Linear Algebraic Theory of Partial Coherence: Discrete Fields and Measures of Partial Coherence

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Abstract

A discrete linear algebraic theory of partial coherence is presented which allows precise mathematical definitions of concepts such as coherence and incoherence. This not only provides new perspectives and insights, but also allows us to employ the conceptual and algebraic tools of linear algebra in applications. We define a scalar degree of partial coherence of an optical field which is zero for complete incoherence and unity for full coherence. The mathematical definitions are related to our physical understanding of the corresponding concepts by considering them in the context of Young's experiment.

1 Introduction

The theory of partial coherence is a well established area of optics [1, 2, 3, 4]. In this paper we will see that formulating the theory in terms of the standard concepts of linear algebra [5] leads to a number of new perspectives. While not containing any new physics, this approach offers new insights and understanding and has the potential to facilitate applications, especially in optical information processing.

We restrict our attention to quasi-monochromatic conditions and do not deal with issues of temporal coherence. We deal with light fields of varying degrees of spatial coherence, as characterized by their mutual intensity (autocorrelation) functions.

In this paper we consider the case of discrete light fields since they lead to a particularly simple matrix-algebraic formulation without the distractions accompanying discussions of continuous function spaces. Once the framework is established, it is not difficult to translate the matrix formalism for discrete fields to a continuous formalism. It may be argued that dealing with discrete fields is artificial since optical fields are continuous fields and most discussions of partial coherence are phrased in classical continuous analysis. However, (i) The discrete formalism presented here captures virtually all of the essential physics, while being more transparent and easily comprehensible. (ii) All continuous fields in essence have only a finite number of degrees of freedom and can be represented by finite discrete vectors; in that sense discrete vectors can be argued to serve the same representational purpose as functions of continuous variables. In fact, some would argue that the discrete representation is superior in that it is less redundant. In any event, discrete and continuous representations are related to each other through sampling and interpolation relations.

In the next two sections we define a number of matrices which characterize the second-order correlations of the field. In section 4 we mathematically define the limits of full coherence and full incoherence in terms of these matrices. In section 5 we define a measure c of the degree of partial coherence which interpolates between these two extremes such that $c = 1$ for full coherence and $c = 0$ for full incoherence. Section 6 extends some of these concepts into the spectral domain, whereas section 7 discusses these concepts in the context of Young's experiment.

2 Mutual intensity matrix

Let $\mathbf{f} = [f(1), f(2), \ldots, f(N)]^{\mathrm{T}}$ be a vector representing a discrete random optical field (mathematically, a finite random sequence). $\mathbf{f}$ may be thought to consist of the representative samples of a function of a continuous variable, although we will not be dealing with sampling issues in this paper. For simplicity, we deal with one-dimensional signals; the extension to two-dimensions being straightforward. We assume quasi-monochromatic conditions and concentrate on the mutual intensity matrix $\mathbf{J_f}$ of $\mathbf{f}$, defined as

$$\mathbf{J_f} = \langle \mathbf{f}\mathbf{f}^{\mathrm{H}} \rangle, \tag{1}$$

where the angle brackets denote ensemble averaging (expectation value) and the superscript H denotes Hermitian transpose (conjugate transpose). We will simply write $\mathbf{J}$ instead of $\mathbf{J_f}$ when there is no room for confusion. Letting $J(m,n)$ denote the elements of $\mathbf{J}$, we note that $J(n,n) = I(n)$ is the intensity of the field and $E = \sum_{k=1}^{N} I(n)$ is the energy. We also define a mean-subtracted version of the mutual intensity, denoted $\mathbf{K_f}$ and defined as

$$\mathbf{K_f} = \langle (\mathbf{f} - \boldsymbol{\mu}_\mathbf{f})(\mathbf{f} - \boldsymbol{\mu}_\mathbf{f})^{\mathrm{H}} \rangle, \tag{2}$$

where $\boldsymbol{\mu}_\mathbf{f} = \langle \mathbf{f} \rangle$ is the mean of $\mathbf{f}$, a vector whose elements are the individual means of the elements of $\mathbf{f}$. Letting $K(m,n)$ denote the elements of $\mathbf{K}$, we note that $K(n,n) = \sigma^2(n)$ is the variance of the field. In the language of the theory of random processes or statistics, the mutual intensity matrix $\mathbf{J}$ is an autocorrelation matrix whereas its mean-subtracted version $\mathbf{K}$ is an autocovariance matrix. The relation between $\mathbf{J}$ and $\mathbf{K}$ is given by $\mathbf{J} = \mathbf{K} + \boldsymbol{\mu}_\mathbf{f}\boldsymbol{\mu}_\mathbf{f}^{\mathrm{H}}$.

Now, the following properties hold for the $\mathbf{J}$ and $\mathbf{K}$ matrices of any field $\mathbf{f}$ (the proofs are omitted where elementary or well known [5, 6]):

1. $\mathbf{J}$ and $\mathbf{K}$ are Hermitian symmetric: $\mathbf{J} = \mathbf{J}^{\mathrm{H}}$ and $\mathbf{K} = \mathbf{K}^{\mathrm{H}}$.

2. As with all Hermitian symmetric matrices, the eigenvalues of $\mathbf{J}$ and $\mathbf{K}$ are all real.

3. $\mathbf{J}$ and $\mathbf{K}$ are positive semi-definite, and as with all such matrices, the eigenvalues are non-negative. Furthermore, $|J(m,n)|^2 \leq |J(n,n)|\,|J(m,m)|$, and likewise for $\mathbf{K}$.

4. Eigenvectors corresponding to distinct eigenvalues will always be orthogonal. Furthermore, as with any Hermitian symmetric matrix, a complete set of orthogonal eigenvectors can always be found even when there are degenerate eigenvalues. Let $\mathbf{U}$

denote the matrix whose columns are the orthonormal eigenvectors. Then, this matrix (known as the modal matrix) will diagonalize the Hermitian symmetric matrix $\mathbf{J}$ or $\mathbf{K}$:

$$\mathbf{J} = \mathbf{U_J} \mathbf{\Lambda_J} \mathbf{U_J^{\mathrm{H}}}, \tag{3}$$

$$\mathbf{K} = \mathbf{U_K} \mathbf{\Lambda_K} \mathbf{U_K^{\mathrm{H}}}, \tag{4}$$

where $\mathbf{\Lambda_J}$ and $\Lambda_{\mathbf{K}}$ are diagonal matrices consisting of the eigenvalues of $\mathbf{J}$ and $\mathbf{K}$ along their diagonals. These expansions are special cases of the singular-value decomposition [5]. They can also be written in the more explicit form:

$$\mathbf{J} = \sum_{k=1}^{N} \lambda_{\mathbf{J}k} \, \mathbf{u}_{\mathbf{J}k} \, \mathbf{u}_{\mathbf{J}k}^{\mathrm{H}}, \tag{5}$$

$$J(m,n) = \sum_{k=1}^{N} \lambda_{\mathbf{J}k} \, u_{\mathbf{J}k}(m) u_{\mathbf{J}k}^{*}(n), \tag{6}$$

where $J(m,n)$ denotes the elements of $\mathbf{J}$, $\lambda_{\mathbf{J}k}$ denotes the kth eigenvalue of $\mathbf{J}$, and $\mathbf{u}_{\mathbf{J}k}(n)$ is the nth element of the kth column of $\mathbf{U_J}$. (A similar expression holds for $K(m,n)$, the elements of $\mathbf{K}$.) This expression is sometimes referred to as the spectral expansion of $J(m,n)$, or since each term in the summation is in the form of an outer product, an outer-product expansion. In an optical context, this expression is also known as the coherent-mode representation [7, 8], as will be clear in section 4.

5. If $\mathbf{J}$ or $\mathbf{K}$ can be expressed as the outer product of two vectors $\mathbf{u}'$ and $\mathbf{u}''$ in the form $\mathbf{u}'\mathbf{u}''^{\mathrm{H}}$, then Hermitian symmetry implies that $\mathbf{u}'$ and $\mathbf{u}''$ must be parallel, so that by appropriate scaling $\mathbf{J}$ or $\mathbf{K}$ can be expressed in self-outer-product form $\mathbf{u}\mathbf{u}^{\mathrm{H}}$. That is, any Hermitian symmetric matrix which can be written in outer-product form can also be written in self-outer-product form.

6. The following are equivalent: (i) $\mathbf{J}$ or $\mathbf{K}$ can be expressed in self-outer-product form (or by virtue of the previous item, merely in outer-product form). (ii) Each row is a multiple of the other rows. (iii) The rank R of the matrix is 1. (iv) The matrix has only one non-zero eigenvalue (equation 6).

7. More generally, the rank of $\mathbf{J}$ (or $\mathbf{K}$) is equal to the number of non-zero eigenvalues [5].

We will see in section 4 that matrices satisfying any of the equivalent conditions stated in item 6 above correspond to fully coherent light. However, we will have to define two more matrices before we start discussing coherence and incoherence.

3 Normalized mutual intensity: the complex coherence matrix

If our interest is more in the relative correlations of various points of the field, but not so much in the absolute intensity of the field, then it is more convenient to work with the following normalized versions of $\mathbf{J}$ and $\mathbf{K}$ respectively:

$$L(m,n) = \frac{\langle f(m)f^*(n)\rangle}{\sqrt{\langle |f(m)|^2\rangle\langle |f(n)|^2\rangle}} = \frac{J(m,n)}{\sqrt{J(m,m)J(n,n)}}, \tag{7}$$

$$M(m,n) = \frac{\langle (f(m)-\mu(m))(f(n)-\mu(n))^*\rangle}{\sqrt{\langle |f(m)-\mu(m)|^2\rangle\langle |f(n)-\mu(n)|^2\rangle}} = \frac{K(m,n)}{\sqrt{K(m,m)K(n,n)}}, \tag{8}$$

where $L(m,n)$ and $M(m,n)$ are the elements of the matrices $\mathbf{L}$ and $\mathbf{M}$, the former which we refer to as the complex coherence matrix. The diagonal elements of both matrices are identically unity. Note that $\mathbf{M}$ is merely the complex coherence matrix of the mean-subtracted field $\mathbf{f}-\boldsymbol{\mu}$. $M(m,n)$ is nothing but the standard statistical correlation factor of the two random variables $f(m)$ and $f(n)$ [6].

Since $\mathbf{L}$ and $\mathbf{M}$ are obtained by normalizing $\mathbf{J}$ and $\mathbf{K}$ in a particular way, they inherit many of their properties:

1. Hermitian symmetry: $\mathbf{L} = \mathbf{L}^{\mathrm{H}}$ and $\mathbf{M} = \mathbf{M}^{\mathrm{H}}$.

2. The eigenvalues of $\mathbf{L}$ and $\mathbf{M}$ are all real.

3. $\mathbf{L}$ and $\mathbf{M}$ are positive semi-definite and their eigenvalues are non-negative. Furthermore, $|L(m,n)| \le 1$ and $|M(m,n)| \le 1$.

4. The eigenvectors can be chosen to be orthonormal, and $\mathbf{L}$ and $\mathbf{M}$ can be decomposed as in equation 3 and equation 5.

5. If $\mathbf{L}$ or $\mathbf{M}$ can be written in outer-product form, it can also be written in self-outer-product form.

6. The following are equivalent: (i) $\mathbf{L}$ or $\mathbf{M}$ can be expressed in self-outer-product form (or in outer-product form). (ii) Each row is a multiple of the other rows. (iii) The rank of the matrix is 1. (iv) The matrix has only one non-zero eigenvalue.

7. The rank R of $\mathbf{L}$ (or $\mathbf{M}$) is equal to the number of non-zero eigenvalues.

Furthermore, we can state the following additional properties:

1. The diagonal entries of both $\mathbf{L}$ and $\mathbf{M}$ are all equal to 1, corresponding to the fact that each point is by definition fully correlated with itself.

2. If $\mathbf{J}$ (or $\mathbf{K}$) has unit rank, then all elements of $\mathbf{L}$ (or $\mathbf{M}$) have unit magnitude. Conversely, if all elements of $\mathbf{L}$ (or $\mathbf{M}$) have unit magnitude, positive semi-definiteness implies that $\mathbf{J}$ (or $\mathbf{K}$) has unit rank.

4 Full coherence and full incoherence

Full coherence and full incoherence (lack of coherence) are the two extreme cases of the continuum of partial coherence. In most texts, these concepts are introduced and discussed for the spatial case through Young's experiment. Under quasi-monochromatic conditions, temporal coherence issues can be set aside and the interference fringe visibility reflects the degree of coherence of the light. Fully coherent light results in maximum fringe depth (greatest visibility) and fully incoherent light results in zero fringe depth (no visibility). Two samples of a coherent light field result in interference effects, whereas samples of an incoherent field do not. For fully coherent light we have complex amplitude superposition, whereas for fully incoherent light we have intensity superposition. The reader is assumed to be already familiar with these elementary concepts [3, 9, 10].

Up to this point we have introduced four different but closely related matrices which characterize the second-order correlations of a random optical field, and discussed the properties of these matrices, most of which followed from Hermitian symmetry and positive semi-definiteness. Now we will provide a mathematical definition of full spatial coherence and full spatial incoherence in terms of these matrices.

Underlying the concept of coherence or incoherence of a field is the statistical correlation of two spatial samples of that field. In general, a field is considered coherent if any two samples of the field are fully correlated, and a field is considered incoherent if any two distinct samples are fully uncorrelated. Two random variables will be said to be fully correlated if they are just as correlated with each other as they are with themselves. Mathematically, this may be expressed by the requirement that the magnitude of their normalized correlation or covariance has the largest possible value of unity. Two random variables will be said to be fully uncorrelated if the magnitude of their normalized correlation or covariance is as small as possible, namely zero.

In practice a field can be said to be effectively coherent or incoherent, if for the purposes of the optical system in question the field behaves as if it was fully coherent or incoherent (for instance, based on finite apertures or finite resolution), but this issue will not concern us here.

Although a number of the results stated below concern the unnormalized matrices $\mathbf{J}$ and $\mathbf{K}$, we will see that the working with the normalized matrices $\mathbf{L}$ and $\mathbf{M}$ has certain advantages. However, the choice between $\mathbf{L}$ and its mean-subtracted version $\mathbf{M}$ is not so clear cut. This depends on precisely how we wish to define coherence and incoherence; we leave this choice of definition to the reader. This issue is further discussed in conjunction with Young's two slit experiment in section 7. For sake of simplicity in presentation, from now on we work with the mutual intensity matrix $\mathbf{J}$ and the complex coherence matrix $\mathbf{L}$. However, we will keep in mind that we may simply replace $\mathbf{J}$ with $\mathbf{K}$ and $\mathbf{L}$ with $\mathbf{M}$ in the following discussion if this is desirable. Choosing to work with the mean-subtracted version results in a different definition of full coherence and full incoherence, but this difference is not an essential one and the concepts converge to the same physical reality when properly interpreted.

4.1 Full incoherence

First we consider incoherent fields. Since any two distinct samples of such a field must be uncorrelated, the mutual intensity matrix $\mathbf{J}$ and its normalized version, the complex coherence matrix $\mathbf{L}$ must be diagonal. In fact, as a consequence of normalization, $\mathbf{L}$ is the identity matrix. (Some of the diagonal elements of $\mathbf{J}$ may be zero because the field is zero at those points, in which case the normalization results in 0/0 indefinite entries; however, interpreting the zero elements as the limit of very small elements allows us to consistently maintain that $\mathbf{L}$ is the identity matrix.) Therefore,

A discrete optical field is fully incoherent if the associated normalized mutual intensity matrix (complex coherence matrix) is the identity matrix:

$$\mathbf{L} = \mathbf{I}, \tag{9}$$

or alternatively if the associated mutual intensity matrix $\mathbf{J}$ is diagonal.

Though trivial, we also note that for the fully incoherent case, the matrix $\mathbf{L}$ is of full rank ($R = N$), and all of its eigenvalues are equal to unity. On the other hand, the $\mathbf{J}$ (or $\mathbf{K}$) matrix for an incoherent field need not be full rank, nor does every full rank matrix correspond to an incoherent field.

We also note that when the matrix $\mathbf{J}$ is equal to a scalar multiple of the identity matrix, the underlying field $\mathbf{f}$ is stationary and referred to as a *white noise* process. When $\mathbf{J}$ is diagonal but not equal to the identity, $\mathbf{f}$ is non-stationary and referred to as a *colored noise* process. When the samples of $\mathbf{f}$ are independent random variables $\mathbf{K}$ will be diagonal, and if full incoherence is defined in terms of the diagonality of $\mathbf{K}$, it follows that $\mathbf{f}$ is an incoherent field. On the other hand, if full incoherence is defined in terms of the diagonality of $\mathbf{J}$, then $\mathbf{f}$ must be a zero-mean process for $\mathbf{J}$ to also be diagonal and thus $\mathbf{f}$ an incoherent field. When the samples are independent and identically distributed, $\mathbf{K}$ will equal the identity matrix, leading to similar consequences, but with the additional feature that we now have a stationary field.

4.2 Full coherence

Now we consider coherent fields. Any two samples of such a field must be fully correlated. As already discussed, this means that the magnitude of their normalized correlation has the largest possible value of unity. Therefore the elements of the complex coherence matrix $\mathbf{L}$ must have unit magnitude:

A discrete optical field is fully coherent if all elements of the associated normalized mutual intensity matrix (complex coherence matrix) have unit magnitude:

$$|L(m,n)| = 1, \qquad m,n = 1,\ldots,N, \tag{10}$$

or alternatively if the associated mutual intensity matrix $\mathbf{J}$ has unit rank.

We have already seen that in this case the matrices $\mathbf{J}$ and $\mathbf{L}$ both have unit rank, are of outer-product form, and consequently have only one non-zero eigenvalue (item 6 in sections 2 and 3 and the following discussion). The sole non-zero eigenvalue of $\mathbf{L}$ is equal to N. (This is easily deduced from the fact that for any square matrix the sum of the eigenvalues is equal to the trace: the sum of the diagonal elements.)

Full coherence can be alternatively defined through the requirement that the rank of $\mathbf{J}$ or $\mathbf{L}$ equal unity, or that $\mathbf{J}$ or $\mathbf{L}$ has only one non-zero eigenvalue, or that $\mathbf{J}$ or $\mathbf{L}$ is of outer-product form.

If we denote the outer-product form of the $\mathbf{L}$ matrix of a coherent field as $\mathbf{L} = \mathbf{u}\mathbf{u}^{\mathrm{H}}$, we see that all elements of $\mathbf{u}$ are also of unit magnitude; that is, the normalized correlation corresponds to the unnormalized correlation of a field normalized to unit magnitude. This means that the result of the normalization is to remove the effects of spatial intensity variation from the field. This is desirable since we are more interested in the relative correlations of pairs of points rather than the variation of intensity across the field.

Deterministic fields (fields which are not random processes) are coherent. Since the ensemble average is superfluous, the matrix $\mathbf{J}$ for such a field is of outer-product form and hence the field is coherent. On the other hand, every coherent field is not necessarily deterministic.

4.3 Stationarity

Another common concept that can be characterized in terms of the matrices defined is *stationarity*. Here we restrict our attention to second-order stationarity, which means that second-order correlations depend only on the separation of the two points in question, and not on their absolute positions. As is common practice, we define stationarity for a finite domain based on periodic boundary conditions:

> A discrete optical field is stationary if the associated mutual intensity matrix is *circulant*:
>
> $$J(m,n) = J((m-n) \bmod N), \qquad m,n = 1,\ldots,N; \tag{11}$$

that is, if the elements on any given circular diagonal are equal to each other. A circular diagonal is one which wraps around in the sense of periodic boundary conditions. If $\mathbf{J}$ is circulant, so is $\mathbf{L}$. An alternative approach would be to define stationarity by requiring that $\mathbf{J}$ be Toeplitz instead of circulant, when circular diagonals are replaced by ordinary diagonals. However, it is standard practice to work with circulant matrices in such discrete settings; they are not only more suitable for analytical formulations but in fact easier to relate to a continuous framework.

5 The degree of partial coherence of a field

Having established the two extremes of full coherence and full incoherence and provided precise definitions for them in terms of their normalized correlation matrices, we now define a scalar measure of the degree of partial coherence of a field. This can be accomplished by interpolating any of the characteristics of the matrices in question. For instance, we saw that the eigenvalues of the matrix $\mathbf{L}$ for incoherent fields are all equal to unity, whereas only one of the eigenvalues of a coherent field is non-zero. Thus a suitable function of the eigenvalues which takes its extremes at these two special cases can serve as a measure of the degree of partial coherence. Alternatively, we saw that the matrix $\mathbf{L}$ for an incoherent field is diagonal, representing maximum concentration around the diagonal, whereas the same matrix for a coherent field has elements with unit magnitude, representing maximum spread. Again, a suitable interpolation will yield a measure of coherence.

There are many ways of constructing such interpolation functions, leading to several definitions of such a measure, some which can be shown to be identical, some monotonically related, and some quite different. Here we will present some of the more obvious candidates for such a measure. Our purpose here is not to offer an exhaustive analysis of these, but to motivate the different possibilities.

All of these measures c' are defined such that their minimum value c'_{min} corresponds to full incoherence and their maximum value c'_{max} corresponds to full coherence (or the other way around). We will employ the following mapping to obtain a final measure c which equals 0 for incoherent light and equals 1 for coherent light:

$$c = \frac{c' - c'_{\mathrm{incoh}}}{c'_{\mathrm{coher}} - c'_{\mathrm{incoh}}}, \tag{12}$$

where c'_{incoh} and c'_{coher} are the extreme values corresponding to the incoherent and coherent cases ($c'_{\mathrm{incoh}} = c'_{\mathrm{min}}$ and $c'_{\mathrm{coher}} = c'_{\mathrm{max}}$ or the other way around).

5.1 Eigenvalue based measures

As already stated, incoherent light is characterized by all unity eigenvalues and coherent light by one non-zero eigenvalue of the matrix $\mathbf{L}$. We also recall that the eigenvalues are all non-negative and their sum is equal to N.

Definition 1: Based on the observation above, the more concentrated the eigenvalues are around the largest one, the more coherent the light, and the more uniformly spread they are, the more incoherent the light. It will be convenient to assume that in the general case the eigenvalues are ordered in decreasing order. Therefore the following measure of the spread of the eigenvalues away from the largest eigenvalue (which has index $n = 1$) will serve as a measure of the degree of partial coherence:

$$c_1' = \frac{1}{N} \sum_{n=1}^{N} (n-1)^2 \lambda_n. \tag{13}$$

When all eigenvalues are unity (incoherent light), we have $c_1' = (N-1)(2N-1)/6$, and when only one eigenvalue is non-zero (coherent light) we have $c_1' = 0$.

Definition 2: If we think of the distribution of eigenvalues as a function of the index n to constitute a kind of generalized spectral distribution, then the above measure essentially corresponds to a spectral spread. The distribution of values of the eigenvalues can also be measured without reference to a particular discrete variable with respect to which they are indexed, merely as the spread amongst a group of numbers; that is, as the variance of the eigenvalues:

$$c_2' = \frac{1}{N} \sum_{n=1}^{N} (\lambda_n - 1)^2, \tag{14}$$

where the 1 subtracted from λ_n is the average value of the eigenvalues. When all eigenvalues are unity, we have $c_2' = 0$, and when only one eigenvalue is non-zero we have $c_2' = N - 1$.

Definition 3: The concept of measuring maximal concentration versus maximally uniform spread brings to mind the concept of entropy which measures maximum order versus disorder. Therefore the following alternative to the previous measure naturally asserts itself:

$$c_3' = -\sum_{n=1}^{N} \frac{\lambda_n}{N} \log\left(\frac{\lambda_n}{N}\right), \tag{15}$$

where we recall $\sum_{n=1}^{N} (\lambda_n / N) = 1$. The base of the logarithm is of no consequence. When all eigenvalues are unity, we have $c_3' = \log N$, and when only one eigenvalue is non-zero we have $c_3' = 0$ (since $z \log z \to 0$ as $z \to 0$).

A similar measure was previously proposed in [11, 12] based on the eigenvalues of the unnormalized mutual intensity matrix $\mathbf{J}$, rather than the normalized complex coherence matrix $\mathbf{L}$ we are employing. Without normalization, this measure does not properly characterize the incoherent limit. Our formulation based on normalized matrices solves this and other problems associated with the mutual intensity matrix.

Although space does not permit us to further discuss the relationships between these measures in this paper, we note that definitions 2 and 3 (and also definition 5 to be given below) can be shown to be monotonically related to each other.

5.2 Matrix spread based measures

We now turn our attention from the eigenvalues to the complex coherence matrix $\mathbf{L}$ itself. When $\mathbf{L}$ is diagonal the light is incoherent and when $\mathbf{L}$ has all unit magnitude entries, the light is coherent. Interpolation between these two extremes leads to the following measures of partial coherence.

Definition 4: We consider the spatial variance (moment of inertia) around the diagonal of the matrix $\mathbf{L}$. Since $|L(m,n)| = |L(n,m)|$ from Hermitian symmetry and assuming periodic boundary conditions so that $L(m, N+n) = L(m,n)$ and $L(N+m, n) = L(m,n)$, we can form the following measure of spread:

$$c_4' = \sum_{n=1}^{N} \sum_{m=1}^{N} (m-n)^2 |L(m,n)|^2. \tag{16}$$

When $\mathbf{L}$ is the unit matrix (incoherent light), we have $c_4' = 0$, and when all elements of $\mathbf{L}$ have unit magnitude (coherent light), we have $c_4' = 2\sum_l^N (N-l)l^2$. It should be noted that c_4' is not strictly a spatial variance since we do not normalize by $\sum_{n=1}^N \sum_{m=1}^N |L(m,n)|^2$. Alternatively, to be truer to the spirit of periodic boundary conditions, it is possible to replace $(m-n)^2$ with $(N-m+n)^2$ when $m-n > N/2$.

Definition 5: Alternatively, we can ignore the spatial distribution of the matrix entries and simply measure the energy of the matrix $\mathbf{L}$. Recalling that the elements of $\mathbf{L}$ are always less than or equal to unity and that the diagonal elements are always equal to unity, this measure will be minimum for the unit matrix (incoherent light) and maximum when all elements have unit magnitude (coherent light):

$$c_5' = \frac{1}{N^2} \sum_{n=1}^{N} \sum_{m=1}^{N} |L(m,n)|^2. \tag{17}$$

When $\mathbf{L}$ is the unit matrix, we have $c_5' = 1/N$, and when all elements of $\mathbf{L}$ have unit magnitude, we have $c_5' = 1$.

These and other measures of partial coherence will be further studied and compared in another paper. Here we satisfy ourselves by noting that there is no definition which is clearly superior in all circumstances; different measures are appropriate for different situations. (For instance, consider the question of whether a mutual intensity function with a narrow main lobe and side lobes, or one with a wider main lobe but no side lobes, is more coherent.)

6 Fourier-domain analysis

In section 4 we stated the definitions of coherence and incoherence in terms of autocorrelations (or autocovariances) of the original field $\mathbf{f}$. Here we wish to state the equivalent conditions for coherence and incoherence in the Fourier domain.

Let us denote the discrete Fourier transform (DFT) of the discrete optical field $\mathbf{f}$ by $\tilde{\mathbf{f}} = \mathbf{F}\mathbf{f}$ where $\mathbf{F}$ is the $N \times N$ unitary discrete Fourier transform matrix defined by $F(m,n) = N^{-1/2}\exp(-i2\pi mn/N)$.

Let us find the mutual intensity matrix $\mathbf{J}_{\tilde{\mathbf{f}}}$ of the Fourier transform of the optical field:

$$\mathbf{J}_{\tilde{\mathbf{f}}} = \langle \mathbf{F}\mathbf{f}(\mathbf{F}\mathbf{f})^{\mathrm{H}} \rangle = \mathbf{F}\mathbf{J}_{\mathbf{f}}\mathbf{F}^{\mathrm{H}}. \tag{18}$$

We observe that the mutual intensity of the Fourier transform is the double Fourier transform of the mutual intensity of the original.

We know that a circulant $\mathbf{J}$ matrix corresponds to a stationary random process. (If the matrices in question represented the kernel of a linear system, a circulant matrix would represent a linear-shift invariant system.) Complex harmonics are known to be the eigenvectors of circulant matrices. Therefore the DFT matrix, whose columns are the complex harmonics, will diagonalize any circulant matrix [5]:

$$\mathbf{\Lambda}_{\mathbf{J}} = \mathbf{F}\mathbf{J}\mathbf{F}^{\mathrm{H}}. \tag{19}$$

First, let us assume that $\mathbf{J}_{\mathbf{f}}$ is circulant, corresponding to a stationary optical field. Then, $\mathbf{J}_{\tilde{\mathbf{f}}}$ will be diagonal. In other words, a stationary optical field is characterized by a circulant matrix in the space domain but by a diagonal matrix in the Fourier domain. Now, let us assume that $\mathbf{J_f}$ is diagonal, corresponding to an incoherent optical field. Then, $\mathbf{J_{\tilde{f}}}$ will be circulant. In other words, an incoherent optical field is characterized by a diagonal matrix in the space domain but by a circulant matrix in the Fourier domain. We may summarize by saying that incoherence and stationarity are Fourier duals (or conjugates).

If we assume that $\mathbf{J_f}$ is of unit rank, then it is possible to show that $\mathbf{J_{\tilde{f}}}$ is also of unit rank. In other words, a coherent field is represented by a unit-rank matrix both in the space and Fourier domains. Since matrices of this form represent coherent fields, we may say that coherence is its own Fourier dual. (This is also consistent with the fact that if a field is deterministic, its Fourier transform will also be deterministic.)

If we restrict our attention only to second-order stationarity and do not pay attention to a possible non-stationarity in the mean, the circulantness of $\mathbf{K}$ can be taken as an alternative definition of stationarity. In this case, the above discussion can also be taken to be valid for $\mathbf{K}$. However, it should be noted that $\mathbf{K_{\tilde{f}}} \neq \mathbf{F}\mathbf{K_f}\mathbf{F}^{\mathrm{H}}$.

7 Young's experiment and correlation versus covariance

The purpose of this section is to consider the superposition of two random variables, which may represent two samples of an optical field, and discuss how the intensity of the superposed field depends on the correlation or covariance of these two random variables. This will lead us to discuss the use of correlation ($\mathbf{J}$ or $\mathbf{L}$) versus covariance ($\mathbf{K}$ or $\mathbf{M}$) in characterizing partially coherent fields.

Traditionally the concepts of spatial coherence and incoherence are motivated in terms of the visibility of fringes in Young's experiment under quasi-monochromatic conditions. The visibility is usually defined as the ratio of the difference between the maximum and minimum intensity, to the sum of the maximum and minimum intensity. Incoherent light is associated with zero fringe visibility whereas coherent light is associated with maximum fringe visibility. Maximum fringe visibility corresponds to maximum constructive/destructive interference where the two complex amplitudes are added or subtracted with full force. Zero visibility corresponds to complete lack of interference where the intensities are added without any cross terms. Therefore, our traditional understanding of coherence and incoherence has to do with the ability or lack of ability to constructively or destructively interfere.

Let the complex amplitude of light represented by these two random variables be denoted by $f_1 = A_1 \exp(i\phi_1)$ and $f_2 = A_2 \exp(i\phi_2)$ where $A_1, A_2, \theta_1, \theta_2$ are real. (In this section we do not use boldface to denote the fields since our discussion is equally valid for samples of a continuous field.) The total observed field is $f = f_1 + f_2 \exp(i\alpha)$, where α is the phase difference associated with the path length difference in Young's experiment. Let us further denote the ensemble-averaged mean of f_1 by μ_1 and that of f_2 by μ_2. Then we can write $f_1 = \mu_1 + g_1$ and similarly for f_2, where g_1 is a zero-mean random variable. Thus, any non-zero mean random variable can be interpreted as the sum of a non-random number plus a zero-mean random variable. The intensity $I = \langle |f_1 + f_2 \exp(i\alpha)|^2 \rangle$ resulting from the superposition of f_1 and f_2 can be written in two alternative but equal forms:

$$I = I_1 + I_2 + 2\sqrt{I_1 I_2}\, |L_{12}| \cos(\angle[L_{12}] - \alpha), \tag{20}$$

$$= |\mu_1|^2 + P_1 + |\mu_2|^2 + P_2 + 2\sqrt{P_1 P_2}\, \left|\mu_1\mu_2^*/\sqrt{P_1P_2} + M_{12}\right| \cos(\angle[\mu_1\mu_2^*/\sqrt{P_1P_2} + M_{12}] - \alpha), \tag{21}$$

where $P_i = \langle |g_i|^2 \rangle$, $I_i = \langle |f_i|^2 \rangle$, and thus $I_i = |\mu_i|^2 + P_i$. We use $\angle[z]$ to denote the argument of z. In deriving the second of these expressions we used a trigonometric formula for the sum of two cosines. The first of these expressions has been expressed in terms of the normalized non-mean-subtracted (correlation) $\mathbf{L}$ matrix entry $L_{12} = \langle f_1 f_2^* \rangle / \sqrt{I_1 I_2}$, and the second has been expressed in terms of the normalized mean-subtracted (covariance) $\mathbf{M}$ matrix entry $M_{12} = \langle g_1 g_2^* \rangle / \sqrt{P_1 P_2}$. The above expressions take their extreme values over α (in terms of which visibility is defined) when the cosine term is equal to ± 1. Thus the visibility, defined as the ratio $(\max - \min)/(\max + \min)$ is given by

$$\text{Visibility} = \frac{2\sqrt{I_1 I_2}\, |L_{12}|}{I_1 + I_2}, \tag{22}$$

$$= \frac{2\sqrt{P_1 P_2}\, |\mu_1\mu_2^*/\sqrt{P_1P_2} + M_{12}|}{|\mu_1|^2 + P_1 + |\mu_2|^2 + P_2}. \tag{23}$$

When the means μ_1, μ_2 are equal to zero, both of the above expressions become identical. In this case $I_i = P_i$ and $M_{12} = L_{12}$. In the case of incoherent light we expect intensity addition and no cross terms (zero visibility), implying $|L_{12}| = |M_{12}| = 0$. In the case of coherent light we expect complex amplitude addition (maximum visibility), implying $|L_{12}| = |M_{12}| = 1$. Therefore we see that the mathematical definitions of coherence and incoherence offered in section 4 are consistent with the traditional understanding of these concepts when the means are zero.

Now we turn our attention to the general non-zero mean case, when equations 20 and 21 (or 22 and 23) are no longer identical. Intensity addition (no interference) means that the resultant intensity is given by $I_1 + I_2$ and therefore that the visibility is 0. If we associate incoherence with intensity addition and zero visibility, it follows that $|L_{12}| = 0$ but the same cannot be said for $|M_{12}|$. If $|M_{12}| = 0$, the resultant intensity is given by $I_1 + I_2 + 2|\mu_1\mu_2|\cos(\angle[\mu_1\mu_2^*] - \alpha)$. We observe that although the correlation of the mean-subtracted part is zero, the deterministic parts corresponding to the non-zero means lead to an interference term which violates strict intensity addition. (In other words, the mean field behaves like a deterministic coherent component, so that a field with non-zero mean cannot be strictly incoherent if we associate incoherence with intensity addition.) However, if we choose to define incoherence with the alternative concept of intensity addition of the mean-subtracted parts only ($P = P_1 + P_2$), treating the means as mere insignificant biases, then for this definition of incoherence it follows that $|M_{12}| = 0$ but $|L_{12}| \neq 0$.

Complex amplitude addition (full interference) means that the extremes of the resultant intensity are given by $I_1 + I_2 \pm 2\sqrt{I_1 I_2}$ and that the visibility is $2\sqrt{I_1 I_2}/(I_1 + I_2)$. If we associate coherence with complex amplitude addition and maximum visibility, it follows that $|L_{12}| = 1$ but the same cannot be said for $|M_{12}|$. If $|M_{12}| = 1$, the normalized correlation of the mean-subtracted parts is unity, but as a whole, complex amplitude addition is violated. However, if we choose to define coherence with the alternative concept of complex amplitude addition of the mean-subtracted parts only ($P = P_1 + P_2 \pm 2\sqrt{P_1 P_2}$), treating the means as mere insignificant biases, then for this definition of coherence it follows that $|M_{12}| = 1$ but $|L_{12}| \neq 1$.

In summary, we have shown that defining coherence and incoherence in terms of the normalized complex coherence matrix $\mathbf{L}$ is appropriate if we expect incoherence to be characterized by intensity addition and coherence to be characterized by complex amplitude addition of the whole fields, including any non-zero mean components. On the other hand it is appropriate to use the mean-subtracted matrix $\mathbf{M}$ if the means of the fields are treated as insignificant biases and are ignored. The use of the mean-subtracted $\mathbf{K}$ and $\mathbf{M}$ is more consistent with the definition of the correlation coefficient of two random variables as commonly used in statistics and probability theory [6]. On the other hand, the mutual intensity $\mathbf{J}$ and the complex coherence $\mathbf{L}$ are more established in optics.

8 Conclusion

In this paper we set the foundations of a linear algebraic theory of partial coherence. While containing no new physics, the presented formulation allows precise definitions of concepts such as coherence and incoherence, offers new insights, and allows us to make the most of the conceptual and algebraic tools of linear algebra. This paper was formulated in the discrete notation of vectors and matrices; continuous function spaces will be dealt with in a separate paper.

We offered several definitions for the degree of partial coherence of a light field c such that $c = 0$ for complete incoherence and $c = 1$ for full coherence. A complete study of the various alternatives and their comparison will be the subject of another paper.

We carefully discussed Young's experiment to clearly show the relationship between our physical understanding of the concept of coherence and the mathematical definitions presented in this paper. In particular, we discussed the relative merits of using correlation or covariance functions as the basis for our definitions.

We believe that the formulation presented will be especially useful in optical information processing applications, since it will allow the precise analytical and numerical formulation of such problems. In particular, in a further paper we will discuss how the present formalism can be applied to the problem of synthesizing light with desired mutual intensity distributions from light with given mutual intensity (previous attempts at dealing with this problem include [13, 14, 15, 16]).

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