# On Optimal Zero-Delay Quantization of Vector Markov Sources

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***Abstract*— For a vector Markov source driven by additive Gaussian noise, we study the existence and structure of optimal quantization policies. The quantizers allowed are the ones which have convex codecells. For the finite horizon problem and bounded cost, we show that an optimal zero-delay quantization policy exists. Then the linear quadratic Gaussian problem is considered as an important extension of the bounded cost assumption for the finite horizon setting. For the infinite horizon setup, the existence of an optimal stationary policy is established among the class of Markov coding policies.**

## I. Introduction

### A. Quantizers as Control Actions

We consider a causal encoding problem where a sensor encodes an observed source to a receiver with zero-delay. The source $\{x_t\}$ to be encoded is an $\mathbb{R}^n$-valued Markov process. The encoder encodes (quantizes) its information $\{x_t\}$ and transmits it to a receiver over a discrete noiseless channel with common input and output alphabet $\mathbb{M} := \{1, 2, \ldots, M\}$, where $M$ is a positive integer, i.e., the encoder quantizes its information.

Formally, the encoder is specified by a *composite quantization policy* $\Pi^{comp}$, which is a sequence of Borel measurable functions $\{Q_t^{comp}, t \geq 0\}$ such that

$$Q_t^{comp} : (\mathbb{R}^n)^{t+1} \times \mathbb{M}^t \to \mathbb{M}.$$

In particular, at time $t$ the encoder transmits $q_t$ generated as

$$q_t = Q_t^{comp}(I_t)$$

where

$$I_t = (x_{[0,t]}, q_{[0,t-1]}), \quad t \geq 1,$$

and $I_0 = \{x_0\}$, where we have used the notation $x_{[0,t]} = (x_0, x_1, \ldots, x_t)$ and $q_{[0,t-1]} = (q_0, \ldots, q_{t-1})$.

The receiver, upon receiving the information from the encoders, generates its decision $u_t$ at time $t$, also causally. An admissible causal receiver policy is a sequence of measurable functions $\gamma = \{\gamma_t\}$ such that

$$\gamma_t : \mathbb{M}^{t+1} \to \mathbb{U}, \qquad t \geq 0$$

where $\mathbb{U}$ denotes the decision space (usually a Borel subset of $\mathbb{R}^n$). Thus

$$u_t = \gamma_t(q_{[0,t]}), \quad t \geq 0.$$

We assume that the encoder and decoder have an agreement on the distribution $\pi_0$ on the initial state $x_0$.

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For a finite horizon setting the goal is to minimize the cost

$$J_{\pi_0}(\Pi^{comp}, \gamma, T) := E_{\pi_0}^{\Pi^{comp}, \gamma}\left[\frac{1}{T}\sum_{t=0}^{T-1} c_0(x_t, u_t)\right], \quad (1)$$

for some $T \geq 1$, where $c_0 : \mathbb{R}^n \times \mathbb{U} \to \mathbb{R}_+$ is a (measurable) cost function and $E_{\pi_0}^{\Pi}\left[\,\cdot\,\right]$ denotes the expectation with initial state distribution $\pi_0$ and under the composite quantization policy $\Pi^{comp}$ and receiver policy $\gamma$.

We have the following assumptions in the source $\{x_t\}$ and the cost function.

**Assumption 1.**
(i) The evolution of the Markov source $\{x_t\}$ is given by

$$x_{t+1} = f(x_t) + w_t, \quad t \geq 0 \quad (2)$$

where $\{w_t\}$ is an independent and identically distributed zero-mean Gaussian vector noise sequence and $f : \mathbb{R}^n \to \mathbb{R}^n$ is measurable.
(ii) $\mathbb{U}$ is compact and $c_0 : \mathbb{R}^n \times \mathbb{U} \to \mathbb{R}_+$ is bounded and continuous (the compactness and boundedness conditions will be relaxed in Section V).
(iii) The initial condition $x_0$ is zero-mean Gaussian.

### B. Revisiting Structural Results

Structural results for the finite horizon control problem described in the previous section have been developed in a number of important publications. Reference [18] summarizes the two main approaches, one adopted by Witsenhausen [20], and the other by Walrand and Varaiya [19]. Teneketzis [18] extended these approaches to the more general setting of non-feedback communication and [22] extended these results to more general state spaces (complete and separable metric spaces, such as $\mathbb{R}^n$). The following state, somewhat informally, these two important structural results.

**Theorem 1.** *[Witsenhausen [20]] For the finite horizon problem, any* causal composite quantization policy *can be replaced without any loss in performance by one which, at time* $t = 1, \ldots, T-1$*, only uses,* $x_t$ *and* $q_{[0,t-1]}$.

For a Polish space $\mathbb{X}$, let $\mathcal{P}(\mathbb{X})$ denote the space of probability measures on $\mathbb{X}$ endowed with the Prohorov metric. Given a composite quantization policy $\Pi^{comp}$, let $\pi_t \in \mathcal{P}(\mathbb{R}^n)$ be the conditional probability measure defined by

$$\pi_t(A) := P(x_t \in A | q_{[0,t-1]})$$

for any Borel set $A$.

Walrand and Varaiya [19] considered sources taking values in a finite set, and obtained the following result. In its current generalized form, the result appears in [22] for $\mathbb{R}^n$-valued sources.

**Theorem 2.** *For a finite horizon problem, any* causal composite quantization policy *can be replaced, without any loss in performance, by one which at any time* $t = 1, \ldots, T-1$ *only uses the conditional probability* $P(dx_{t-1}|q_{[0,t-1]})$ *and the state* $x_t$*. This can be expressed as a quantization policy which only uses* $(\pi_t, t)$ *to generate a quantizer* $Q_t : \mathbb{R}^n \to \mathbb{M}$*, where the quantizer* $Q_t$ *uses* $x_t$ *to generate the quantization output as* $q_t = Q_t(x_t)$ *at time* $t$*.*

The main difference between the two structural results above is the following: In the setup of Theorem 1, the encoder's memory space is not fixed and keeps expanding as the decision horizon $T$. In the setup of Theorem 2, the memory space of an optimal encoder is fixed. In general, the space of probability measures is a very large one; however, it may be the case that different quantization outputs may lead to the same conditional probability measure on the state process, leading to a reduction in the required memory. Furthermore, setup in Theorem 2 allows one to apply the theory of Markov Decision Processes.

In this paper, we will show that under the stated assumptions on the Markov process and the cost function and some additional assumptions on the admissible quantization policies, there always exists a policy considered in Theorem 2 that minimizes the finite horizon cost (1). For the infinite horizon problem (10), we show that an optimal stationary quantization policy exists among those considered in Theorem 2.

The rest of the paper is organized as follows. The next section gives a brief review of the literature. Section II contains background material on quantizers considered in this paper. In Section III a controlled Markov chain is constructed for our problem. Section IV establishes the existence of optimal policies for the finite horizon case for bounded cost functions. Section V considers the case with linear systems and quadratic costs. Section VI considers the more involved infinite horizon case. Section VII contains concluding discussions.

### C. Literature Review

Existence of optimal quantizers for a one-stage cost problem has been investigated, among other works, by [1], [16], and [21]. For optimal vector quantization in a multi-stage problem, Borkar et al. [6] investigated existence results. Recently [2] considered the average cost optimality equation for causal coding of i.i.d. sources with finite lookahead. To our knowledge, the existence of optimal quantizers for a finite horizon setting has not been considered in the literature for the setup considered in this paper.

For the infinite horizon setting, [6] provided a stochastic control formulation for the optimal quantization problem with an entropy constraint. In doing so, they considered a coding policy in which the admissible quantizers $Q_t$ were restricted to ones using the so-called nearest neighbor encoding rule and having their granular region contained in a fixed compact set for all time stages. Furthermore, there are additional regularity conditions on the dynamics of the system. Our approach differs from [6] in the relaxed structural assumptions we make. We allow quantizers with convex codecells (to be defined later), which is a more general condition than assuming the nearest neighbor encoding rule.

Other relevant work include [5] which considered optimization over probability measures for causal and non-causal settings, and [18], [13] which considered zero-delay coding of Markov sources in various setups.

## II. Space of Quantizer Actions

In this section, we define the space of quantizers considered in the paper. Our construction builds on [21].

**Definition 1.** An $M$-cell quantizer $Q$ on $\mathbb{R}^n$ is a (Borel) measurable mapping $Q : \mathbb{R}^n \to \{1, \ldots, M\}$. We let $\mathcal{Q}$ denote the collection of all $M$-cell quantizers on $\mathbb{R}^n$.

Note that each $Q \in \mathcal{Q}$ is uniquely characterized by its *quantization cells* (or bins) $B_i = \{x : Q(x) = i\}$, $i = 1, \ldots, M$ which form a measurable partition of $\mathbb{R}^n$.

**Remark 1.**
(a) We allow for the possibility that some of the cells of the quantizer are empty.
(b) In source coding theory, a quantizer is a mapping $Q : \mathbb{R}^n \to \mathbb{R}^n$ with a finite range. Thus $Q$ is defined by a partition and a reconstruction value in $\mathbb{R}^n$ for each cell in the partition. That is, for given cells $\{B_1, \ldots, B_M\}$ and reconstruction values $\{c_1, \ldots, c_M\} \subset \mathbb{R}^n$, we have $Q(x) = c_i$ if and only if $x \in B_i$. In our definition, we do not include the reconstruction values.

A quantizer $Q$ with cells $\{B_1, \ldots, B_M\}$ can also be characterized as a stochastic kernel $Q$ from $\mathbb{R}^n$ to $\{1, \ldots, M\}$) defined by

$$Q(i|x) = 1_{\{x \in B_i\}}, \quad i = 1, \ldots, M$$

We will endow the quantizers with a topology induced by such a stochastic kernel interpretation. If $P$ is a probability measure on $\mathbb{R}^n$ and $Q$ is a stochastic kernel from $\mathbb{R}^n$ to $\{1, \ldots, M\}$, then $PQ$ denotes the resulting joint probability measure on $\mathbb{R}^n \times \{1, \ldots, M\}$.

Let $\mathcal{P}(\mathbb{R}^{\mathbb{N}})$ denote the family of all probability measures on $(\mathbb{R}^N, \mathcal{B}(\mathbb{R}^N))$ for some $N \in \mathbb{N}$. A sequence $\{\mu_n, n \in \mathbb{N}\}$ in $\mathcal{P}(\mathbb{R}^N)$ is said to converge to $\mu \in \mathcal{P}(\mathbb{R}^N)$ *weakly* if

$$\int_{\mathbb{R}^N} c(x)\mu_n(dx) \to \int_{\mathbb{R}^N} c(x)\mu(dx)$$

for every continuous and bounded $c : \mathbb{R}^N \to \mathbb{R}$.

For two probability measures $\mu, \nu \in \mathcal{P}(\mathbb{R}^N)$, the *total variation* metric is defined by

$$\|\mu - \nu\|_{TV} = 2 \sup_{B \in \mathcal{B}(\mathbb{R}^N)} |\mu(B) - \nu(B)|.$$

A sequence $\{\mu_n\}$ is said to converge to $\mu \in \mathcal{P}(\mathbb{R}^N)$ in total variation if $\|\mu_n - \mu\|_{TV} \to 0$.

**Definition 2** ([21])**.** A quantizer sequence $Q_n$ converges to $Q$ weakly at $P$ ($Q_n \to Q$ weakly at $P$) if $PQ^n \to PQ$ weakly. Similarly, $Q_n$ converges to $Q$ in total variation at $P$ ($Q_n \to Q$ at $P$ in total variation at $P$) if $PQ^n \to PQ$ in total variation.

The class of quantization policies which admit the structure suggested in Theorem 2 is an important one, we henceforth define

$$\Pi_W := \bigg\{\Pi^{comp} = \{Q_t^{comp}, t \geq 0\} : \exists \gamma^0 : \mathcal{P}(\mathbb{R}^n) \to \mathcal{Q}$$
$$Q_t^{comp}(I_t) = (\gamma_t^0(\pi_t))(x_t), \forall I_t \bigg\}, \tag{3}$$

to represent this class of policies.

We restrict the the set of quantizers considered by only allowing quantizers having convex cells.

**Assumption 2.** Let $\mathcal{Q}_c$ denote the set of all quantizers in $\mathcal{Q}$ having convex cells, i.e., $Q \in \mathcal{Q}_c$ if and only if $Q^{-1}(i)$ is a (possible empty) convex subset of $\mathbb{R}^n$ for all $i = 1, \dots, M$. We assume that all quantizers $Q_t$ are from $\mathcal{Q}_c$.

Let $\Pi_W^C$ denote the set of all composite quantization policies $\Pi_W$ (defined in (3)) which in addition satisfy the condition that all quantizers $Q_t$, $t \geq 0$ have convex cells (i.e., $Q_t \in \mathcal{Q}_c$ for all $t \geq 0$).

As discussed in [11], by the separating hyperplane theorem, there exist pairs of complementary closed half spaces $\{(H_{i,j}, H_{j,i}) : 1 \leq i,j \leq M, i \neq j\}$ such that for all $i = 1, \dots, M$, $B_i \subset \bigcap_{j\neq i} H_{i,j}$. Since $\bar{B}_i := \bigcap_{j\neq i} H_{i,j}$ is a closed convex polytope for each $i$, if the probability measure $P$ has a density function, then one has $P(\bar{B}_i \setminus B_i) = 0$ for all $i = 1, \dots, M$. One can thus obtain a ($P$–a.s) representation of $Q$ by the $M(M-1)/2$ hyperplanes $h_{i,j} = H_{i,j} \cap H_{j,i}$. One can represent such a hyperplane $h$ by a vector $(a_1, \dots, a_m, b) \in \mathbb{R}^{n+1}$ with $\sum_k |a_k|^2 = 1$ such that $h = \{x \in \mathbb{R}^n : \sum_i a_i x_i = b\}$, thus obtaining a parametrization over $\mathbb{R}^{(M-1)(n+1)}$ of all quantizers in $\mathcal{Q}_c$.

**Remark 2.** We note that the assumption of convex codecells is adopted for technical reasons and it may be the case that there is a loss in optimality in restricting the analysis to convex codecells.

## III. Controlled Markov Chain Construction

Suppose we use a quantizer policy in $\Pi_W$. Properties of conditional probability lead to the following expression for $\pi_t(dx)$:

$$\frac{\int \pi_{t-1}(dx_{t-1}) P(q_{t-1}|\pi_{t-1}, x_{t-1}) P(dx|x_{t-1}, u_{t-1})}{\int \int \pi_{t-1}(dx_{t-1}) P(q_{t-1}|\pi_{t-1}, x_{t-1}) P(dx|x_{t-1}, u_{t-1})}.$$

Here, $P(q_{t-1}|\pi_{t-1}, x_{t-1})$ is determined by the quantizer policy. The following follows from the proof of Theorem 2.5 of [22].

**Theorem 3.** *The sequence of conditional measures and quantizers $\{(\pi_t, Q_t)\}$ form a controlled Markov process in $\mathcal{P}(\mathbb{R}^n) \times \mathcal{Q}$.*

The following is a key lemma.

**Lemma 1.** *For all $t \geq 1$, $\pi_t(dx)$ is absolutely continuous with respect to the Lebesgue measure, i.e., it has a probability density function, which we will also denote by $\pi_t$ by an abuse of notation. The density function $\pi_t$ is uniformly continuous for every $t$ and the sequence $\{\pi_t\}$ is a uniformly bounded and uniformly equicontinuous family.*

## IV. Existence of Optimal Policies: Finite Horizon Setting

For any quantization policy in $\Pi_W$ and any $T \geq 1$ we have

$$\inf_{\gamma} J_{\pi_0}(\Pi^{comp}, \gamma, T) = E_{\pi_0}^{\Pi^{comp}} \left[ \frac{1}{T} \sum_{t=0}^{T-1} c(\pi_t, Q_t) \right],$$

where

$$c(\pi_t, Q_t) = \sum_{i=1}^{M} \inf_{u \in \mathbb{U}} \int_{Q_t^{-1}(i)} \pi_t(dx) c_0(x, u).$$

**Theorem 4.** *Under Assumption 1, an optimal receiver policy exists.*

**Proof.** At any given time an optimal receiver will minimize $\int P(dx_t|q_{[0,t]}) c(x_t, u_t)$. The existence of a minimizer then follows from Theorem 3.1 in [21]. □

We have the following result on the existence of optimal quantizers for the finite horizon setting.

**Theorem 5.** *For any $T \geq 1$, under Assumption 1, there exists a policy in $\Pi_W^C$ such that*

$$\inf_{\Pi^{comp} \in \Pi_W^C} \inf_{\gamma} J_{\pi_0}(\Pi^{comp}, \gamma, T) \tag{4}$$

*is achieved. Letting $J_T^T(\cdot) = 0$ and*

$$J_0^T(\pi_0) := \min_{\Pi^{comp} \in \Pi_W^C, \gamma} J_{\pi_0}(\Pi^{comp}, \gamma, T),$$

*the dynamic programming recursion*

$$T J_t^T(\pi_t) = \min_{Q \in \mathcal{Q}_c} \Big( c(\pi_t, Q_t) + T E[J_{t+1}^T(\pi_{t+1})|\pi_t, Q_t] \Big)$$

*holds for all $t = 0, 1, \dots, T-1$.*

**Proof.** The following lemmas provide the essential steps in the proof.

**Lemma 2.** *(a) Let $\{\mu_n\}$ be a sequence of density functions on $\mathbb{R}^n$ which are uniformly equicontinuous and uniformly bounded and assume $\mu_n \to \mu$ weakly. Then $\mu_n \to \mu$ in total variation.*
*(b) Let $\{Q_n\}$ be a sequence in $\mathcal{Q}_c$ such that $Q_n \to Q$ weakly at $P$ for some $Q \in \mathcal{Q}_c$. If $P$ admits a density, then $Q_n \to Q$ in total variation at $P$.*
*(c) Let $P_n Q_n \to PQ$ weakly, where $\{Q_n\}$ is a sequence in $\mathcal{Q}_c$ and $Q \in \mathcal{Q}_c$. Suppose further that $P_n \to P$ in total variation where $P$ admits a density. Then $P_n Q_n \to PQ$ in total variation.*

Let $\mathcal{S} \subset \mathcal{P}(\mathbb{R}^n)$ be the set of reachable states for $\pi_t$ under any composite coding policy. Note that by Lemma 1, the collection of densities in $\mathcal{S}$ is uniformly bounded and equicontinuous. We have the following supporting results.

**Lemma 3.** *$\mathcal{Q}_c$ is compact in total variation at any input $\pi$ admitting a density in the sense that any sequence in $\mathcal{Q}_c$ has a subsequence that converges in total variation to some $Q \in \mathcal{Q}_c$ at $\pi$.*

**Lemma 4.** *$c(\pi, Q)$ is continuous on $\mathcal{S} \times \mathcal{Q}_c$.*

We can now apply backward induction. At $t = T-1$ we have

$$J_{T-1}^T(\pi) = \inf_Q c(\pi, Q).$$

By Lemma 4 and the compactness of the set of quantizers $\mathcal{Q}_c$ (Lemma 3) there exists an optimal quantizer that achieves the infimum.

**Lemma 5.** *Let $F : \mathcal{S} \times \mathcal{Q}_c \to \mathbb{R}$ be continuous on $\mathcal{S} \times \mathcal{Q}_c$ in the sense that $\pi_n Q_n \to \pi Q$ in total variation implies that $F(\pi_n, Q_n) \to F(\pi, Q)$. Then $\inf_{Q \in \mathcal{Q}_c} F(\pi, Q)$ is achieved by some $Q$ in $\mathcal{Q}_c$ and $\min\limits_Q F(\pi, Q)$ is continuous in $\pi$.*

As a consequence of Lemmas 4 and 5, $J_{T-1}^T(\pi)$ is continuous in $\pi$.

Consider $t = T-2$. We want to show that the minimization problem

$$J_{T-2}^T(\pi) = \min_Q \left( \frac{1}{T} c(\pi, Q) + E\big[J_{T-1}^T(\pi_{T-1}) | \pi_{T-2} = \pi, Q_{T-2} = Q\big] \right) \quad (5)$$

has a solution and $J_{T-2}^T(\pi)$ is continuous in $\pi$.

Define the conditional probability distributions given by

$$\begin{aligned} &\hat{\pi}(m, \pi, Q)(C) \\ &\quad := P(x_{t+1} \in C | \pi_t = \pi, Q_t = Q, q_t = m) \\ &\quad = \frac{1}{\pi(B_m)} \int_C \left( \int \pi(dx) 1_{\{x \in B_m\}} \phi(z - f(x)) \right) dz \end{aligned}$$

where $B_m$, $m = 1, \dots, M$ are the cells of $Q$ (if $\pi(B_m) = 0$, then $\hat{\pi}(m, \pi, Q)$ is set arbitrarily). Note that

$$\begin{aligned} &E\big[J_{T-1}^T(\pi_{T-1}) | \pi_{T-2} = \pi, Q_{T-2} = Q\big] \\ &\quad = \sum_{m=1}^M J_{T-1}^T\big(\hat{\pi}(m, \pi, Q)\big) \pi\big(Q^{-1}(m)\big) \end{aligned} \quad (6)$$

where

$$\pi\big(Q^{-1}(m)\big) = P(q_{T-2} = m | \pi_{T-2} = \pi, Q_{T-2} = Q).$$

The following lemma implies that if $\pi_n Q_n \to \pi Q$, then

$$\begin{aligned} &J_{T-1}^T\big(\hat{\pi}(m, \pi_n, Q_n)\big) \pi_n\big(Q_n^{-1}(m)\big) \\ &\qquad \to J_{T-1}^T\big(\hat{\pi}(m, \pi, Q)\big) \pi\big(Q^{-1}(m)\big) \end{aligned} \quad (7)$$

for all $m$.

**Lemma 6.** *If $\pi_n Q_n \to \pi Q$ in total variation, then for every $m = 1, \dots, M$ with $\pi(B_m) > 0$, we have $\hat{\pi}(m, \pi_n, Q_n) \to \hat{\pi}(m, \pi, Q)$ in total variation.*

It follows that if $\hat{\pi}(m, \pi, Q) > 0$ for some $m$, then Lemma 6 implies that $\hat{\pi}(m, \pi, Q_n) \to \hat{\pi}(m, \pi, Q)$ in total variation as $\pi_n Q_n \to \pi Q$; hence (7) holds in this case. If $\pi\big(Q^{-1}(m)\big) = 0$, then by the boundedness of the cost (7) holds again. In view of (6), $E\big[J_{T-1}^T(\pi_{T-1}) | \pi_{T-2} = \pi, Q_{T-2} = Q\big]$ is continuous in $(\pi, Q)$.

We obtain that both expressions on the right side of (5) are continuous in the sense of Lemma 5. The the same lemma implies that the minimization problem (5) has a solution and $J_{T-2}^T(\pi)$ is continuous in $\pi$. The recursion applies for all further stages $t = T-3, \dots, 0$, establishing the result. Details will be given in [24]. □

## V. Linear Systems with Quadratic Costs

Linear systems driven by Gaussian noise are important in many applications in control, estimation and signal processing. Furthermore, as observed in [23] (see also [15]), for jointly optimal quantization and control for linear systems under quadratic cost criteria and under mild technical assumptions, the total cost can be separated into an estimation component (for a control-free source) and a control component. The results in this section address the estimation component.

In view of this motivation, in this section, we consider linear sources under quadratic criteria.We modify Assumption 1 as follows.

**Assumption 3.**
(i) The evolution of the Markov source $\{x_t\}$ is given by

$$x_{t+1} = Ax_t + w_t, \quad t = 0, 1, 2, \dots$$

where $\{w_t\}$ is an independent and identically distributed Gaussian noise sequence and $A$ is a square matrix.
(ii) $c_0(x, u) = \|x - u\|^2 = (x-u)'(x-u)$ (where $(x-u)'$ denotes the transpose of $(x-u)$) and $u \in \mathbb{R}^n$.
(iii) $x_0$ admits a zero-mean Gaussian density.

We note that for every given quantizer action,

$$\int \|x\|^2 P(dx_t | q_{[0,t]})\, dx < \infty,$$

so a unique optimal receiver policy exists and is given by

$$\gamma_t(q_{[0,t]}) = \int x P(dx_t | q_{[0,t]}). \quad (8)$$

**Theorem 6.** *Under Assumption 3, for any $T \geq 1$ there exists a policy in $\Pi_W^C$ such that*

$$\inf_{\Pi^{comp} \in \Pi_W} \inf_\gamma J_{\pi_0}(\Pi^{comp}, \gamma, T) \quad (9)$$

*is achieved. Letting $J_T^T(\cdot) = 0$ and*

$$J_0^T(\pi_0) := \min_{\Pi^{comp} \in \Pi_W, \gamma} J_{\pi_0}(\Pi^{comp}, \gamma, T),$$

*the dynamic programming recursion*

$$T J_t^T(\pi_t) = \min_{Q \in \mathcal{Q}_c} \Big( c(\pi_t, Q_t) + T E[J_{t+1}^T(\pi_{t+1}) | \pi_t, Q_t] \Big)$$

*holds for all $t = 0, 1, \dots, T-1$.*

**Proof.** The proof differs from that of Theorem 5 in that the cost is unbounded. A rather technical uniform integrability argument allows the generalization; details will be given in [24]. □

## VI. Infinite Horizon setting

To facilitate the analysis in the infinite horizon setting, we add the following assumption.

**Assumption 4.** There exists a unique invariant probability measure $\pi$ for the Markov chain $\{x_t\}$ such that under the invariant probability measure $\pi$, $E_\pi[\|x\|^2] < \infty$. (Note that uniqueness comes naturally due to the irreducibility of $\{x_t\}$ caused by the presence of Gaussian noise in (2))

For the infinite horizon setting, one goal may be to analyze a discounted cost problem where the objective is the computation of

$$\inf_{\Pi^{comp}} \inf_{\gamma} J^\beta(\Pi^{comp}, \gamma)$$

for some $\beta \in (0,1)$, where

$$J^\beta(\Pi^{comp}, \gamma) = \lim_{T\to\infty} E_{\pi_0}^{\Pi^{comp},\gamma}\bigg[\sum_{t=0}^{T-1} \beta^t c_0(x_t, u_t)\bigg].$$

The solution to this problem follows from the discussion in Section IV. In particular, it is well known that the value iteration algorithm (see for example [12]) converges to an optimal solution since the cost function is bounded and the measurable selection hypothesis are applicable in view of Theorem 5.

The more challenging case is the average cost problem where a policy achieving

$$\inf_{\Pi^{comp}} \inf_{\gamma} J(\Pi^{comp}, \gamma) \tag{10}$$

is sought, where

$$J(\Pi^{comp}, \gamma) = \limsup_{T\to\infty} E_{\pi_0}^{\Pi^{comp},\gamma}\bigg[\frac{1}{T}\sum_{t=0}^{T-1} c_0(x_t, u_t)\bigg].$$

For this infinite horizon setting, the structural results in Theorem 1 and Theorem 2 are not available in the literature.

To facilitate the analysis, we first consider various classes of composite quantization policies.

### *A. Performance of Classes of Quantization Policies*

We will consider two classes of policies.

*1) Markov Quantizer Policies ($\Pi_M^C$):* This class contains quantizer policies that are Markov and which belong to $\Pi_W^C$. Such a Markov composite quantizer policy is a sequence of mappings such that at time $t \geq 0$, the policy causally maps $\pi_t \in \mathcal{P}(\mathbb{R}^n)$ to $\mathcal{Q}_c$ to generate the quantizer $Q_t$.

A more general class of Markov policies contain *randomized* ones, which assign a probability measure $P_{\mathcal{Q}_c,t,\pi_t}$ at time $t$ on $\mathcal{B}(\mathcal{Q}_c)$, in the following sense:

$$\begin{aligned}&P^\Pi(Q(x) = q | q_{[0,t-1]}, Q_{[0,t-1]}, \pi_{[0,t]}) \\ &\quad = \int_{\mathcal{Q}_c}\int_{\mathbb{R}^n} P_{\mathcal{Q}_c,\pi_t,t}(dQ_t) 1_{\{Q_t(x)=q\}} \pi_t(dx).\end{aligned}$$

Assigning a probability measure on $\mathcal{Q}_c$ is well-defined due to the hyper-plane parametrization of quantizers with convex codecells in Section II.

It follows from an argument in [9] or [7, Thm. 1] that without any loss, we can express the joint measure using a uniform random variable supported on a space of quantizers in the following sense. With $r$ denoting a realization of an independent uniformly distributed random variable with support $[0,1]$ and a Borel measurable mapping $Q^*$ that maps, for every $t$, $[0,1] \times \mathcal{P}(\mathbb{R})$ into $\mathcal{Q}_c$ (hence, $Q^*(r,\pi,t) \in \mathcal{Q}_c$), we have that for every $D \in \mathcal{B}(\mathbb{R}^n)$ and $q \in \mathbb{M}$

$$\begin{aligned}&\int_{\mathcal{Q}_c}\int_D P_{\mathcal{Q}_c,\pi_t,t}(dQ_t) 1_{\{Q_t(x)=q\}} \pi_t(dx) \\ &\qquad = \int_{[0,1]}\int_D U(dr) 1_{\{(Q^*(r,\pi_t,t))(x)=q\}} \pi_t(dx).\end{aligned}$$

Hence, if randomization is allowed, we assume that the randomization information is shared between the encoder and the decoder, that is

$$I_t^r = \{q_{[0,t-1]}, r_{[0,t-1]}, (Q^*(r,\pi,t)_{[0,t-1]}\},$$

and $\pi_t(dx)$ denotes the conditional probability measure on $\mathbb{R}^n$ such that

$$\pi_t(A) := P(x_t \in A | q_{[0,t-1]}, r_{[0,t-1]}), \quad A \in \mathcal{B}(\mathbb{R}^n).$$

*2) Stationary Quantizer Policies ($\Pi_S^C$):* A further restrictive class of quantizer policies are those which are stationary. Such policies are Markov (hence are in $\Pi_M^C$), but they do not depend on time. Thus, the function $Q^*(r,\pi,t)$ considered earlier does not depend on $t$.

### *B. Linear Programming Approach under Policies Admitting Separation*

In the following, we adopt the convex analytic approach of [4] (see [3] for a detailed discussion). Here we only present the essential steps. Let $\mathcal{F}_t$ denote the filtration generated by the information at the controller. Define a $\mathcal{F}_t$ measurable empirical occupation measure for all $D \in \mathcal{B}(\mathcal{P}(\mathbb{R}^n) \times \mathcal{Q}_c)$:

$$v_t(D) = \frac{1}{t}\bigg(\sum_{s=0}^{t-1} 1_{((\pi_s, Q_s)\in D)}\bigg).$$

Define $\mathcal{G}$ to be the set of ergodic occupation measures on $\mathcal{B}(\mathcal{P}(\mathbb{R}^n) \times \mathcal{Q}_c)$, which satisfy the following for all continuous and bounded functions $g : \mathcal{P}(\mathbb{R}^n) \times \mathcal{Q}_c \to \mathbb{R}$:

$$\begin{aligned}&\bigg\{v : \int g(\pi', Q') v(d\pi', dQ') \\ &\quad = \int \Big(\int g(\pi, Q) P^\Pi(d\pi, dQ | \pi', Q')\Big) v(d\pi', dQ')\bigg\}\end{aligned}$$

where $P^\Pi(d\pi, dQ|\pi', Q')$ stands for the transition kernel under a quantizer policy $\Pi$ which is stationary and possibly randomized.

With $\Pi \in \Pi_M^C$, it follows that any weak limit of $v_T(\cdot)$ is in the set $\mathcal{G}$.

If $v_T \to v^*$ weakly, for continuous $c$, we have that

$$\liminf_{k\to\infty} \int_{\mathcal{P}\times\mathcal{Q}_c} c(\pi,Q) v_{n_k}(d\pi,dQ) \geq \int_{\mathcal{P}\times\mathcal{Q}_c} c(\pi,Q) v^*(d\pi,dQ).$$

Hence, we look for a minimizing $v^* \in \mathcal{G}$, which provides a uniform lower bound under any policy. Thus, we solve the linear program of the minimization of

$$\int_{\mathcal{P}\times\mathcal{Q}_c} c(\pi,Q) v(d\pi,dQ),$$

such that $v \in \mathcal{G}$.

**Theorem 7.** *Under Assumptions 1 and 4, there exists an optimal quantization policy in $\Pi_M^C$. Without any loss, an optimal quantization policy is stationary (and possibly randomized; hence it is in $\Pi_S^C$)) provided that the initial state is picked according to the optimal stationary measure.*

**Theorem 8.** *Under the setup of Theorem 7, there exists an optimal quantization policy which is stationary and deterministic.*

## VII. Concluding Remarks

In this paper we established the existence of optimal quantization policies. The key ingredient of our analysis was the characterization of quantizers as a subset of the space of stochastic kernels, endowed with a weak convergence topology on the product space of the input and the output.

For the infinite horizon setting, one weakness possible in our result is that we restrict the initial condition to live in a particular set. In some applications, the encoder has freedom on where to start the belief. Therefore, this may be a realistic assumption.

The machinery presented here will be useful in the optimal quantized control of a linear system driven by unbounded noise. In particular, if the dual effect of the control policies can be decoupled from the estimation error [23], [17], [15] then the design here can be used to establish existence of optimal policies.

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