter. Recall that by the Kalman Filter (see [14]) with

$$\Sigma_{0|-1} = E[x_0 x_0']$$

and for $t \geq 0$,

$$\begin{aligned}\Sigma_{t+1|t} &= A\Sigma_{t|t-1}A' + W \\ &\quad -(A\Sigma_{t|t-1}C')(C\Sigma_{t|t-1}C' + V)^{-1}(C\Sigma_{t|t-1}A'),\end{aligned}$$

the following recursion holds for $t \geq 0$ and with $\tilde{m}_{-1} = 0$:

$$\begin{aligned}\bar{m}_t &= A\bar{m}_{t-1} + Bu_{t-1} \\ &\quad + \Sigma_{t|t-1}C'(C\Sigma_{t|t-1}C' + V)^{-1}(CA(x_{t-1} - \tilde{m}_{t-1}) \\ &\quad + v_t).\end{aligned}$$

Now, note that the cost

$$\inf_{\Pi^{comp}} \inf_{\gamma} J(\Pi^{comp}, \gamma, T) \tag{12}$$

with

$$J(\Pi^{comp}, \gamma, T) = \frac{1}{T} E_{\nu_0}^{\Pi^{comp},\gamma}[\sum_{t=0}^{T-1} x_t'Qx_t + u_t'Ru_t],$$

can be written equivalently as

$$\begin{aligned}J(\Pi^{comp}, \gamma, T) &= \frac{1}{T} E_{\nu_0}^{\Pi^{comp},\gamma}[\sum_{t=0}^{T-1} \bar{m}_t'Q\bar{m}_t + u_t'Ru_t] \\ &\quad + \frac{1}{T}\sum_{t=0}^{T-1}(x_t - \bar{m}_t)'Q(x_t - \bar{m}_t),\end{aligned}$$

since the quadratic error $(x_t - \bar{m}_t)'Q(x_t - \bar{m}_t)$ is independent of the coding or the control policy (and only depends on the estimation performance at the encoder).

Thus, we have that the process $(\bar{m}_t, \Sigma_{t+1|t})$ and $u_t$ form a controlled Markov chain and we can invoke Theorem 2.2:

Any causal quantizer, can be, without any loss replaced with one in $\Pi_W$ (where the state is now $(\bar{m}_t, \Sigma_{t+1|t})$ instead of $x_t$) as a consequence of Theorem 2.2. Furthermore, any quantizer in $\Pi_W$ can be replaced without any loss with a predictive quantizer with the new state $\bar{m}_t$, as a consequence of Lemma 3.1 applied to the new state with identical arguments: Observe that the past control actions do not affect the evolution of $\Sigma_{t+1|t}$.

We then have the following result.

**Theorem 5.1:** For the minimization problem (12), the optimal control policy is given by $u_t = L_t E[x_t | q_{[0,t]}]$, where

$$L_t = -(R + B' P_{t+1} B)^{-1} B' K_{t+1} A,$$

and

$$P_t = A_t' K_{t+1} B (R + B' K_{t+1} B)^{-1} B' K_{t+1} A,$$
$$K_t = A_t' K_{t+1} A_t - P_t + Q,$$

with $K_T = P_{T-1} = 0$.

Given the optimal control policy, the following result is obtained.

**Theorem 5.2:** For the minimization problem (12), the optimal cost it given by $\frac{1}{T} J_0(\Pi^{comp}, T)$, where $J_0(\Pi^{comp}, T)$ is given by

$$\begin{aligned} & E[x_0' K_0 x_0] \\ & + E[(x_0 - E[x_0|\mathcal{I}_0^c])'(Q + A'K_1A)(x_0 - E[x_0|\mathcal{I}_0^c])] \\ & + \sum_{t=1}^{T-1} E[(x_t - E[x_t|\mathcal{I}_t^c])'(Q + A'K_{t+1}A - K_t) \\ & \qquad \times (x_t - E[x_t|\mathcal{I}_t^c])] \\ & + \sum_{t=0}^{T-1} E[(x_t - \bar{m}_t)'Q(x_t - \bar{m}_t) + w_t' K_{t+1} w_t]. \quad (13) \end{aligned}$$

Now that we have separated the cost terms, and given that we can use a predictive encoder without any loss, we consider the quantization for a control-free system and the following result essentially follows from Theorem 4.1 in [30].

**Theorem 5.3:** For the minimization of the cost in (3), any *causal composite quantization policy* can be replaced, without any loss in performance, by an encoder which only uses the output of the Kalman Filter and the information available at the receiver. Furthermore, any causal coder can be replaced with one which only uses the conditional probability on $\bar{m}_t$, $P(d\bar{m}_t | q_{[0,t-1]})$, and the realization $(\bar{m}_t, \Sigma_{t|t-1}, t)$ at time $t$. (see Figure 2). ⋄

## VI. Conclusion

In this paper, joint optimization of encoding and control policies have been obtained for the Linear Quadratic Gaussian problem, and it has been established that separation of estimation and control applies, an optimal quantizer exists under technical assumptions on the space of policies considered and optimal control policy is linear in its conditional estimate. Results have been extended to the partially observed case, where the structure of optimal coding and control policies is presented.

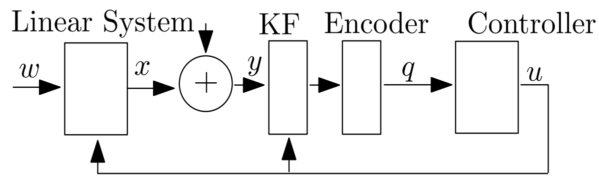


Fig. 2: Separation of Estimation and Quantization: When the source is Gaussian, generated by the linear system (1), the cost is quadratic, and the observation channel is Gaussian, the separated structure of the encoder above involving a Kalman Filter (KF) is optimal. Here, the encoder is a predictive encoder without any loss.

As a side result, towards obtaining the main results of the paper, structural results in the literature for optimal causal (zero-delay) quantization of Markov sources is extended to systems driven by control.

## References


[1] E. A. Abaya and G. L. Wise. Convergence of vector quantizers with applications to optimal quantization. *SIAM J. on Applied Mathematics*, 44:183–189, 1984.

[2] R. Bansal and T. Başar. Stochastic team problems with nonclassical information revisited: When is an affine law optimal? *IEEE Transactions Automat. Contr.*, 32:554–559, 1987.

[3] R. Bansal and T. Başar. Simultaneous design of measurement and control strategies in stochastic systems with feedback. *Automatica*, 45:679–694, September 1989.

[4] V. S. Borkar and S. K. Mitter. Lqg control with communication constraints. Kluwer Academic Publishers, Boston, 1997.

[5] V. S. Borkar, S. K. Mitter, and S. Tatikonda. Optimal sequential vector quantization of Markov sources. *SIAM J. Control and Optimization*, 40:135–148, 2001.

[6] V.S. Borkar, S.K. Mitter, A. Sahai, and S. Tatikonda. Sequential source coding: An optimization viewpoint. In *Proceedings of the IEEE Conference on Decision and Control*, pages 1035–1042, December 2005.

[7] R. E. Curry. *Estimation and Control with Quantized Measurements*. MIT Press, Cambridge, MA, 1969.

[8] H. Ebeid and T. P. Coleman. Source coding with feedforward using the posterior matching scheme. In *Proceedings of the IEEE Int. Symp. on Information Theory (ISIT)*, Austin, Texas, U.S.A, 2010. Austin.

[9] T. R. Fischer. Optimal quantized control. *IEEE Transactions on Automatic Control*, 27:996–998, August 1982.

[10] M. Fu. Lack of separation principle for quantized linear quadratic Gaussian control. *IEEE Transactions on Automatic Control, to appear*.

[11] G. Gabor, G. Szekeres, and Z. Györfi. On the Gaarder-Slepion tracking system conjecture. *IEEE Transactions Information Theory*, 37:1165–1168, July 1991.

[12] M. Gastpar, B. Rimoldi, and M. Vetterli. To code, or not to code: Lossy source-channel communication revisited. *IEEE Transactions on Information Theory*, 49:1147–1158, May 2003.

[13] A. György and T. Linder. Codecell convexity in optimal entropy-constrained vector quantization. *IEEE Transactions Information Theory*, 49:1821–1828, July 2003.

[14] H.J. Kushner. *Introduction to Stochastic Control Theory*. Holt, Rinehart and Winston, New York, 1972.

[15] T. Linder and R. Zamir. Causal coding of stationary sources and individual sequences with high resolution. *IEEE Transactions on Information Theory*, 52:662–680, February 2006.

[16] A. Mahajan and D. Teneketzis. On the design of globally optimal communication strategies for real-time noisy communication with noisy feedback. *IEEE Journal on Selected Areas in Communications*, 28:580–595, May 2008.

[17] A. Mahajan and D. Teneketzis. Optimal design of sequential real-time communication systems. *IEEE Transactions on Information Theory*, 55:5317–5338, November 2009.

[18] A. S. Matveev and A. V. Savkin. *Estimation and Control over Communication Networks*. Birkhäuser, Boston, 2008.

[19] G. Nair, F. Fagnani, S. Zampieri, and J. R. Evans. Feedback control under data constraints: an overview. *Proceedings of the IEEE*, pages 108–137, 2007.

[20] A. Nayyar and D. Teneketzis. On the structure of real-time encoders and decoders in a multi-terminal communication system. *IEEE Transactions on Information Theory*, 57:6196–6214, September 2011.

[21] D. L. Neuhoff and R. K. Gilbert. Causal source codes. *IEEE Transactions on Information Theory*, 28:701–713, September 1982.

[22] R.P. Phelps. *Lectures on Choquet's theorem*. Van Nostrand, New York:, 1966.

[23] D. Pollard. Quantization and the method of $k$-means. *IEEE Transactions Information Theory*, 28:199–205, 1982.

[24] S. Tatikonda, A. Sahai, and S. Mitter. Stochastic linear control over a communication channels. *IEEE Transactions on Automatic Control*, 49:1549–1561, September 2004.

[25] D. Teneketzis. On the structure of optimal real-time encoders and decoders in noisy communication. *IEEE Transactions on Information Theory*, 52:4017–4035, September 2006.

[26] R. Venkataramanan and S. S. Pradhan. Source coding with feed-forward: Rate-distortion theorems and error exponents for a general source. *IEEE Transactions Information Theory*, 53:2154–2179, June 2007.

[27] J. C. Walrand and P. Varaiya. Optimal causal coding-decoding problems. *IEEE Transactions on Information Theory*, 19:814–820, November 1983.

[28] H. S. Witsenhausen. On the structure of real-time source coders. *Bell Syst. Tech. J*, 58:1437–1451, July/August 1979.

[29] S. Yüksel. Jointly optimal lqg quantization and control policies for multi-dimensional linear Gaussian sources. *submitted*.

[30] S. Yüksel. On optimal causal coding of partially observed Markov sources in single and multi-terminal settings. *IEEE Transactions Information Theory, to appear (available on arXiv)*.

[31] S. Yüksel and T. Linder. On optimal zero-delay quantization of vector Markov sources. In *Proceedings of the IEEE Conference on Decision and Control, Hawaii*, December 2012.

[32] S. Yüksel and T. Linder. Optimization and convergence of observation channels in stochastic control. *SIAM J. on Control and Optimization*, 50:864–887, 2012.

[33] S. Yüksel and S. Tatikonda. A counterexample in distributed optimal sensing and control. *IEEE Transactions on Automatic Control*, 54:841–844, April 2009.