

Stationarity and Ergodicity of Stochastic Non-Linear Systems Controlled over Communication Channels

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Abstract—This paper is concerned with the following problem: Given a stochastic non-linear system controlled over a noisy channel, what is the largest class of channels for which there exist coding and control policies so that the closed loop system is stochastically stable? Stochastic stability notions considered are stationarity, ergodicity or asymptotic mean stationarity. We do not restrict the state space to be compact, for example systems considered can be driven by unbounded noise. Necessary and sufficient conditions are obtained for a large class of systems and channels. A generalization of Bode’s Integral Formula for a large class of non-linear systems and information channels is obtained.

I. INTRODUCTION

Consider an N -dimensional controlled non-linear system described by the discrete-time equations

$$x_{n+1} = f(x_n, u_n, w_n), \quad (1)$$

for a (Borel measurable) function f , with $\{w_t\}$ being an independent and identically distributed (i.i.d) system noise process.

This system is connected over a noisy channel with a finite capacity to a controller, as shown in Figure 1. The controller has access to the information it has received through the channel. A source coder maps the source symbols, state values, to corresponding channel inputs. The channel inputs are transmitted through a channel; we assume that the channel is a discrete channel with input alphabet \mathcal{M} and output alphabet \mathcal{M}' .

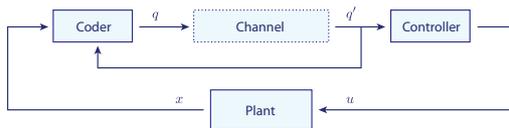


Figure 1: Control over a noisy channel with feedback.

We refer by a *Coding Policy* Π , a sequence of functions $\{\gamma_t^e, t \geq 0\}$ which are causal such that the channel input at time t , $q_t \in \mathcal{M}$, under Π^{comp} is generated by a function of its local information, that is,

$$q_t = \gamma_t^e(\mathcal{I}_t^e),$$

where $\mathcal{I}_t^e = \{x_{[0,t]}, q'_{[0,t-1]}\}$ and $q_t \in \mathcal{M}$, the channel input alphabet given by $\mathcal{M} := \{1, 2, \dots, M\}$, for $0 \leq t \leq T - 1$.

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Here, we have the notation for $t \geq 1$: $x_{[0,t-1]} = \{x_s, 0 \leq s \leq t - 1\}$. The channel maps q_t to q'_t in a stochastic fashion so that $P(q'_t | q_t, q_{[0,t-1]}, q'_{[0,t-1]})$ is a conditional probability measure on \mathcal{M}' for all $t \in \mathbb{Z}_+$. If this expression is equal to $P(q'_t | q_t)$, the channel is said to be a memoryless channel, that is, the past variables do not affect the channel output q'_t given the current channel input q_t .

The receiver/controller, upon receiving the information from the channel, generates its decision at time t , also causally: An admissible causal controller policy is a sequence of functions $\gamma = \{\gamma_t\}$ such that

$$\gamma_t : \mathcal{M}'^{t+1} \rightarrow \mathbb{R}^m, \quad t \geq 0,$$

so that $u_t = \gamma_t(q'_{[0,t]})$. We call such encoding and control policies, *causal* or *admissible*.

In the networked control literature, the goal in the encoder and controller design is typically either to optimize the system according to some performance criterion or stabilize the system. For stabilization, linear systems have been studied extensively where the goal has been to identify conditions so that the controlled state is stochastically stable, as we review briefly later.

This paper is concerned with necessary and sufficient conditions on information channels in a networked control system for which there exist coding and control policies such that the controlled system is stochastically stable in one or more of the following senses: (i) The state $\{x_t\}$ and the coding and control parameters lead to a stable (positive Harris recurrent) Markov chain and (ii) $\{x_t\}$ is asymptotically stationary, or asymptotically mean stationary (AMS) and satisfies Birkhoff’s sample path ergodic theorem, (iii) $\{x_t\}$ is ergodic.

A. Literature review

Due to space constraints, we are unable to provide a detailed account of the literature. We refer the reader to the full paper for a detailed literature review [35]. In the literature, the study of non-linear systems have typically considered noise-free controlled systems controlled over discrete noiseless channels. Many of the studies on control of non-linear systems over communication channels have focused on constructive schemes (and not on converse theorems), primarily for noise-free sources and channels. For noise-free systems, it typically suffices to only consider a sufficiently small *invariant* neighborhood of an equilibrium point to obtain stabilizability conditions. Entropy based arguments can be used to obtain converse results: The entropy, as a measure of uncertainty growth, of a

dynamical system has two related interpretations: A topological (distribution-free / geometric) one and a measure-theoretic (probabilistic) one. The distribution-free entropy notion for a dynamical system taking values in a compact metric space is concerned with the time-normalized number of distinguishable paths/orbits by some finite $\epsilon > 0$ the system's paths can take values in as the time horizon increases and $\epsilon \rightarrow 0$. With such a distribution-free setup [23] studied the stabilization of deterministic systems controlled over discrete noiseless finite capacity channels: The topological entropy gives a measure of the number of distinct control inputs needed to make a compact set invariant for a noise-free system. [23] extends the notion of topological entropy to controlled dynamical systems, and develops the notion of feedback entropy or invariance entropy [6], see also [5] for related results. For a comprehensive discussion of such a geometric interpretation of entropy in controlled systems, see [13]. The results for deterministic systems pose questions on set stability which are not sufficient to study stochastic setups. Stochasticity also allows for control over general noisy channels, and thus applicable to establish connections with information theory (we note that a distribution-free counterpart for such studies requires one to investigate zero-error capacity formulations, however many practical channels including erasure channels, have zero zero-error capacity). On the other hand, the measure-theoretic (also known as Kolmogorov – Sinai or metric) entropy is more relevant to information-theoretic as well as random noise-driven stochastic contexts since in this case, one considers the *typical* distinguishable paths/orbits of a dynamical system and not all of the sample paths a dynamical system may take (and hence the topological entropy typically provides upper bounds on the measure-theoretic entropy).

The stability criteria outlined earlier have been studied extensively for linear systems of the form

$$x_{t+1} = Ax_t + Bu_t + Gw_t, \quad (2)$$

where $x_t \in \mathbb{R}^N$ is the state at time t , $u_t \in \mathbb{R}^m$ is the control input, and $\{w_t\}$ is a sequence of zero-mean i.i.d. \mathbb{R}^d -valued Gaussian random vectors. Here, (A, B) and (A, G) are controllable pairs. Assume that all eigenvalues $\{\lambda_i, 1 \leq i \leq N\}$ of A are unstable, that is have magnitudes greater than or equal to 1. There is no loss here since if some eigenvalues are stable, by a similarity transformation, the unstable modes can be decoupled from the stable ones and one can instead consider a lower dimensional system; stable modes are already stochastically stable. For noise-free linear systems controlled over discrete-noiseless channels, Wong and Brockett [31], Baillieul [1]; and more generally, Tatikonda and Mitter [28] (see also [27]) and Nair and Evans [22] have obtained the minimum lower bound needed for stabilization over a class communication channels under various assumptions on the system noise and channels; sometimes referred to as a *data-rate theorem*. This theorem states that for stabilizability under information constraints, in the mean-square sense, a minimum average rate per time stage needed for stabilizability has to be at least $\sum_i \log_2(|\lambda_i|)$. The particular notion of stochastic stability is crucial in character-

izing the conditions on the channels; see [19], [26], and [17]. Towards generating a solution approach for systems driven by unbounded noise, [33] and [37] developed a martingale-method for establishing stochastic stability, which later led to a random-time state-dependent drift criterion arriving at the existence of an invariant distribution possibly with moment constraints, extending the earlier deterministic state dependent results in [20]. [34] considered discrete noisy channels, possibly with memory, with noiseless feedback.

In the information theory literature, for non-stationary linear Gaussian sources Gray and Hashimoto (see [10], [12] and [11]) and Berger [3] have obtained rate-distortion theoretic results which are in agreement with *data-rate* results in networked control (see [36] for a detailed review). These findings, although for non-causal codes, reveal that the information requirements for unstable linear systems do not come from a restriction because of causality in coding, but due to the inherent differential entropy growth rate of the sources.

The following definition (see Definition 3.1 in [34]) will be useful in the analysis later in the paper.

Definition 1. Channels are said to be of **Class A** type, if

- they satisfy the following Markov chain condition:

$$q'_t \leftrightarrow q_t, q_{[0,t-1]}, q'_{[0,t-1]} \leftrightarrow \{x_0, w_t, t \geq 0\}, \quad (3)$$

- their capacity with feedback is given by:

$$C = \lim_{T \rightarrow \infty} \max_{\{P(q_t|q_{[0,t-1]}, q'_{[0,t-1]}), 0 \leq t \leq T-1\}} \frac{1}{T} I(q_{[0,T-1]} \rightarrow q'_{[0,T-1]}), \quad (4)$$

where the directed mutual information is defined by

$$I(q_{[0,T-1]} \rightarrow q'_{[0,T-1]}) = \sum_{t=1}^{T-1} I(q_{[0,t]}; q'_t | q'_{[0,t-1]}) + I(q_0; q'_0).$$

Memoryless channels belong to this class; for such channels, feedback does not increase the capacity [7]. Such a class also includes finite state stationary Markov channels which are indecomposable [24]. Further examples can be found in [29] and in [8].

Theorem 1. [36] [34] Consider a multi-dimensional linear system with all eigenvalues unstable, that is $|\lambda_i| \geq 1$ for $i = 1, \dots, N$. For such a system controlled over a **Class A** type noisy channel with feedback, if the channel capacity satisfies

$$C < \sum_i \log_2(|\lambda_i|),$$

(i) there does not exist a stabilizing coding and control scheme with the property $\liminf_{T \rightarrow \infty} \frac{1}{T} h(x_T) \leq 0$, (ii) the system cannot be made AMS or ergodic.

Theorem 2. [36] [34] Consider a multi-dimensional system with a diagonalizable matrix A controlled over a discrete memoryless channel. If the Shannon capacity of the channel satisfies

$$C > \sum_{|\lambda_i| > 1} \log_2(|\lambda_i|),$$

there exists a stabilizing scheme which makes the process $\{x_t\}$ AMS. Furthermore, if the channel is a discrete noiseless channel or an erasure channel, this condition implies the existence of a policy leading to stationarity and ergodicity for $\{x_t\}$.

II. SUBLINEAR ENTROPY GROWTH AND A GENERALIZATION OF BODE'S INTEGRAL FORMULA FOR NON-LINEAR SYSTEMS

In the paper, instead of a general \mathbb{R}^N -valued non-linear state model (1), we will consider non-linear systems of the form

$$x_{n+1} = f(x_n, w_n) + Bu_n, \quad (5)$$

$$x_{n+1} = f(x_n) + Bu_n + w_n, \quad (6)$$

$$x_{n+1} = f(x_n, u_n) + w_n \quad (7)$$

In all of the models above, x_n is the \mathbb{R}^N -valued state, w_n is the \mathbb{R}^p -valued noise variable, u_n is \mathbb{R}^m valued and w_n assumed to be an independent noise process with $w_n \sim \nu$.

We assume throughout that f is continuously differentiable in the state variable. For a possibly non-linear differentiable function $f: \mathbb{R}^N \rightarrow \mathbb{R}^m$, the Jacobian matrix of f is an $n \times m$ matrix function consisting of partial derivatives of f such that

$$J(f)(i, j) = \frac{\partial(f(x))_i}{\partial x_j}, \quad 1 \leq i \leq m, 1 \leq j \leq n.$$

We will often have the following assumption.

Assumption 1. In the models considered above $f(\cdot, w): \mathbb{R}^N \rightarrow \mathbb{R}^N$ is invertible for every realization of w .

In the following $|J(f)|$ will denote the absolute value of the determinant of the Jacobian. Furthermore, with $f_w(x) = f(x, w)$, we define $J(f(x, w)) := J(f_w(x))$.

Assumption 2. There exist $L_1, M_1 \in \mathbb{R}$ so that for all x, w

$$L_1 \leq \log_2(|J(f(x, w))|) \leq M_1$$

Theorem 3. Consider the networked control problem over a Class A channel. (i) Let f have the form in (5), (ii) Assumptions 1 and 2 hold, and (iii) x_0 have finite differential entropy. Let $\pi_t(A) := P(x_t \in A)$ for Borel A . a) If there is an admissible coding and control policy such that

$$\liminf_{t \rightarrow \infty} h(x_t | q'_{[0, t-1]})/t \leq 0,$$

or

$$\liminf_{t \rightarrow \infty} h(x_t)/t \leq 0,$$

it must be that

$$C \geq \liminf_{T \rightarrow \infty} \frac{1}{T} \sum_{t=1}^T \int \pi_t(dx) \left(\int \nu(dw) \log_2(|J(f(x, w))|) \right)$$

b) If there is an admissible coding and control policy such that

$$\limsup_{t \rightarrow \infty} h(x_t | q'_{[0, t-1]})/t \leq 0,$$

or

$$\limsup_{t \rightarrow \infty} h(x_t)/t \leq 0,$$

it must be that

$$C \geq \limsup_{T \rightarrow \infty} \frac{1}{T} \sum_{t=1}^T \int \pi_t(dx) \left(\int \nu(dw) \log_2(|J(f(x, w))|) \right)$$

In either case, if $L := \inf_{x, w} \log_2 |J(f(x, w))|$, then $C \geq L$.

For a proof, see [35].

Remark 1. The condition $\limsup_{t \rightarrow \infty} h(x_t)/t \leq 0$ is a weak condition: A process whose second moment grows subexponentially s.t. $\limsup_{T \rightarrow \infty} \frac{\log(E[x_T^2])}{T} \leq 0$, satisfies this condition.

The proof of Theorem 3 reveals an interesting connection with and generalization of Bode's Integral Formula (and what is known as the *waterbed effect*) [4] [21] to non-linear systems, which we state formally in the following. The result also suggests that an appropriate generalization for non-linear systems is through an information theoretic approach that recovers Bode's original result for the linear case as we discuss further below. Some earlier discussions under different conditions were reported in [38], [9] and [16].

Theorem 4. (i) Let f have the form in (5), (ii) Assumption 1 hold, and (iii) x_0 have finite differential entropy. Under any admissible policy with $\limsup_{t \rightarrow \infty} h(x_t)/t \leq 0$, it must be that

$$\begin{aligned} & \limsup_{T \rightarrow \infty} \frac{1}{T} I(q_{[0, T-1]} \rightarrow q'_{[0, T-1]}) \\ & \geq \limsup_{T \rightarrow \infty} \frac{1}{T} \sum_{t=0}^{T-1} \int \pi_t(dx) \int \nu(dw) \log_2(|J(f(x, w))|) \end{aligned}$$

Remark 2. [Reduction to Bode's Integral Formula for Linear Systems and Gaussian Noise] If the system considered is linear with all open-loop eigenvalues unstable, the channel is an additive noise channel so that $q'_t = q_t + v_t$ for some stationary Gaussian noise, and time-invariant control policies are considered leading to a stable system, then with the more common notation of $y_t = q'_t$, the right hand side of the statement in Theorem 4 would be the sum of the unstable eigenvalues of the linear system matrix. For a stationary Gaussian process [see [7], page 274] the entropy rate can be written as

$$\frac{1}{2} \log(2\pi e) + \int_{-1/2}^{1/2} \frac{1}{2} \log(S(f)) df$$

with S denoting the spectral density of the process. With

$$I(q'_t; q_{[0, t]} | q'_{[0, t-1]}) = h(q'_t | q'_{[0, t-1]}) - h(v_t | v_{[0, t-1]}),$$

(8) the left hand side of (9) reduces to the difference between the entropy rate of the process q'_t (i.e., $\lim_{t \rightarrow \infty} h(q'_t | q'_{[0, t-1]})$) and that of the stationary noise process v_t . Then, the left hand side of (9) equals $\int_{-1/2}^{1/2} \frac{1}{2} \log(\frac{S_y(f)}{S_v(f)}) df$, which then is equal to the integral of the log-sensitivity function (corresponding to the transfer function from the disturbance process v_t to the output process q'_t). This leads to the celebrated Bode's Integral Formula. \diamond

III. ASYMPTOTIC MEAN STATIONARITY AND ERGODICITY

In the following, we build on, but significantly modify the approaches in [36] and [18] to account for non-linearity of the system. Consider the system (6), under a given control policy, controlled over a channel.

Assumption 3. *We assume*

$$\begin{aligned} M &:= \sup_{x \in \mathbb{R}^N} \log_2 |J(f(x))| < \infty, \\ L &:= \inf_{x \in \mathbb{R}^N} \log_2 |J(f(x))| > -\infty. \end{aligned}$$

Proposition 1. *Consider the system (6) controlled over a Class A type noisy channel with feedback with $h(x_0) < \infty$ under Assumption 3. If $C < L$,*

$$\limsup_{T \rightarrow \infty} P(|x_T| \leq b(T)) \leq \frac{M - (L - C)}{M},$$

for all $b(T) > 0$ such that $\lim_{T \rightarrow \infty} \frac{1}{T} \log_2(b(T)) = 0$.

An implication of this result follows.

Theorem 5. *Consider the system (6) controlled over a Class A type noisy channel with feedback, and let Assumption 1 hold. If, under some causal encoding and controller policy, the state process is AMS, then the channel capacity C must satisfy $C \geq L$.*

For a proof, see [35]. We recover the following result for linear systems in [36] as a special case.

Corollary 1. *For the linear case with $f(x) = Ax$ with eigenvalues $|\lambda_i| \geq 1$, $C \geq \sum_k \log_2(|\lambda_i|)$ is a necessary condition for the AMS property under any admissible coding and control policy.*

Remark 3. In information theory, a well-established result is that for noiseless coding of information stable sources (this includes all finite state stationary and ergodic sources) over a class of information stable noisy channels (which includes the channels we consider here), an asymptotically noise-free recovery is possible if the channel capacity is greater than the source entropy through the use of non-causal codes, see e.g. [30] [14]. However, for the problem we consider (i) the source is non-stationary and open-loop unstable, (ii) the encoding is causal, and (iii) the source process space is not finite-alphabet. Nonetheless, we see that the invariance properties of the source process appear in the rate bounds that we obtain. \diamond

IV. STATIONARITY AND POSITIVE HARRIS RECURRENCE UNDER STRUCTURED (STATIONARY) POLICIES

In many applications, one uses a state-space formulation for coding and control policies. In the following, we will consider stationary update rules which have the form that

$$\begin{aligned} q_t &= \gamma^e(x_t, m_t) \\ u_t &= \gamma^d(m_t, q'_t), \\ m_t &= \eta(m_{t-1}, q'_{t-1}), \end{aligned} \quad (9)$$

for functions γ^e , γ^d , and η . In the form above, m is a \mathbb{S} -valued memory or *quantizer state* variable. A large class of adaptive encoding policies have the form above. This includes, delta modulation, differential pulse coded modulation (DPCM), adaptive differential pulse coded modulation (ADPCM), Goodman-Gersho type adaptive quantizers, as well as the coding schemes used for stabilization of networked control systems under fixed-rate codes [33].

In this section, instead of asymptotic mean stationarity, we will consider the more stringent condition of (asymptotic) stationarity of the controlled source process. For ease in presentation we will assume that m_t takes values in a countable set, even though the extension to more general spaces is possible.

Lemma 1. *If the channel is memoryless, the process (x_t, m_t) forms a Markov chain.*

For this Markov chain, let $\pi_t(B) = E[1_{\{(x_t, m_t) \in B\}}]$ for all Borel B . With a slight abuse of notation, assume that π_t has a density which is denoted by the same letter. We assume here that the channel is memoryless.

Theorem 6. *Suppose that the encoding, control and the memory update laws are given by (9). (i) Let f have the form (5) or (6), (ii) Assumptions 1 and 2 hold, (iii) $h(x_0) < \infty$. For the positive Harris recurrence of the process x_t, m_t , which implies the existence of a unique invariant measure (ergodicity) π , with density also to be denoted by π , it must be that*

$$C \geq \int \pi(dx) \left(\int \nu(dw) \log_2(|J(f(x, w))|) \right), \quad (10)$$

provided that $(\int \nu(dw) \log_2(|J(f(x, w))|))$ is π -integrable and $\limsup_{t \rightarrow \infty} \frac{1}{t} h(x_t) \leq 0$.

For a proof, see [35].

Remark 4. The preceding theorem affirms an information theoretic and ergodic theoretic result that the Shannon capacity of the channel should be greater than the entropy rate of a dynamical system. Likewise, the Lyapunov exponent-like expression of the averages of $\log_2(|J(f(\cdot, w_t))|)$ is present in the converse bound. We note that for a noise-free system with a random initial condition, Pesin's formula [32] provides a characterization of the measure-theoretic entropy through Lyapunov exponents. \diamond

V. DISCRETE NOISELESS CHANNELS AND A POLICY LEADING TO STATIONARITY AND ERGODICITY

In this section, we provide achievability results and a stabilizing coding/control policy. As discussed earlier, the study of non-linear systems have typically considered noise-free controlled systems; e.g. [2], [15], and [25]. As also noted earlier, for noise-free systems, it typically suffices to only consider a sufficiently small *invariant* neighborhood of an equilibrium point to obtain stabilizability conditions which is not necessarily the case when the system is driven by an additive noise process. We consider such an example in the following.

Theorem 7. Consider a non-linear system of the form (7), where $\{w_t\}$ is a sequence of zero-mean Gaussian random vectors and there exists a control function $\kappa(z)$ such that $|f(x, \kappa(z))|_\infty \leq |a||x - z|_\infty$ for all $x, z \in \mathbb{R}^N$, with $\kappa(0) = 0$. For the stationarity and ergodicity of $\{x_t\}$ (and thus with a unique invariant probability measure), it suffices that $C > N \log_2(|a|) + 1$.

The proof of this result builds on the construction of an adaptive quantizer, a sequence of stopping times which allow for the transmission of information from the granular region of the adaptive quantizer, and the application of random-time stochastic drift criteria [37]. We refer the reader to [35].

Remark 5. The approach adopted in the proof of Theorem 7 applies for more general channels (such as erasure channels or discrete memoryless channels) subject to more tedious bounds. \diamond

VI. CONCLUSION

In this paper, conditions on information channels leading to stochastic stability of non-linear systems controlled over noisy channels have been investigated. Stochastic stability notions considered were asymptotic mean stationarity, ergodicity and stationarity. Results for linear systems are recovered as a special case.

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