potential for analysis and design of very complicated fractional-order control systems with time delay.

The future direction in this research is to make more efforts on changing of the orders of FOPID controller. Furthermore, the choosing of the controller providing the optimal control in the global stability region can be investigated. To achieve this, it is needed to obtain the curves of the important time domain specifications such as settling time and maximum overshoot. Because the frequency and time domain performances will be met on a single plane, the designer can easily decide about choosing the controller parameters according to the desired performance.

REFERENCES

[1] K. J. Astrom and T. Hagglund, *Advanced PID Control*. Research T. Park, NC: ISA, 2005.

[2] M. Johnson, J. Croweand, and M. H. Moradi, *PID Control: New Identification and Design Methods*. London, U.K.: Springer, 2005.

[3] A. Datta, M.-T. Ho, and S. P. Bhattacharyya, *Structure and Synthesis of PID Controllers*. London, U.K.: Springer-Verlag, 2000.

[4] C. Hwang and Y.-C. Cheng, "A numerical algorithm for stability testing of fractional delay systems," *Automatica*, vol. 42, no. 5, pp. 825–831, 2006.

[5] I. Podlubny, "Fractional-order systems and $PI^{\lambda}D^{\mu}$-controllers," *IEEE Trans. Autom. Control*, vol. 44, no. 1, pp. 208–214, 1999.

[6] J. F. Leu, S. Y. Tsay, and C. Hwang, "Design of optimal fractional-order PID controllers," *J. Chinese Inst. Chem. Eng.*, vol. 33, no. 2, pp. 193–202, 2002.

[7] C. A. Monje, B. M. Vinagre, V. Feliu, and Y. Q. Chen, "On auto-tuning of fractional order $PI^{\lambda}D^{\mu}$ controllers," in *Proc. FDA 2006 Fractional Derivatives and Appl.*, Porto, Jul. 19–21, 2006.

[8] C. Hwang, J.-F. Leu, and S.-Y. Tsay, "A note on time-domain simulation of feedback fractional-order systems," *IEEE Trans. Autom. Control*, vol. 47, no. 4, pp. 625–631, 2002.

[9] S. Manabe, "Early development of fractional order control," in *Proc. ASME 2003 Design Eng. Tech. Conf.*, Chicago, IL, Sep. 2–6, 2003.

[10] G. J. Silva, A. Datta, and S. P. Bhattacharyya, "New results on the synthesis of PID controllers," *IEEE Trans. Autom. Control*, vol. 47, no. 2, pp. 241–252, 2002.

[11] N. Tan, "Computation of stabilizing lag/lead controller parameters," *Comp. Elect. Eng.*, vol. 29, pp. 835–849, 2003.

[12] M. T. Soylemez, N. Munro, and H. Baki, "Fast calculation of stabilizing PID controllers," *Automatica*, vol. 39, pp. 121–126, 2003.

[13] S. E. Hamamci and N. Tan, "Design of PI controllers for achieving time and frequency domain specifications simultaneously," *ISA Trans.*, vol. 45, no. 4, pp. 529–543, 2006.

[14] J. I. Neimark, "D-decomposition of the space of quasi-polynomials (on the stability of linearized distributive systems)," in *Amer. Math. Soc. (AMS) Transl.*, ser. 2. Providence, R. I.: AMS Soc., 1973, vol. 102, Ten papers in analysis, pp. 95–131.

[15] P. J. Lawrenson and S. R. Bowes, "Generalization of D-decomposition techniques," *Proc. Inst. Elect. Eng.*, vol. 116, pp. 1463–1470, 1969.

[16] J. McKay, "The D-partition method applied to systems with dead time and distributed lag," *Meas. Control*, vol. 3, pp. 293–294, 1970.

[17] S. L. Cheng and C. Hwang, "On stabilization of time-delay unstable systems using PID controllers," *J. Chinese Inst. Chem. Eng.*, vol. 30, no. 2, pp. 123–140, 1999.

[18] Y. M. Borushko and V. M. Vartanyan, "Automated synthesis of automated control systems using regions of specified quality," *Soviet J. Automation Inf. Sci.*, vol. 21, no. 3, pp. 64–66, 1988.

[19] A. T. Shenton and Z. Shafiei, "Relative stability for control systems with adjustable parameters," *J. Guid., Control, Dyn.*, vol. 17, pp. 304–310, 1994.

[20] C.-H. Chang and K.-W. Han, "Gain margins and phase margins for control systems with adjustable parameters," *J. Guid., Control, Dyn.*, vol. 13, no. 3, pp. 404–408, 1990.

[21] Y.-C. Cheng and C. Hwang, "Stabilization of unstable first-order time-delay systems using fractional-order PD controllers," *J. Chinese Inst. Eng.*, vol. 29, no. 2, pp. 241–249, 2006.

[22] R. E. Bellman and K. L. Cooke, *Differential-Difference Equations*. New York: Academic, 1963.

[23] K. Ogata, *Modern Control Engineering*. Englewood Cliffs, NJ: Prentice-Hall, 1970.

Optimal Signaling Policies for Decentralized Multicontroller Stabilizability Over Communication Channels

Serdar Yüksel and Tamer Başar

***Abstract*—In this note, we study the problem of distributed control over communication channels, where a number of distributed stations collaborate to stabilize a linear system. We quantify the rate requirements and obtain optimal signaling, coding and control schemes for decentralized stabilizability in such multicontroller systems. We show that in the absence of a centralized decoder at the plant, there is in general a rate loss in decentralized systems as compared to a centralized system. This result is in contrast with the absence of rate loss in the stabilization of multisensor systems. Furthermore, there is rate loss even if explicit channels are available between the stations. We obtain the minimum data rates needed in terms of the open-loop system matrix and the connectivity graph of the decentralized system, and obtain the optimal signaling policies. We also present constructions leading to stability. In addition, we show that if there are dedicated channels connecting the controllers, rate requirements become more lenient, and as a result strong connectivity is not required for decentralized stabilizability. We determine the minimum number of such external channels leading to a stable system, in case strong connectivity is absent.**

***Index Terms*—Cooperative control, decentralized stabilization, distributed control, information theory.**

I. INTRODUCTION AND LITERATURE REVIEW

Decentralization in control systems has become ubiquitous with the use of digital and wireless channels such as the Internet or dedicated bus lines [in a controller area network (CAN)] in control systems. Some typical applications include environmental detection, rescue operations, traffic management, formation control, and aerospace applications; see for instance [1]. In such remote control problems, one major concern is the characterization of the minimum amount of information transfer needed for a satisfactory performance. This information transfer could be between various components of a networked control system, as depicted in Fig. 1. Such networks and channels bring up many challenges, since they involve two disciplines which are still in their infancy, namely the decentralized control and the multiterminal information theories.

Decentralized stabilization has attracted considerable interest in the literature [2], [3]. One of the accomplishments in this domain is the introduction of *decentralized fixed modes* [4] and graph-theoretic characterization of stability [3] (see also [2]).

In distributed control problems, it is possible for the controllers to communicate through the plant [5]. The process of communicating via the plant is known as *signaling*. Most of the control literature on signaling is concerned with indirect aspects of signaling in optimal control

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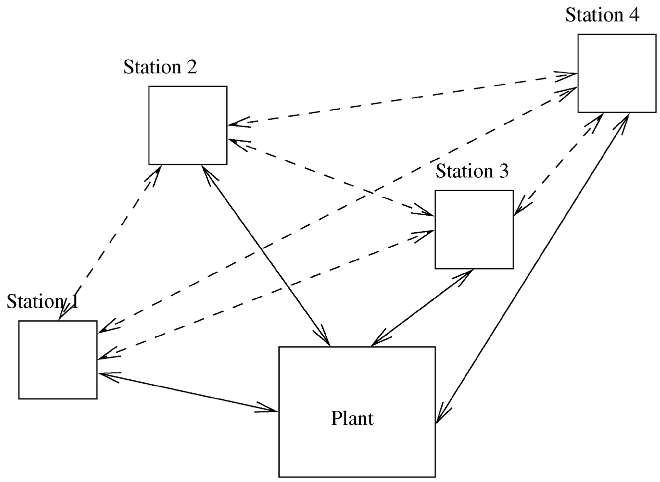

Fig. 1. Distributed control system. Under information structure A (IS A) controllers use local information. Under IS B, there can be external communication between controllers: dashed lines depict the possible communication links between the stations.

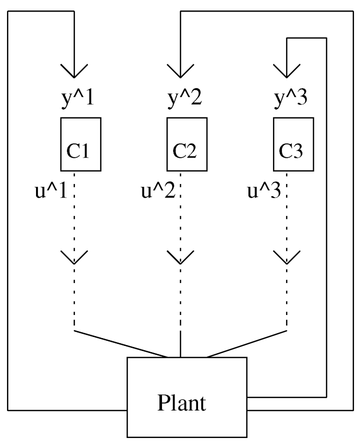

Fig. 2. Multicontroller system structure: There does not exist a centralized decoder. In a multisensor case, there exists a centralized decoder.

problems: [6] showed that for an LQR problem, when there is no incentive for signaling, then certain convexity properties are preserved leading to optimality of linear policies. [7] provided such a condition when there is no incentive for signaling. This paper, however, provides a new framework to the signaling problem, when there is an incentive to do so.

As regards to communication theoretic issues, most of the efforts in the literature have been under the assumption that the system exhibits either the multisensor structure or the multicontroller structure. One important difference between the multicontroller and the multisensor setups is the following (see Fig. 2): In a multisensor structure, there exists a centralized controller which assembles the observations from multiple sensors, generates an estimate (as in a fusion center) and computes the control. However, in a multicontroller setup, there is no centralized decoder at the plant. This is due to the fact that in a realistic scenario, the plant should be merely acting on the control signals received, for otherwise there would not be any need for the transmission of control over a communication channel. It should be observed, however, that the plant can still have local feedback control, and the discussion here is with regard to the control signals sent over to a remote location. If the plant is able to do filtering with regard to the controller actions, then the results with regard to the multisensor setup will be applicable. As an example, consider a robot vehicle which is being remotely controlled. The remote controller can develop an estimate on the vehicle's position, using the system dynamics, past received observations and the previously transmitted control signals. However, the vehicle should be designed so that it can act on any of the commands generated by the remote controllers, such as reducing acceleration, changing direction and so forth, even when the commands are random and hence unpredictable.

For distributed systems exhibiting the multisensor structure, due to the assumption of a centralized decoder, one can use Slepian–Wolf coding theorem to arrive at the rate requirements. [8] shows the absence of rate loss in noiseless multisensor systems with a centralized controller. We note that the Slepian–Wolf framework is not required if one allows the use of time-varying policies, an approach we adopt in this paper. Also see the references in [9], and [12], [13], [11] and [10].

1) Main Results: This paper deals with distributed systems exhibiting the multicontroller structure. The main contributions can be summarized as follows.

1) For multicontroller decentralized systems, we obtain the minimal sum-rate and present control, signaling and sensing schemes which minimize the sum-rate required for decentralized stabilizability. We also show that there is in general a strict rate loss in multicontroller decentralized systems, as compared to a centralized system. This is in contrast with multisensor decentralized systems.
2) Explicit channels between the stations might improve the communication requirements. If explicit channels exist between controllers, strong connectivity is not required for decentralized stabilizability. This note obtains the minimum number of such channels.

A. Problem Formulation

We consider the class of multistation n-dimensional discrete-time LTI systems

$$
\begin{aligned}
x_{t+1} &= Ax_t + \sum_{j=1}^{L} B^j u_t^j \\
y_t^i &= C^i x_t, \qquad 1 \leq i \leq L
\end{aligned} \tag{1}
$$

where $(A, [B^1|B^2|\ldots|B^L])$ is controllable and $(A, [(C^1)^T|(C^2)^T|\ldots|(C^L)^T]^T)$ is observable, but the individual pairs may not enjoy this, such as (A, B^i) may not be controllable or (A, C^i) may not be observable, for $1 \leq i \leq L$. Here, $x_t \in R^n$, is the state of the system, $u_t^i \in R^{m_i}$ is the control applied by station i, and $y_t^i \in R^{p_i}$ is the observation available at station i at time t. Without any loss of generality, we assume the system matrix A to be in Jordan form. The initial state x_0 is a random vector with a continuous probability density function over a bounded support set, $\mathcal{X}_0 \subset R^n$. We consider two Information Structures (ISs), IS A, and IS B. In the first one, there is no external communication between controllers. In the second structure, stations can communicate via external channels.

1) IS A: Under this structure, the information available to station i at time t is

$$I_t^i = \{y_{[0,t]}^i, u_{[0,t-1]}^i\}$$

where, $u_{[0,t-1]}^i$ denotes the sequence $\{u_0^i, u_1^i, \ldots, u_{t-1}^i\}$ and $y_{[0,t]}^i = \{y_0^i, y_1^i, \ldots, y_t^i\}$.

2) IS B: Under this information structure, the information available to station i at time t becomes

$$I_t^i = \{y_{[0,t]}^i, u_{[0,t-1]}^i, Z_{[0,t-1]}^i\}$$

where $Z^i_{[0,t-1]}$ denotes the information provided to station i by other stations through external channels up to time $t-1$:

$$Z^i_{[0,t-1]} = \{z_s^{m,i}, 0 \le s \le t-1,\ m \ne i\}$$

where $z_s^{m,i}$ denotes the message transmitted from station m to station i at time s.

The control signals $u^i, \forall i$, and the external message signals, $z^{i,j}, \forall i, j, j \ne i$, are coded and decoded over discrete noiseless channels with finite capacity. Hence, the applied control and transmitted messages follow a coding, binary representation, and a decoding process. We assume fixed-rate encoding, that is, the rate is defined as the (base-2) logarithm of the number of symbols to be transmitted: The coding process of the controller at station i is measurable with respect to the sigma-algebra generated by I_t^i to $\{1, 2, \ldots, W_t^i\}$, which is the quantizer codebook at station i at time t. Hence, at each time t, station i sends $\log_2(W_t^i)$ bits over the channel to the plant. Under IS B, the external signaling process of the controller at station i to station m is a mapping from $\sigma(I_t^i)$ to $\{1, 2, \ldots, W_t^{i,m}\}$. Hence, at each time stage, t, station i sends $\log_2(W_t^{i,m})$ bits over an external channel to station m.

Finally, we describe the information available at the plant. The plant knows the codebooks used by each controller, to be able to perform decoding. The decoder output at the plant with regard to the information received from station i at time t, which we again denote as $u_t^i \in R^{m_i}$, is generated through a (memoryless) mapping from $\{1, 2, \ldots, W_t^i\}$ to R^{m_i}. In addition to the quantizer policy and codebook, the plant also knows when the controllers at each station choose to signal information or to apply control. This is needed to ensure that the plant is capable of negating the effect of signaling.

3) Problem Statement: Under IS A, let $\mathcal{R}_\mathcal{A}$ denote the set of average (over time horizon) rates on L channels which lead to (decentralized) stabilization, i.e.,

$$\mathcal{R}_\mathcal{A} = \Big\{R^i, i \in 1, 2, \ldots, L : \exists \{u^1_{[0,\infty)}, u^2_{[0,\infty)}, \ldots, u^L_{[0,\infty)}\}, \lim_{T\to\infty} E[|x_T|^2] = 0\Big\}$$

where $R^i = \limsup_{T\to\infty} (1/T)\sum_{t=1}^T \log_2(W_t^i)$. Define $R_A := \min_{\mathcal{R}_\mathcal{A}}\{\sum_{i=1}^L R^i\}$, such that decentralized stabilization is possible. Under IS B, let $\mathcal{R}_\mathcal{B}$ denote the set of average (over time horizon) rates on L^2 channels which lead to (decentralized) stabilization, i.e.

$$\mathcal{R}_\mathcal{B} = \Big\{R^i, R^{i,j}, i \ne j \in 1, 2, \ldots, L : \exists \{u^1_{[0,\infty)}, u^2_{[0,\infty)}, \ldots, u^L_{[0,\infty)}\}, \lim_{T\to\infty} E[|x_T|^2] = 0\Big\}$$

where $R^i = \limsup_{T\to\infty} (1/T)\sum_{t=1}^T \log_2(W_t^i)$, and $R^{i,j} = \limsup_{T\to\infty} (1/T)\sum_{t=1}^T \log_2(W_t^{i,j})$.

Define $R_B := \min_{\mathcal{R}_\mathcal{B}}\{\sum_{i=1}^L R^i + \sum_{i=1}^L \sum_{j\ne i} R^{i,j}\}$, such that decentralized stabilization is possible. We are interested in the minimum achievable average sum rates, R_A and R_B, under both information structures. ◇

We refer to [14] for background on the relevant information theoretic preliminaries. We next introduce some relevant graph-theoretic notions: A *directed graph* $\mathcal{G}$ consists of a set of vertices, $\mathcal{V}$, and a set of directed edges, $(a,b) \in \mathcal{E}$, such that $a, b \in \mathcal{V}$. A path in $\mathcal{G}$ of length d consists of a sequence of d directed edges such that each edge is connected. A graph in which there exists a path from any node to any other is a strongly connected graph. We define the minimum distance between two sets of nodes $S_1, S_2 \subset \mathcal{G}$ as $d(S_1, S_2) = \min\{d(i,j), i \to j, i \in S_1, j \in S_2\}$, where $d(i,j)$ denotes the number of paths between node i and j, with the trivial case being $d(i,i) = 0$ for all nodes.

II. Decentralized Stabilizability

We denote the controllability matrix for station i by $\mathcal{C}^i$, where

$$\mathcal{C}^i = [\, B^i | AB^i | \ldots | A^{n-1}B^i \,]$$

and the observability matrix for station j by $\mathcal{O}^j$, which is given by

$$\mathcal{O}^j = [\, (C^j)^T | (C^jA)^T | \ldots | (C^jA^{n-1})^T \,]^T .$$

We let N^i denote the unobservable subspace of station i, and K^i denote its controllable subspace. In other words, N^i is the null-space of $\mathcal{O}^i$, and K^i is the range space of $\mathcal{C}^i$. We define O^i to be the subspace orthogonal to N^i, and call it the observable subspace. Such subspaces can be computed via the Hautus-Rosenbrock test. We define a *mode* of a linear system as an eigenvector corresponding to an (open-loop) eigenvalue. The set of unstable modes is the set of eigenvectors of the system matrix corresponding to unstable eigenvalues. We let $\{x^i\}$ be the subspace of the eigenvectors corresponding to λ_i. These notions naturally extend to generalized eigenvectors and to complex eigenvalues.

We next review the notion of connectivity. If $N^j \not\supset K^i$, then station i can affect the observations of station j, and thus, communicate to station j via control [5], which we capture through the notation $i \to j$. In this case, station i is said to be *connected* to station j. If every station is connected to every other station, possibly via other stations, the system is said to be strongly connected. Through communication via the plant, the controllable subspace can be expanded and the unobservable subspace can be shrunk [5]. The following is from [3]:

Decentralized stabilization in a multicontroller setting is possible, if the system is jointly controllable, jointly observable, and strongly connected. Such a stabilization is in general possible through time-varying controls.

The reader is referred to [2] for conditions of decentralized stabilization with a more restrictive set of controllers (such as, time-invariant, output feedback controllers, for which the absence of decentralized fixed modes is required in addition to joint controllability, observability). Hereafter, while performing the rate analysis, we will assume that the system is decentrally stabilizable in the sense of [3], in the absence of any structural restriction on the class of controllers.

III. Decentralized Stabilization of Multicontroller Systems Under IS A

One of the main results of this note is the following.

Theorem 3.1: Consider IS A. Assume that unrestricted decentralized stabilization is possible, that is, the system is jointly controllable, jointly observable, and strongly connected. Suppose the eigenvalues corresponding to each of the (possibly generalized) eigenvectors are ordered as $\{n_1, n_2, \ldots, n_n\}$, and $\{x^{n_1}, x^{n_2}, \ldots, x^{n_n}\}$ denote the ordered eigenvectors. Let $\mathcal{N}$ be the set of such orderings, and for $i \ge 1$, $M_i := \mathrm{span}(x^{n_1}, x^{n_2}, \ldots, x^{n_n})$. Then, a lower bound on the sum rate required, R_A, between the controllers and the plant for stabilizability is given by

$$\min_{\{n_1,n_2,\ldots,n_n\}\in\mathcal{N}} \left\{ \sum_{|\lambda_i|>1} (1+\eta_{M_i})\Big(\log_2(|\lambda_i|)\Big)\right\} \tag{2}$$

where

$$\eta_{M_i} := \min_{\mathcal{K}\subset\{1,2,\ldots,L\},m} \left\{ \sum_{j\in\mathcal{K}} d(j,m) : j \to m, \{x^i\} \subset \cup_{j\in\mathcal{K}} O^j \cup O^m \cup M_i\{x^i\} \subset K^m \cup M_i \right\} \tag{3}$$

is the minimum number of paths connecting the stations which can observe mode i to the stations which can control it given that $(x^{1,2,\dots,i-1})$ at the controllers is set to zero. ⋄

The following example helps to illustrate the preceding result. Consider the two-controller system

$$x_{t+1} = Ax_t + \sum_{j=1}^{2} B^j u_t^j, \qquad t \geq 0$$
$$y_t^i = C^i x_t, \qquad i = 1,2 \tag{4}$$

with $A = \mathrm{diag}(2,2)$, $B^1 = [0 \quad 1]^T$, $C^1 = [1 \quad 1]$, $B^2 = [1 \quad 0]^T$, $C^2 = [1 \quad -1]$.

Here, neither of the stations can recover the modes of the system independently; however the system is decentrally stabilizable (even by linear policies). If the system were centralized the average rate needed would be 2 bits. In the decentralized case, however, the minimum achievable average sum rate under the information structure for the controllers and the plant is 3 bits.

Toward the proof of Theorem 3.1, we present a number of lemmas, the first one being related to connectivity.

Lemma 3.1: $i \rightarrow j$ if and only if $C^j(A)^l B^i \neq 0$, for at least one l, $1 \leq l \leq n$.

Proof:

i) Suppose that $C^j(A)^l B^i \neq 0$ for at least one l. This implies the existence of a control u such that $\mathcal{O}^j \mathcal{C}^i u \neq 0$.

ii) The observation at station j, in the presence of controls from station i, is

$$y_t^j = C^j(A)^t x_0 + \sum_{k=0}^{t-1} C^j(A)^{t-k-1} B^i u_k^i.$$

If all the terms $C^j(A)^l B^i$ are zero, for $1 \leq l \leq n$, then via the Cayley–Hamilton theorem, $C^j A^l B^i = 0$ for all $l \in \mathrm{Z}^+$. Thus, we have $y_t^j = C^j(A)^t x_0$. Hence, the control of station i does not affect the observation of station j.

⋄

Lemma 3.2: For a sequence of scalar continuous random variables $\{v_t, t = 1,2,\dots\}$ converging to zero in the mean-square sense, $\lim_{t\to\infty} E[v_t^2] = 0$, the (differential) entropy has the property that $\limsup_{t\to\infty} H(v_t) = -\infty$.

Proof: Fix an $\epsilon > 0$. Then, there exists a t_0, such that for every $t > t_0$, $E[v_t^2] \leq \epsilon$. For a zero-mean random variable with a fixed second moment, the entropy is maximized by a Gaussian random variable with the same second moment. Hence, for every $t \geq t_0$, $H(v_t) \leq (1/2)\log(2\pi E[v_t^2]) \leq (1/2)\log(2\pi\epsilon)$, and the result follows, since ϵ can be taken to be arbitrarily small, and $\lim_{\epsilon\to 0}\log(2\pi\epsilon) = -\infty$. ⋄

Lemma 3.3: Suppose that the initial state has a continuous probability density function with a bounded support set. To make a particular mode, x^i, corresponding to an unstable eigenvalue λ_i mean-square stable, the average rate of information that needs to be transmitted from the initial state to the control station is $\log_2(|\lambda_i|)$. The information that needs to be transmitted from the stations which can observe the mode to the station(s) equipped with the capability of controlling the mode is lower bounded by $\log_2(|\lambda_i|) - \limsup_{t\to\infty}(1/t)I(x_0^i; y_0^{S_i t}|x_0^{[n_1,n_i-1]})$, where y^{S_i} denotes the observations at the set of nodes satisfying $S^i = \{m : \{x^i\} \subset (K^m \cup M_i)\}$.

Proof: Let $\mathbf{x}^{\boldsymbol{\lambda}}$ be the eigenspace of λ, i.e., the space spanned by the eigenvectors corresponding to eigenvalue λ. Then, the dynamics of the ith mode can be written as

$$x_t^i = \lambda_i^t x_0^i + \left(\sum_k 1_{x^k \in \mathbf{x}^{\boldsymbol{\lambda}}, k \neq i} f(x_0^k)\right) + B^{S_i}(u_0^{S_i t})$$

where $1_{(.)}$ denotes the indicator function, $f(.)$ is some function that depends on the Jordan form of the system matrix, and the superscript S_i denotes the stations that can control mode i, i.e., $x^i \in K^m$ for station m in S_i, $u_0^{S_i t}$ denotes the control sequence applied by such stations, and $B^{S_i}(.)$ denotes the mapping from the set of applied controls to the state. The evolution equation above follows from the fact that modes sharing a similar eigenvalue are decoupled from other modes and the intradependency of these modes does not involve cross terms, given the initial values. Let $O_i = \{m : \{x^i\} \subset (\bigcup O^m) \cup M_i\}$. We have

$$\begin{aligned} H(x_t^i) &= H\left(\lambda_i^t x_0^i + \left(\sum_k 1_{x^k \in \mathbf{x}^{\boldsymbol{\lambda}}, k\neq i} f(x_0^k)\right) + B^{S_i} u_t^{S_i}\right) \\ &\geq H(\lambda_i^t x_0^i + f(u_0^{S_i t}) | u_0^{S_i t}, x^k \in \mathbf{x}^{\boldsymbol{\lambda}}, k \neq i) \\ &= H(\lambda_i^t x_0^i | u_0^{S_i t}) \\ &= t\log_2(|\lambda_i|) + H(x_0^i | u_0^{S_i t}). \end{aligned} \tag{5}$$

The first inequality above follows from the fact that conditioning does not increase the entropy. It now follows from (5), by rearranging terms, that $H(x_0^i) - H(x_0^i|u_0^{S_i t}) \geq H(x_0^i) + t\log_2(|\lambda_i|) - H(x_t^i)$. Since the initial state has finite entropy, the sequence $\{x_t^i\}$ converges to zero in the mean-square sense, and, by Lemma 3.2, $\lim_{t\to\infty} H(x_t^i) = -\infty$, it follows that

$$\begin{aligned} &\liminf_{t\to\infty} \frac{1}{t}\{H(x_0^i) - H(x_0^i|u_0^{S_i t})\} \\ &\quad \geq \liminf_{t\to\infty} \frac{1}{t}\{t\log_2(|\lambda_i|) + H(x_0^i) - H(x_t^i)\} \\ &\quad \geq \log_2(|\lambda_i|). \end{aligned} \tag{6}$$

We have that $u_0^{S_i t}$ is a causal function of the information available $(y_0^{S_i t}, y_0^{O_i t})$ and thus by the data processing inequality it follows that $I(x_0; u_0^{S_i t}) \leq I(x_0; y_0^{S_i t}, y_0^{O_i t})$ and $\mathrm{liminf}_{t\to\infty}(1/t)I(x_0^i; y_0^{S_i t}, y_0^{O_i t}) \geq \log_2(|\lambda_i|)$. Further, it follows that $\mathrm{limsup}_{t\to\infty}(1/t)I(x_0^i; y_0^{S_i t})$ is finite and

$$\begin{aligned} &\mathrm{liminf}_{t\to\infty} \frac{1}{t} I(x_0^i; y_0^{O_i t} | y_0^{S_i t}) \\ &\quad \geq \log_2(|\lambda_i|) - \mathrm{limsup}_{t\to\infty} \frac{1}{t} I(x_0^i; y_0^{S_i t}) \\ &\quad \geq \log_2(|\lambda_i|) - \mathrm{limsup}_{t\to\infty} \frac{1}{t}\{I(x_0^i; y_0^{S_i t}|x_0^{[n_1,n_i-1]}) \\ &\qquad + I(x_0^i; x_0^{[n_1,n_i-1]})\}. \end{aligned}$$

Due to linearity

$$I(x_0^i; y_0^{S_i t}|x_0^{[n_1,n_i-1]}) = I(x_0^i; y_0^{S_i t}|x_0^{[n_1,n_i-1]} = 0).$$

Result follows, since $\lim_{t\to\infty}(1/t)I(x_0^i; x_0^{[n_1,n_i-1]}) = 0$. ⋄

Lemma 3.4: A total average rate of at least

$$\eta_{M_i}\left(\log_2(|\lambda_i|) - \lim_{t\to\infty} \frac{1}{t} I(x_0^i; y_0^{S_i t}|x_0^{[n_1,n_i-1]})\right)$$

bits is needed for the stabilizability of the ith mode.

Proof: Let $O_i = \{m : \{x^i\} \subset (\bigcup O^m) \cup M_i\}$. Consider the paths from O_i to S_i. Let $z_0^{S_i t}$ denote the signaling outputs received by

S_i up to time t, from S_i. Following a conditional version of the data processing inequality, we have

$$\begin{aligned}&\frac{1}{t}I(x_0^i; z_0^{S_i t}, y_0^{S_i t}|x_0^{[n_1,n_i-1]})\\&=\frac{1}{t}H(x_0^i|x_0^{[n_1,n_i-1]})-H(x_0^i|z_0^{S_i t}, y_0^{S_i t}, x_0^{[n_1,n_i-1]})\\&\le\frac{1}{t}H(x_0^i|x_0^{[n_1,n_i-1]})-H(x_0^i|z_0^{S_i t}, u_0^{O_i t}, y_0^{S_i t}, x_0^{[n_1,n_i-1]})\\&=\frac{1}{t}H(x_0^i|x_0^{[n_1,n_i-1]})-H(x_0^i|u_0^{O_i t}, y_0^{S_i t}, x_0^{[n_1,n_i-1]})\\&=\frac{1}{t}I(x_0^i; u_0^{O_i t}|y_0^{S_i t}, x_0^{[n_1,n_i-1]})+I(x_0^i; y_0^{S_i t}|x_0^{[n_1,n_i-1]}).\end{aligned}$$

The inequality follows from the fact that conditioning does not increase the entropy. The next equality follows from the fact that $z_0^{S_i t}$ is a deterministic function of $u_0^{O_i t}$. Other lines follow from basic properties of mutual information [14]. Hence

$$\liminf_{t\to\infty}\frac{1}{t}I(x_0^i; u_0^{O_i t}|y_0^{S_i t}, x_0^{[n_1,n_i-1]}) \ge \log_2(|\lambda_i|)-\limsup_{t\to\infty}\frac{1}{t}I(x_0^i; y_0^{S_i t}|x_0^{[n_1,n_i-1]}).$$

Upon obtaining the minimum number of paths in a connectivity graph, a repeated application of the above arguments (which follows from a conditional version of the data processing inequality) to each of η_{M_i} paths leads to the desired result. ◇

We note that the above admits a max-flow min-cut interpretation (Fig. 3) over a temporal graph. One can approach the problem as information transfer over a network, where the rate of information flow across any cut is less than the mutual information between the inputs on either side of the cut conditioned on the inputs on the other side of the cut.

Lemma 3.5: $\limsup_{t\to\infty}(1/t)I(x_0^i; y_0^{mt}|x_0^{[n_1,n_i-1]})$ is equal to either $\log_2(|\lambda_i|)$ or 0. It is 0 if $\{x^i\}\not\subset O^m\cup M_i$.

Proof: Without any loss of generality, we have y^m as a linear combination of modes with the same eigenvalue. If $\{x^i\}\not\subset(O^l\cup M_i)$, then

$$\begin{aligned}&\limsup_{t\to\infty}\frac{1}{t}I(x_0^i; y_0^{mt}|x_0^{[n_1,n_i-1]})\\&\quad=\limsup_{t\to\infty}\frac{1}{t}\{t\log_2(|\lambda_i|)-H(y_0^m|x_0^i, x_0^{[n_1,n_i-1]})\\&\qquad-t\log_2(|\lambda_i|)+H(y_0^m|x_0^{[n_1,n_i-1]})\}=0.\end{aligned}\tag{7}$$

The other case can be analyzed in a similar fashion. ◇

Proof of Theorem 3.1: If there exists a station for a mode i such that $\{x^i\}\subset(O^i\cap K^i)\bigcup M_i$, then there is no need to relay any information through the plant. Otherwise the controllers should signal information through the plant. Following Lemma 3.3, the information should be at least $\log_2(|\lambda_i|)$ for each mode. The signaling has to be through the controllers that are connected, following Lemma 3.1. Lemmas 3.4 and 3.5 show that the minimum number of information bits has to be transmitted over the connected paths. Once one mode is stabilized, one can proceed on to the next mode. As is observed in Lemma 3.3, one can set the preceding modes to zero, to obtain a further lower bound. This sequential analysis is applicable to all the unstable modes. Upon obtaining a minimizing path for the entire process, the following rate is obtained as a lower bound:

$$\min_{\{n_1,n_2,\ldots,n_n\}\in\mathcal{N}}\left\{\sum_{|\lambda_i|>1}\eta_{M_i}\left(\log_2(|\lambda_i|)\right)\right\}\tag{8}$$

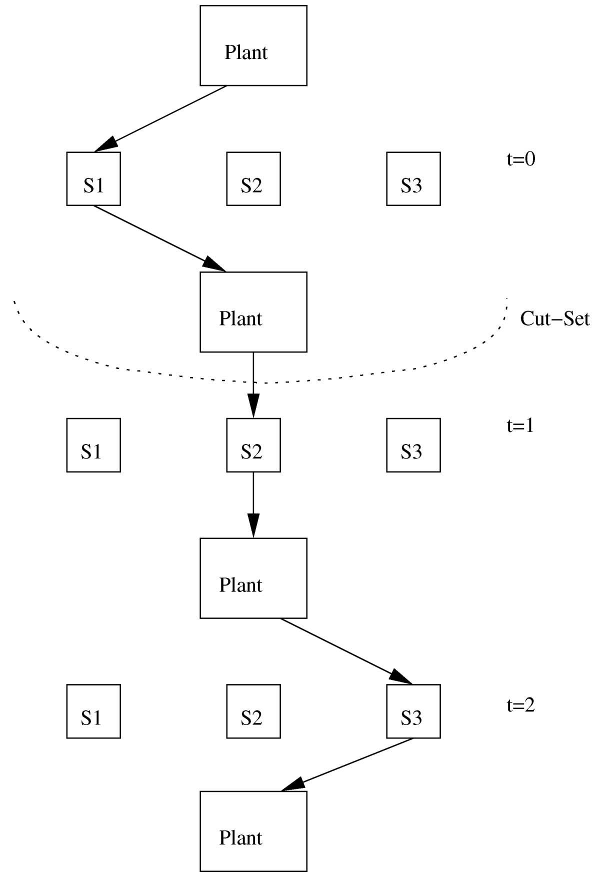

Fig. 3. A max-flow min-cut interpretation. In the figure, station 1 observes a mode, relays it to station 2, which further relays it to station 3.

where $\mathcal{N}$ and η_{M_i} are defined in the statement of Theorem 3.1. This completes the proof. ◇

We now state a result on the tightness of the lower bound provided in Theorem 3.1.

Theorem 3.2: There exist stabilizing coding and control policies whose sum rate is arbitrarily close to the lower bound provided in (9). Hence, this bound is asymptotically achievable. ◇

Toward the proof, we again first state and prove a lemma.

Lemma 3.6: Station l can compute $C^lA^nx_n$, at time n.

Proof: At time n, the station will have access to $x_0^{O_l}$. One can write

$$\begin{aligned}&C^lA^nx_n\\&=C^l\left(\sum_{i=0}^{n-1}\alpha_iA^i\right)x_n=C^l\left(\sum_{i=0}^{n-1}\alpha_iA^i\right)A^nx_0\\&=C^l\left(\sum_{i=0}^{n-1}\alpha_iA^{i+n}\right)x_0=C^l\left(\sum_{i=0}^{n-1}\alpha_i'A^i\right)x_0=\sum_{i=0}^{n-1}\alpha_i'y_i^l\end{aligned}$$

where $\alpha_i, 1\le i\le n$, and $\alpha_i', 1\le i\le n$ can be obtained via the Cayley–Hamilton theorem. This completes the proof. ◇

Proof of Theorem 3.2: Without any loss of generality, suppose that there exists only one station, station l, that can control a mode i, and only one station, $k\ne l$, that can observe a mode x^i and $k\to l$. Then, the information on mode x^i is to be sent to station l through the plant. Suppose x_0^i is to be sent to station l. Sensor k recovers x_0^i at a time no later than n. It then quantizes x_0^i uniformly. Station k sets

$u_t^k = q_t(x_0^i)$, where q_t denotes the quantization operation at time t. In this case

$$\begin{aligned} x_{n+1} &= Ax_n + B^k q_n(x_0^i) \\ y_{n+1}^l &= C^l(Ax_{n+1} + B^k q_n(x_0^i)). \end{aligned}$$

Assembling the observations $y_{n+1}^l, y_{n+2}^l, \ldots, y_{2n}^l$, and following the fact that $C^l(A)^m B^k \neq 0$ for at least one $m, 1 \leq m \leq n$, and Lemma 3.6 , the quantized output $q_n(x_0^i)$ can be recovered at a time no later than $2n$. Sensor l can recover the quantized information $q_n(x_0^i)$, which it subsequently sends to station l. Via this information, the estimate at time $2n$, $\hat{x^i}_0(2n)$, can be computed. Let $p > 0$ be an integer. If an average quantization rate of $R = n \log_2 |\lambda_i| + \epsilon$, for some $\epsilon > 0$ is used, then the estimation error $x_0^i - \hat{x^i}_0(pn)$ approaches zero at a rate faster than $1/(|\lambda_i|)^{pn}$.

The plant undoes the signaling, since it is assumed to know the control protocol. The controller can drive the estimated value to zero in at most n time stages. Finally, we need to consider multiple transmissions. The remaining controllers can be designed to be idle, while a particular mode is being relayed by the plant. Such a sequential scheme ensures convergence. ϵ can be taken arbitrarily small via adjusting the time-stages. ◇

IV. Decentralized Stabilization of Multicontroller Systems Under IS B

We now consider IS B, that is there are external links between the stations. Under IS B, the communication requirements might be relaxed, since the incentive for signaling through the plant disappears. This observation leads to the following theorem, which we state without a proof.

Theorem 4.1: Consider information structure B. Let $\mathcal{N}$ be the set of orderings of the eigenvectors, and let M_i be defined as in Theorem 3.1. The minimum sum rate, R_B, required for decentralized stabilizability is given by

$$\min_{\{n_1, n_2, \ldots, n_n\} \in \mathcal{N}} \left\{ \sum_{|\lambda_i| > 1} \log_2(|\lambda_i|)(1 + 1_{\{x^{n_i}\} \not\subset M_i'}) \right\} \quad (9)$$

where, for $i \geq 1$

$$M_i' = \left(\bigcup_{l=1}^{L} (K^l \cap O^l) \right) \bigcup M_i.$$

There exist stabilizing policies whose sum rate is arbitrarily close to this rate. ◇

We note, however, that in case such dedicated channels were to exist, then one would not need strong connectivity.

Corollary 4.1: Suppose there are external channels between all of the stations. In this case, for the existence of a stabilizing decentralized design, it suffices for the system to be jointly controllable and jointly observable.

The presence of external channels can reduce the sum rate. If the system itself is not strongly connected, then one needs to artifically make the system strongly connected through the communication channels for stabilizability.

Theorem 4.2: Suppose the connectivity graph of the stations is such that cardinality of the set of disconnected station clusters $\mathcal{A}$, $|\mathcal{A}|$, is greater than one, that is, there exist at least two sets $\mathcal{S}_1, \mathcal{S}_2 \subset \mathcal{A}$ such that there does not exist any path connecting any node in $\mathcal{S}_1$ to any node in $\mathcal{S}_2$. Then, for stabilizability, there needs to be at most $|\mathcal{A}|$ external channels connecting any node within each cluster, to any other node in an another cluster.

Proof: The $|\mathcal{A}|$ links can be connected between the clusters to ensure a cyclic connectivity graph, so that there exists a path from any station to any other one. This ensures strong connectivity. ◇

Signaling has the effect of having the number of paths η_{M_i} at least two, and greater than two if more stations are involved. One could bypass the multipath transfers via external channels, and hence, the average sum rate performance improves if one adds external links between stations, which lie on the minimum sum-rate connections with $\eta_{M_i} > 2$. This leads to the following result.

Theorem 4.3: The minimum number of external channels needed to achieve the optimal sum-rate is given by adding direct links betwen the stations which lie on the control path for modes with $\eta_{M_i} > 2$. If $\eta_{M_i} = 2$, the addition of an external channel does not improve the performance. ◇

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References

[1] J. A. Fax and R. M. Murray, "Information flow and cooperative control of vehicle formations," *IEEE Trans. Automat. Control*, vol. 49, pp. 1465–1476, Sep. 2004.

[2] J. P. Corfmat and A. S. Morse, "Decentralized control of linear multivariable systems," *Automatica*, vol. 11, pp. 479–497, Sep. 1976.

[3] B. D. O. Anderson and J. B. Moore, "Time-varying feedback decentralized control," *IEEE Trans. Automat. Control*, vol. 26, pp. 1133–1139, Oct. 1981.

[4] S. Wang and E. J. Davison, "On the stabilization of decentralized control. systems," *IEEE Trans. Automat. Control*, vol. 18, pp. 473–478, Oct. 1973.

[5] H. Kobayashi, H. Hanafusa, and T. Yoshikawa, "Controllability under decentralized information structure," *IEEE Trans. Automat. Control*, vol. 23, pp. 182–188, Apr. 1978.

[6] A. Rantzer, "Linear quadratic team theory revisited," in *Proc. IEEE ACC*, Minneapolis, Jun. 2006.

[7] M. Rotkowitz, R. Cogill, and S. Lall, "A simple condition for the convexity of optimal control over networks with delays," in *Proc. IEEE CDC/ECC*, Sevilla, Spain, Dec. 2005.

[8] S. Tatikonda, "Some scaling properties of large distributed control systems," in *Proc. 42nd IEEE CDC*, Maui, Hawaii, Dec. 2003, pp. 3142–3147.

[9] S. Yüksel and T. Başar, "Communication constraints for decentralized stabilizability with time-invariant policies," *IEEE Trans. Autom. Control*, vol. 52, no. 6, pp. 1060–1066, Jun. 2007.

[10] S. Yüksel and T. Başar, "Minimum rate coding for LTI systems over noiseless channels," *IEEE Trans. Automat. Control*, vol. 51, no. 12, pp. 1878–1887, Dec. 2006.

[11] S. Jiang and P. Voulgaris, "Design of distributed controllers with constrained and noisy links," *Int. J. Robust Nonlinear Contr.*, vol. 16, pp. 891–912, 2006.

[12] C. Zhang and G. Dullerud, "Decentralized control with communication bandwidth constraints," in *Proc. IEEE ACC*, New York, NY, Jun. 2007.

[13] C. Langbort and V. Gupta, "Minimal interconnection topology in distributed control design," in *IEEE ACC*, Minneapolis, MN, Jun. 2006, pp. 845–850.

[14] T. M. Cover and J. A. Thomas, *Elements of Information Theory*. New York: Wiley, 1991.