Chapter 1

Holomorphic and real analytic calculus

In this chapter we develop basic analysis in the holomorphic and real analytic settings. We do this, for the most part, simultaneously, and as a result some of the ways we do things are a little unconventional, especially when compared to the standard holomorphic treatments in, for example, texts like [Fritzsche and Grauert 2002, Gunning and Rossi 1965, Hörmander 1973, Krantz 1992, Laurent-Thiébaut 2011, Range 1986, Taylor 2002]. We assume, of course, a thorough acquaintance with real analysis such as one might find in [Abraham, Marsden, and Ratiu 1988, Chapter 2] and single variable complex analysis such as one might find in [Conway 1978].

1.1 Holomorphic and real analytic functions

Holomorphic and real analytic functions are defined as being locally prescribed by a convergent power series. We, therefore, begin by describing formal (i.e., not depending on any sort of convergence) power series. We then indicate how the usual notion of a Taylor series gives rise to a formal power series, and we prove Borel's Theorem which says that, in the real case, all formal power series arise as Taylor series. This leads us to consider convergence of power series, and then finally to consider holomorphic and real analytic functions.

Much of what we say here is a fleshing out of some material from Chapter 2 of [Krantz and Parks 2002], adapted to cover both the holomorphic and real analytic cases simultaneously.

1.1.1 $\mathbb{F}^n$

Because much of what we say in this chapter applies simultaneously to the real and complex case, we shall adopt the convention of using the symbol $\mathbb{F}$ when we wish to refer to one of $\mathbb{R}$ or $\mathbb{C}$. We shall use $|x|$ to mean the absolute value (when $\mathbb{F} = \mathbb{R}$) or the complex modulus (when $\mathbb{F} = \mathbb{C}$). In like manner, we shall denote by $\bar{x}$ the complex conjugate of x if $\mathbb{F} = \mathbb{C}$ and $\bar{x} = x$ if $\mathbb{F} = \mathbb{R}$. We use

$$\langle \boldsymbol{x}, \boldsymbol{y} \rangle = \sum_{j=1}^{n} x_j \bar{y}_j$$

to denote the standard inner product and

$$\|\boldsymbol{x}\| = \Big(\sum_{j=1}^{n} |x_j|^2\Big)^{1/2}$$

to denote the standard norm on $\mathbb{F}^n$ in both cases.

We shall need notions of balls and disks in $\mathbb{F}^n$. Balls are defined in the usual way. Given $\boldsymbol{x}_0 \in \mathbb{F}^n$ and $r \in \mathbb{R}_{>0}$, the ***open ball*** with radius r and centre $\boldsymbol{x}_0$ is

$$\mathsf{B}^n(r, \boldsymbol{x}_0) = \{\boldsymbol{x} \in \mathbb{F}^n \mid \|\boldsymbol{x} - \boldsymbol{x}_0\| < r\},$$

with

$$\overline{\mathsf{B}}^n(r, \boldsymbol{x}_0) = \{\boldsymbol{x} \in \mathbb{F}^n \mid \|\boldsymbol{x} - \boldsymbol{x}_0\| \le r\}$$

similarly denoting the ***closed ball***. For $\boldsymbol{x}_0 \in \mathbb{F}^n$ and for $\boldsymbol{r} \in \mathbb{R}^n_{>0}$, denote

$$\mathsf{D}^n(\boldsymbol{r}, \boldsymbol{x}_0) = \{\boldsymbol{x} \in \mathbb{F}^n \mid |x_j - x_{0j}| < r_j,\ j \in \{1, \dots, n\}\},$$

which we call the ***open polydisk*** of radius $\boldsymbol{r}$ and centre $\boldsymbol{x}_0$. The closure of the open polydisk, interestingly called the ***closed polydisk***, is denoted by

$$\overline{\mathsf{D}}^n(\boldsymbol{r}, \boldsymbol{x}_0) = \{\boldsymbol{x} \in \mathbb{F}^n \mid |x_j - x_{0j}| \le r_j,\ j \in \{1, \dots, n\}\}.$$

If $r \in \mathbb{R}_{>0}$ then we denote $\hat{r} = (r, \dots, r) \in \mathbb{R}^n_{>0}$ so that $\mathsf{D}^n(\hat{r}, \boldsymbol{x}_0)$ and $\overline{\mathsf{D}}^n(\hat{r}, \boldsymbol{x}_0)$ denote polydisks whose radii in all components are equal.

If $\mathcal{U} \subseteq \mathbb{F}^n$ is open, if $A \subseteq \mathcal{U}$, and if $\boldsymbol{f} \colon \mathcal{U} \to \mathbb{F}^m$ is continuous, we denote

$$\|\boldsymbol{f}\|_A = \sup\{\|\boldsymbol{f}(\boldsymbol{x})\| \mid \boldsymbol{x} \in A\}. \tag{1.1}$$

A ***domain*** in $\mathbb{F}^n$ is a nonempty connected open set.

A map $\boldsymbol{f} \colon \mathcal{U} \to \mathbb{F}^m$ from an open subset $\mathcal{U} \subseteq \mathbb{F}^n$ to $\mathbb{F}^m$ will be of ***class*** $\mathbf{C}^{\mathbf{r}}$, $r \in \mathbb{Z}_{\ge 0} \cup \{\infty\}$, if it is of class C^r in the real variable sense, noting that $\mathbb{C}^n \simeq \mathbb{R}^{2n}$.

1.1.2 Multi-index and partial derivative notation

A ***multi-index*** is an element of $\mathbb{Z}^n_{\ge 0}$. For a multi-index I we shall write $I = (i_1, \dots, i_n)$. We introduce the following notation:

1. $|I| = i_1 + \dots + i_n$;
2. $I! = i_1! \cdots i_n!$;
3. $\boldsymbol{x}^I = x_1^{i_1} \cdots x_n^{i_n}$ for $\boldsymbol{x} = (x_1, \dots, x_n) \in \mathbb{F}^n$.

Note that the elements $\boldsymbol{e}_1, \dots, \boldsymbol{e}_n$ of the standard basis for $\mathbb{F}^n$ are in $\mathbb{Z}^n_{\ge 0}$, so we shall think of these vectors as elements of $\mathbb{Z}^n_{\ge 0}$ when it is convenient to do so.

The following property of the set of multi-indices will often be useful.

1.1.1 Lemma (Cardinality of sets of multi-indices) *For* $\mathrm{n} \in \mathbb{Z}_{>0}$ *and* $\mathrm{m} \in \mathbb{Z}_{\geq 0}$,

$$\operatorname{card}\{\mathrm{I} \in \mathbb{Z}_{\geq 0}^{\mathrm{n}} \mid |\mathrm{I}| = \mathrm{m}\} = \binom{\mathrm{n}+\mathrm{m}-1}{\mathrm{n}-1}.$$

Proof We begin with an elementary lemma.

1 Sublemma *For* $\mathrm{n} \in \mathbb{Z}_{>0}$ *and* $\mathrm{m} \in \mathbb{Z}_{\geq 0}$, $\sum_{\mathrm{j}=0}^{\mathrm{m}} \binom{\mathrm{n}+\mathrm{j}-1}{\mathrm{n}-1} = \binom{\mathrm{n}+\mathrm{m}}{\mathrm{n}}$.

Proof Recall that for $j,k \in \mathbb{Z}_{\geq 0}$ with $j \leq k$ we have

$$\binom{k}{j} = \frac{k!}{j!(k-j)!}.$$

We claim that if $j,k \in \mathbb{Z}_{>0}$ satisfy $j \leq k$ then

$$\binom{k}{j} + \binom{k}{j-1} = \binom{k+1}{j}.$$

This is a direct computation:

$$\begin{aligned}\frac{k!}{j!(k-j)!} + \frac{k!}{(j-1)!(k-j+1)!} &= \frac{(k-j+1)k!}{(k-j+1)j!(k-j)!} + \frac{jk!}{j(j-1)!(k-j+1)!} \\ &= \frac{(k-j+1)k! + jk!}{j!(k-j+1)!} \\ &= \frac{(k+1)k!}{j!((k+1)-j)!} = \frac{(k+1)!}{j!((k+1)-j)!} = \binom{k+1}{j}.\end{aligned}$$

Now we have

$$\begin{aligned}\sum_{j=0}^{m} \binom{n+j-1}{n-1} &= 1 + \sum_{j=1}^{m} \binom{n+j-1}{n-1} = 1 + \sum_{j=1}^{m} \binom{n+j}{n} - \sum_{j=1}^{m} \binom{n+j-1}{n} \\ &= 1 + \binom{n+m}{n} - \binom{n}{n} = \binom{n+m}{n},\end{aligned}$$

as desired. ▼

We now prove the lemma by induction on n. For $n = 1$ we have

$$\operatorname{card}\{j \in \mathbb{Z}_{\geq 0} \mid j = m\} = 1 = \binom{m}{0},$$

which gives the conclusions of the lemma in this case. Now suppose that the lemma holds for $n \in \{1, \dots, k\}$. If $I \in \mathbb{Z}_{\geq 0}^{k+1}$ satisfies $|I| = m$, then write $I = (i_1, \dots, i_k, i_{k+1})$ and take

$I' = (i_1, \dots, i_k)$. If $i_{k+1} = j \in \{0, 1, \dots, m\}$ then $|I'| = m - j$. Thus

$$\begin{aligned}\operatorname{card}\{I \in \mathbb{Z}_{\geq 0}^{k+1} \mid |I| = m\} &= \sum_{j=0}^{m} \operatorname{card}\{I' \in \mathbb{Z}_{\geq 0}^{n} \mid |I'| = m - j\} \\ &= \sum_{j=0}^{m} \binom{k+m-j+1}{k-1} = \sum_{j'=0}^{m} \binom{k+j'-1}{k-1} \\ &= \binom{k+m}{k} = \binom{(k+1)+m-1}{(k+1)-1},\end{aligned}$$

using the sublemma in the penultimate step. This proves the lemma by induction. ■

Multi-index notation is also convenient for representing partial derivatives of multivariable functions. Let us start from the ground up. Let $(\boldsymbol{e}_1, \dots, \boldsymbol{e}_n)$ be the standard basis for $\mathbb{F}^n$ and denote by $(\boldsymbol{\alpha}_1, \dots, \boldsymbol{\alpha}_n)$ the dual basis for $(\mathbb{F}^n)^*$. Let $\mathcal{U} \subseteq \mathbb{F}^n$ and let $f \colon \mathcal{U} \to \mathbb{F}$. The ***$\mathbb{F}$-derivative*** of f at $\boldsymbol{x}_0 \in \mathcal{U}$ is the unique $\mathbb{F}$-linear map $\boldsymbol{D}f(\boldsymbol{x}_0) \in \mathrm{L}(\mathbb{F}^n; \mathbb{F})$ for which

$$\lim_{\boldsymbol{x} \to \boldsymbol{x}_0} \frac{f(\boldsymbol{x}) - f(\boldsymbol{x}_0) - \boldsymbol{D}f(\boldsymbol{x}_0)(\boldsymbol{x} - \boldsymbol{x}_0)}{\boldsymbol{x} - \boldsymbol{x}_0} = 0,$$

provided that such a linear map indeed exists. Higher $\mathbb{F}$-derivatives are defined recursively. The kth $\mathbb{F}$-derivative of f at $\boldsymbol{x}_0$ we denote by $\boldsymbol{D}^k f(\boldsymbol{x}_0)$, noting that $\boldsymbol{D}^k f(\boldsymbol{x}_0) \in \mathrm{L}^k_{\mathrm{sym}}(\mathbb{F}^n; \mathbb{F})$ is a symmetric k-multilinear map [see Abraham, Marsden, and Ratiu 1988, Proposition 2.4.14]. If $\boldsymbol{D}^k f(\boldsymbol{x})$ exists for every $\boldsymbol{x} \in \mathcal{U}$ and if the map $\boldsymbol{x} \mapsto \boldsymbol{D}^k f(\boldsymbol{x})$ is continuous, f is of ***$\mathbb{F}$-class C^k*** or ***k-times continuously $\mathbb{F}$-differentiable***. This slightly awkward and nonstandard notation will be short-lived, and is a consequence of our trying to simultaneously develop the real and complex theories.

We denote by $\mathrm{L}^k(\mathbb{F}^n; \mathbb{F})$ the set of k-multilinear maps with its usual basis

$$\{(\boldsymbol{\alpha}_{j_1} \otimes \cdots \otimes \boldsymbol{\alpha}_{j_k}) \mid j_1, \dots, j_k \in \{1, \dots, n\}\}.$$

Thinking of $\boldsymbol{D}f(\boldsymbol{x}_0)$ as a multilinear map, forgetting about its being symmetric, we write

$$\boldsymbol{D}^k f(\boldsymbol{x}_0) = \sum_{j_1, \dots, j_k = 1}^{n} \frac{\partial^k f}{\partial x_{j_1} \cdots \partial x_{j_k}}(\boldsymbol{x}_0) \boldsymbol{\alpha}_{j_1} \otimes \cdots \otimes \boldsymbol{\alpha}_{j_k}.$$

Let us introduce another way of writing the $\mathbb{F}$-derivative. For $j_1, \dots, j_k \in \{1, \dots, n\}$, define $I \in \mathbb{Z}^n_{\geq 0}$ by letting $i_m \in \mathbb{Z}_{\geq 0}$ be the number of times $m \in \{1, \dots, n\}$ appears in the list of numbers $j_1, \dots, j_k$. We recall from Section F.2.3 the product $\odot$ between $A \in \mathrm{L}^k_{\mathrm{sym}}(\mathbb{F}^n; \mathbb{F})$ and $B \in \mathrm{L}^l_{\mathrm{sym}}(\mathbb{F}^n; \mathbb{F})$ given by

$$A \odot B = \frac{(k+l)!}{k!\,l!} \operatorname{Sym}_{k+l}(A \otimes B),$$

where, for $C \in \mathrm{L}^m_{\mathrm{sym}}(\mathbb{F}^n; \mathbb{F})$,

$$\operatorname{Sym}_m(C)(\boldsymbol{v}_1, \dots, \boldsymbol{v}_m) = \frac{1}{m!} \sum_{\sigma \in \mathfrak{S}_m} C(\boldsymbol{v}_{\sigma(1)}, \dots, \boldsymbol{v}_{\sigma(m)}).$$

Then, as discussed in Section F.2.4, we can also write

$$\boldsymbol{D}^k f(\boldsymbol{x}_0) = \sum_{\substack{i_1,\dots,i_n\in\mathbb{Z}_{\geq 0}\\ i_1+\dots+i_n=k}} \frac{1}{i_1!\cdots i_n!}\frac{\partial^k f}{\partial x_1^{i_1}\cdots\partial x_n^{i_n}}(\boldsymbol{x}_0)\boldsymbol{\alpha}_1^{i_1}\odot\cdots\odot\boldsymbol{\alpha}_n^{i_n}$$

For $I = (i_1,\dots,i_n)\in\mathbb{Z}_{\geq 0}^n$, we write

$$\boldsymbol{D}^I f(\boldsymbol{x}_0) = \frac{\partial^{|I|} f}{\partial x_1^{i_1}\cdots\partial x_n^{i_n}}(\boldsymbol{x}_0).$$

We may also write this in a different way:

$$\boldsymbol{D}^I f(\boldsymbol{x}_0) = \frac{\partial^{|I|} f}{\underbrace{\partial x_1\cdots\partial x_1}_{i_1\ \text{times}}\cdots\underbrace{\partial x_n\cdots\partial x_n}_{i_n\ \text{times}}}(\boldsymbol{x}_0).$$

Indeed, because of symmetry of the $\mathbb{F}$-derivative, for any collection of numbers $j_1,\dots,j_{|I|}\in\{1,\dots,n\}$ for which k occurs i_k times for each $k\in\{1,\dots,n\}$, we have

$$\boldsymbol{D}^I f(\boldsymbol{x}_0) = \frac{\partial^{|I|} f}{\partial x_{j_1}\cdots\partial x_{j_{|I|}}}(\boldsymbol{x}_0).$$

In any case, we can also write

$$\boldsymbol{D}^k f(\boldsymbol{x}_0) = \sum_{\substack{I\in\mathbb{Z}_{\geq 0}^n\\ |I|=k}} \frac{1}{I!}\boldsymbol{D}^I f(\boldsymbol{x}_0)\boldsymbol{\alpha}_1^{i_1}\odot\cdots\odot\boldsymbol{\alpha}_n^{i_n}$$

We shall freely interchange the various partial $\mathbb{F}$-derivative notations discussed above, depending on what we are doing.

1.1.3 Formal power series

To get started with our discussion of holomorphicity and analyticity, it is useful to first engage in a little algebra so that we can write power series without having to worry about convergence.

1.1.2 Definition (Formal power series with finite indeterminates) Let $\boldsymbol{\xi} = \{\xi_1,\dots,\xi_n\}$ be a finite set and denote by $\mathbb{Z}_{\geq 0}^{\boldsymbol{\xi}}$ the set of maps from $\boldsymbol{\xi}$ into $\mathbb{Z}_{\geq 0}$. The set of ***$\mathbb{F}$-formal power series*** with indeterminates X is the set of maps from $\mathbb{Z}_{\geq 0}^{\boldsymbol{\xi}}$ to $\mathbb{F}$, and is denoted by $\mathbb{F}[[\boldsymbol{\xi}]]$. •

There is a concrete way to represent $\mathbb{Z}_{\geq 0}^{\boldsymbol{\xi}}$. Given $\phi\colon\boldsymbol{\xi}\to\mathbb{Z}_{\geq 0}$ we note that $\phi(\boldsymbol{\xi})$ is uniquely determined by the n-tuple

$$(\phi(\xi_1),\dots,\phi(\xi_n))\in\mathbb{Z}_{\geq 0}^n.$$

Such an n-tuple is nothing but an n-multi-index. Therefore, we shall identify $\mathbb{Z}_{\geq 0}^{\xi}$ with the set $\mathbb{Z}_{\geq 0}^{n}$ of multi-indices.

Therefore, rather than writing $\alpha(\phi)$ for $\alpha \in \mathbb{F}[[\boldsymbol{\xi}]]$ and $\phi \in \mathbb{Z}_{\geq 0}^{n}$, we shall write $\alpha(I)$ for $I \in \mathbb{Z}_{\geq 0}^{n}$. Using this notation, the $\mathbb{F}$-algebra operations are defined by

$$\begin{aligned}(\alpha+\beta)(I) &= \alpha(I)+\beta(I),\\ (a\alpha)(I) &= a(\alpha(I)),\\ (\alpha\cdot\beta)(I) &= \sum_{\substack{I_1,I_2\in\mathbb{Z}_{\geq 0}^{n}\\ I_1+I_2=I}} \alpha(I_1)\beta(I_2),\end{aligned}$$

for $a \in \mathbb{F}$ and $\alpha, \beta \in \mathbb{F}[[\xi_1,\ldots,\xi_n]]$. We shall identify the indeterminate ξ_j, $j \in \{1,\ldots,n\}$, with the element α_j of $\mathbb{F}[[\boldsymbol{\xi}]]$ defined by

$$\alpha_j(I) = \begin{cases} 1, & I = \boldsymbol{e}_j,\\ 0, & \text{otherwise},\end{cases}$$

where $(\boldsymbol{e}_1,\ldots,\boldsymbol{e}_n)$ is the standard basis for $\mathbb{F}^n$, thought of as an element of $\mathbb{Z}_{\geq 0}^{n}$. One can readily verify that, using this identification, the k-fold product of ξ_j is

$$\xi_j^k(I) = \begin{cases} 1, & I = k\boldsymbol{e}_j,\\ 0, & \text{otherwise}.\end{cases}$$

Therefore, it is straightforward to see that if $\alpha \in \mathbb{F}[[\boldsymbol{\xi}]]$ then

$$\alpha = \sum_{I=(i_1,\ldots,i_n)\in\mathbb{Z}_{\geq 0}^{n}} \alpha(I)\xi_1^{i_1}\cdots\xi_n^{i_n}. \tag{1.2}$$

Adopting the notational convention $\boldsymbol{\xi}^I = \xi_1^{i_1}\cdots\xi_n^{i_n}$, the preceding formula admits the compact representation

$$\alpha = \sum_{I\in\mathbb{Z}_{\geq 0}^{n}} \alpha(I)\boldsymbol{\xi}^I.$$

We can describe explicitly the units in the ring $\mathbb{F}[[\boldsymbol{\xi}]]$, and give a formula for the inverse for these units.

1.1.3 Proposition (Units in $\mathbb{F}[[\boldsymbol{\xi}]]$) *A member $\alpha \in \mathbb{F}[[\boldsymbol{\xi}]]$ is a unit if and only if $\alpha(\mathbf{0}) \neq 0$. Moreover, if α is a unit, then we have*

$$\alpha^{-1}(\mathrm{I}) = \frac{1}{\alpha(\mathbf{0})}\sum_{\mathrm{k}=0}^{\infty}\Big(1-\frac{\alpha(\mathrm{I})}{\alpha(\mathbf{0})}\Big)^{\mathrm{k}}$$

for all $\mathrm{I} \in \mathbb{Z}_{\geq 0}^{\mathrm{n}}$.

Proof First of all, suppose that α is a unit. Thus there exists $\beta \in \mathbb{F}[[\boldsymbol{\xi}]]$ such that $\alpha\cdot\beta = 1$. In particular, this means that $\alpha(\mathbf{0})\beta(\mathbf{0}) = 1$, and so $\alpha(\mathbf{0})$ is a unit in $\mathbb{F}$, i.e., is nonzero.

Next suppose that $\alpha(\mathbf{0}) \neq 0$. To prove that α is a unit we use the following lemma.

1 Lemma *If $\beta \in \mathbb{F}[[\xi]]$ satisfies $\beta(\mathbf{0}) = 0$ then $(1-\beta)$ is a unit in $\mathbb{F}[[\xi]]$ and*

$$(1-\beta)^{-1}(\mathrm{I}) = \sum_{\mathrm{k}=0}^{\infty} \beta^{\mathrm{k}}(\mathrm{I})$$

for all $\mathrm{I} \in \mathbb{Z}^{\mathrm{n}}_{\geq 0}$.

Proof First of all, we claim that $\sum_{k=0}^{\infty} \beta^k$ is a well-defined element of $\mathbb{F}[[\xi]]$. We claim that $\beta^k(I) = 0$ whenever $|I| \in \{0, 1, \dots, k\}$. We can prove this by induction on k. For $k = 0$ this follows from the assumption that $\beta(\mathbf{0}) = 0$. So suppose that $\beta^k(I) = 0$ for $|I| \in \{0, 1, \dots, k\}$, whenever $k \in \{0, 1, \dots, r\}$. Then, for $I \in \mathbb{Z}^n_{\geq 0}$ satisfying $|I| \in \{0, 1, \dots, r+1\}$, we have

$$\begin{aligned}\beta^{r+1}(I) = (\beta \cdot \beta^r)(I) &= \sum_{\substack{I_1, I_2 \in \mathbb{Z}^n_{\geq 0} \\ I_1 + I_2 = I}} \beta(I_1)\beta^r(I_2) \\ &= \beta(\mathbf{0})\beta^r(I) + \sum_{\substack{I' \in \mathbb{Z}^n_{\geq 0} \\ I - I' \in \mathbb{Z}^n_{\geq 0} \setminus \mathbf{0}}} \beta(I')\beta^r(I - I') = 0,\end{aligned}$$

using the definition of the product in $\mathbb{F}[[\xi]]$ and the induction hypothesis. Thus we indeed have $\beta^k(I) = 0$ whenever $|I| \in \{0, 1, \dots, k\}$. This implies that, if $I \in \mathbb{Z}^n_{\geq 0}$, then the sum $\sum_{k=0}^{\infty} \beta^k(I)$ is finite, and the formula in the statement of the lemma for $(1-\beta)^{-1}$ at least makes sense. To see that it is actually the inverse of $1 - \beta$, for $I \in \mathbb{Z}^n_{\geq 0}$ we compute

$$(1-\beta) \cdot \Big(\sum_{k=0}^{\infty} \beta^k\Big)(I) = \sum_{k=0}^{\infty} \beta^k(I) - \sum_{k=1}^{\infty} \beta^k(I) = 1,$$

as desired. ▼

Proceeding with the proof, let us define $\beta = 1 - \frac{\alpha}{\alpha(\mathbf{0})}$ so that $\beta(\mathbf{0}) = 0$. By the lemma, $1 - \beta$ is a unit. Since $\alpha = \alpha(0)(1-\beta)$ it follows that α is also a unit, and that $\alpha^{-1} = \alpha(\mathbf{0})^{-1}(1-\beta)^{-1}$. The formula in the statement of the proposition then follows from the lemma above. ■

Note that one of the consequences of the proof of the proposition is that the expression given for α^{-1} makes sense since the sum is finite for a fixed $I \in \mathbb{Z}^n_{\geq 0}$.

1.1.4 Formal Taylor series

One can see an obvious notational resemblance between the representation (1.2) and power series in the usual sense. A common form of power series is the Taylor series for an infinitely $\mathbb{F}$-differentiable function about a point. In this section we flesh this out by assigning to an infinitely $\mathbb{F}$-differentiable map a formal power series in a natural way. Throughout this section we let $\boldsymbol{\xi} = \{\xi_1, \dots, \xi_n\}$ so $\mathbb{F}[[\boldsymbol{\xi}]]$ denotes the $\mathbb{F}$-formal power series in these indeterminates.

We let $(\boldsymbol{e}_1, \dots, \boldsymbol{e}_n)$ be the standard basis for $\mathbb{F}^n$. We might typically denote the dual basis for $(\mathbb{F}^n)^*$ by $(\boldsymbol{e}^1, \dots, \boldsymbol{e}^n)$, but notationally, in this section, it is instead convenient to denote the dual basis by $(\boldsymbol{\alpha}_1, \dots, \boldsymbol{\alpha}_n)$.

Let $\boldsymbol{x}_0 \in \mathbb{F}^n$ and let $\mathcal{U}$ be a neighbourhood of $\boldsymbol{x}_0 \in \mathbb{F}^n$. We suppose that $f\colon \mathcal{U} \to \mathbb{F}^m$ is infinitely $\mathbb{F}$-differentiable. We let $\boldsymbol{D}^k f(\boldsymbol{x}_0)$ be the kth $\mathbb{F}$-derivative of f at $\boldsymbol{x}_0$, noting that this is an element of $\mathrm{L}^k_{\mathrm{sym}}(\mathbb{F}^n;\mathbb{F}^m)$ as discussed in Section 1.1.2. As we saw in our discussion in Section 1.1.2, writing this in the basis for $\mathrm{L}^k_{\mathrm{sym}}(\mathbb{F}^n;\mathbb{F})$ gives

$$\boldsymbol{D}^k f(\boldsymbol{x}_0) = \sum_{a=1}^{m} \sum_{\substack{I\in\mathbb{Z}^n_{\geq 0}\\ |I|=k}} \frac{1}{I!}\boldsymbol{D}^I f_a(\boldsymbol{x}_0)(\boldsymbol{\alpha}_1^{i_1}\odot\cdots\odot\boldsymbol{\alpha}_n^{i_n})\otimes\boldsymbol{e}_a.$$

Thus, to $f \in \mathrm{C}^\infty(\mathcal{U})$ we can associate an element $\alpha_f(\boldsymbol{x}_0) \in \mathbb{F}[[\xi_1,\dots,\xi_n]]\otimes\mathbb{F}^m$ by

$$\alpha_f(\boldsymbol{x}_0) = \sum_{a=1}^{m} \sum_{I\in\mathbb{Z}^n_{\geq 0}} \frac{1}{I!}\boldsymbol{D}^I f_a(\boldsymbol{x}_0)(\xi_1^{i_1}\cdots\xi_n^{i_n})\otimes\boldsymbol{e}_a.$$

One can (somewhat tediously) verify using the high-order Leibniz Rule (which we prove as Lemma A.2.2 below) that this map is a homomorphism of $\mathbb{F}$-algebras. That is to say,

$$\alpha_{af}(\boldsymbol{x}_0) = a\alpha_f(\boldsymbol{x}_0),\quad \alpha_{f+g}(\boldsymbol{x}_0) = \alpha_f(\boldsymbol{x}_0)+\alpha_g(\boldsymbol{x}_0),\quad \alpha_{fg}(\boldsymbol{x}_0) = \alpha_f(\boldsymbol{x}_0)\alpha_g(\boldsymbol{x}_0).$$

We shall call $\alpha_f(\boldsymbol{x}_0)$ the ***formal Taylor series*** of f at $\boldsymbol{x}_0$.

To initiate our discussions of convergence, let us consider $\mathbb{R}$-valued functions for the moment, just for simplicity. The expression for $\alpha_f(\boldsymbol{x}_0)$ is reminiscent of the ***Taylor series*** for f about $\boldsymbol{x}_0$:

$$\sum_{k=0}^{\infty} \sum_{I\in\mathbb{Z}^n_{\geq 0}} \frac{1}{I!}\boldsymbol{D}^I f(\boldsymbol{x}_0)(\boldsymbol{x}-\boldsymbol{x}_0)^I.$$

This series will generally not converge, even though as small children we probably thought that it did converge for infinitely differentiable functions. The situation regarding convergence is, in fact, as dire as possible, as is shown by the following theorem of Borel [1895]. Actually, Borel only proves the case where $n = 1$. The proof we give for arbitrary n follows [Mirkil 1956].

1.1.4 Theorem (Borel) *If $\mathbf{x}_0 \in \mathbb{R}^n$ and if $\mathcal{U}$ is a neighbourhood of $\mathbf{x}_0$, then the map $\mathrm{f} \mapsto \alpha_{\mathrm{f}}(\mathbf{x}_0)$ from $\mathrm{C}^\infty(\mathcal{U})$ to $\mathbb{R}[[\xi]]$ is surjective.*

Proof Let us define $h\colon \mathbb{R}\to\mathbb{R}$ by

$$h(x) = \begin{cases} 0, & x\in(-\infty,-2],\\ \mathrm{e}\cdot\mathrm{e}^{-1/(1-(x+1)^2)}, & x\in(-2,-1),\\ 1, & x\in[-1,1],\\ \mathrm{e}\cdot\mathrm{e}^{-1/(1-(x-1)^2)}, & x\in(1,2),\\ 0, & x\in[2,\infty). \end{cases}$$

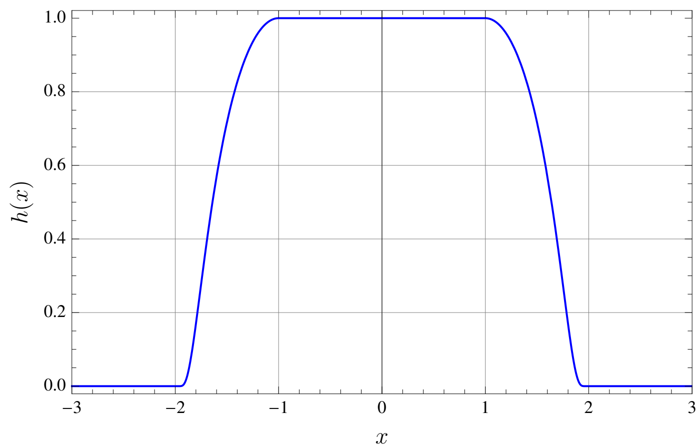

Figure 1.1 The bump function

As is well-known, cf. [Abraham, Marsden, and Ratiu 1988, Page 82] and Example 1.1.5, the function h is infinitely differentiable. We depict the function in Figure 1.1.

Let $\alpha \in \mathbb{R}[[\xi]]$. Without loss of generality we assume that $x_0 = \mathbf{0}$. Let $r \in \mathbb{R}_{>0}$ be such that $\mathsf{B}^n(r, \mathbf{0}) \subseteq \mathcal{U}$. We recursively define a sequence $(f_j)_{j \in \mathbb{Z}_{\geq 0}}$ in $\mathrm{C}^\infty(\mathcal{U})$ as follows. We take $f_0 \in \mathrm{C}^\infty(\mathcal{U})$ such that $f_0(\mathbf{0}) = \alpha(\mathbf{0})$ and such that $\mathrm{supp}(f_0) \subseteq \overline{\mathsf{B}}^n(r, \mathbf{0})$, e.g., take

$$f_0(\boldsymbol{x}) = \alpha(\mathbf{0}) h(\tfrac{2}{r}\|\boldsymbol{x}\|).$$

Now suppose that $f_0, f_1, \dots, f_k$ have been defined and define $g_{k+1}\colon \mathcal{U} \to \mathbb{R}$ to a homogeneous polynomial function in $x_1, \dots, x_n$ of degree $k+1$ so that, for every multi-index $I = (i_1, \dots, i_n)$ for which $|I| = k+1$, we have

$$\boldsymbol{D}^I g_{k+1}(\mathbf{0}) = \alpha(I) - \boldsymbol{D}^I f_0(\mathbf{0}) - \dots - \boldsymbol{D}^I f_k(\mathbf{0}).$$

Note that $\boldsymbol{D}^I g_{k+1}(\mathbf{0}) = 0$ if $|I| \in \{0, 1, \dots, k\}$ since in these case $\boldsymbol{D}^I g_{k+1}(\boldsymbol{x})$ will be a homogeneous polynomial of degree $k + 1 - m$. Next let

$$\bar{g}_{k+1}(\boldsymbol{x}) = g_{k+1}(\boldsymbol{x}) h(\tfrac{2}{r}\|\boldsymbol{x}\|)$$

so that, since the function $\boldsymbol{x} \mapsto h(\frac{2}{r}\|\boldsymbol{x}\|)$ is equal to 1 in a neighbourhood of $\mathbf{0}$,

$$\boldsymbol{D}^I \bar{g}_{k+1}(\mathbf{0}) = \alpha(I) - \boldsymbol{D}^I f_0(\mathbf{0}) - \dots - \boldsymbol{D}^I f_k(\mathbf{0})$$

and $\boldsymbol{D}^I \bar{g}_{k+1}(\mathbf{0}) = 0$ if $|I| \in \{0, 1, \dots, k+1\}$. Also $\mathrm{supp}(\bar{g}_{k+1}) \subseteq \overline{\mathsf{B}}^n(r, \mathbf{0})$. Next let $\lambda \in \mathbb{R}_{>0}$. If we define $h_{\lambda,k+1}(\boldsymbol{x}) = \lambda^{-k-1} \bar{g}_{k+1}(\lambda \boldsymbol{x})$ then

$$\boldsymbol{D}^I h_{\lambda,k+1}(\boldsymbol{x}) = \lambda^{-k-1+m} \boldsymbol{D}^I \bar{g}_{k+1}(\lambda \boldsymbol{x}).$$

Thus, if $|I| \in \{0, 1, \dots, k\}$, then we can choose λ sufficiently large that $|\boldsymbol{D}^I h_{\lambda,k+1}(\boldsymbol{x})| < 2^{-k-1}$ for every $\boldsymbol{x} \in \mathsf{B}^n(r, \mathbf{0})$. With λ so chosen we take

$$f_{k+1}(\boldsymbol{x}) = h_{\lambda,k+1}(\boldsymbol{x}).$$

This recursive definition ensures that, for each $k \in \mathbb{Z}_{\geq 0}$, f_k has the following properties:

1. $\operatorname{supp}(f_k) \subseteq \overline{\mathsf{B}}^n(r, \mathbf{0})$;
2. $\boldsymbol{D}^I f_k(\mathbf{0}) = \alpha(I) - \boldsymbol{D} f_0(\mathbf{0}) - \dots - \boldsymbol{D}^I f_k(\mathbf{0})$ whenever $I = (i_1, \dots i_n)$ satisfies $|I| = k$;
3. $\boldsymbol{D}^I f_k(\mathbf{0}) = 0$ if $|I| \in \{0, 1, \dots, k-1\}$;
4. $|\boldsymbol{D}^I f_k(\boldsymbol{x})| < 2^{-k}$ whenever $|I| \in \{0, 1, \dots, k-1\}$ and $\boldsymbol{x} \in \mathsf{B}^n(r, \mathbf{0})$.

We then define $f(\boldsymbol{x}) = \sum_{k=0}^{\infty} f_k(\boldsymbol{x})$.

From the second property of the functions f_k, $k \in \mathbb{Z}_{\geq 0}$, above we see that $\alpha = \alpha_f$. It remains to show that f is infinitely differentiable. We shall do this by showing that the sequences of partial sums for all partial $\mathbb{F}$-derivatives converge uniformly. The partial sums we denote by

$$F_m(\boldsymbol{x}) = \sum_{k=0}^{m} f_k(\boldsymbol{x}).$$

Since all functions in our series have support contained in $\overline{\mathsf{B}}^n(r, \mathbf{0})$, this is tantamount to showing that, for all multi-indices I, the sequences $(\boldsymbol{D}^I F_m)_{m \in \mathbb{Z}_{\geq 0}}$ are Cauchy sequences in the Banach space $\mathrm{C}^0(\overline{\mathsf{B}}^n(r, \mathbf{0}); \mathbb{R})$ of continuous $\mathbb{R}$-valued functions on $\overline{\mathsf{B}}^n(r, \mathbf{0})$ equipped with the norm

$$\|g\|_\infty = \sup\{|g(\boldsymbol{x})| \mid \boldsymbol{x} \in \overline{\mathsf{B}}^n(r, \mathbf{0})\};$$

see [Hewitt and Stromberg 1975, Theorem 7.9].

Let $\epsilon \in \mathbb{R}_{>0}$ and let $I = (i_1, \dots, i_n)$ be a multi-index. Let $N \in \mathbb{Z}_{>0}$ be such that

$$\sum_{j=l+1}^{m} \frac{1}{2^j} < \epsilon$$

for every $l, m \geq N$, this being possible since $\sum_{j=1}^{\infty} 2^{-j} < \infty$. Then, for $l, m \geq \{N, |I|\}$ with $m > l$ and $\boldsymbol{x} \in \mathsf{B}^n(r, \mathbf{0})$, we have

$$\left|\boldsymbol{D}^I F_l(\boldsymbol{x}) - \boldsymbol{D}^I F_m(\boldsymbol{x})\right| = \left|\sum_{j=l+1}^{m} \boldsymbol{D}^I f_j(\boldsymbol{x})\right| \leq \sum_{j=l+1}^{m} \left|\boldsymbol{D}^I f_j(\boldsymbol{x})\right| \leq \sum_{j=l+1}^{m} \frac{1}{2^j} < \epsilon,$$

showing that $(\boldsymbol{D}^I F_m)_{m \in \mathbb{Z}_{>0}}$ is a Cauchy sequence in $\mathrm{C}^0(\overline{\mathsf{B}}^n(r, \mathbf{0}); \mathbb{R})$, as desired. ■

Thus any possible coefficients in a formal power series can arise as the Taylor coefficients for an infinitely differentiable function. Of course, an arbitrary power series

$$\sum_{k=0}^{\infty} \sum_{I=(i_1,\dots,i_k)} \alpha(I)(x_1 - x_{01})^{i_1} \dots (x_n - x_{0n})^{i_n}$$

may well only converge when $\boldsymbol{x} = \boldsymbol{x}_0$. Not only this, but even when the Taylor series does converge, it may not converge to the function producing its coefficients.

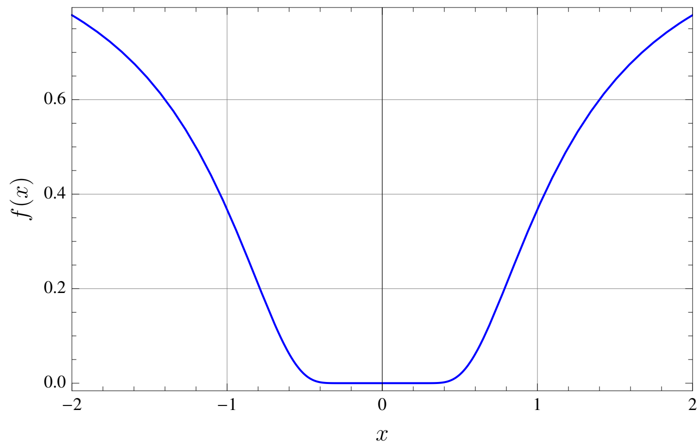

Figure 1.2 Everyone's favourite smooth but not analytic function

1.1.5 Example (A Taylor series not converging to the function giving rise to it) We define $f\colon \mathbb{R} \to \mathbb{R}$ by

$$f(x) = \begin{cases} \mathrm{e}^{-\frac{1}{x^2}}, & x \neq 0, \\ 0, & x = 0, \end{cases}$$

and in Figure 1.2 we show the graph of f. We claim that the Taylor series for f is the zero $\mathbb{R}$-formal power series. To prove this, we must compute the derivatives of f at $x = 0$. The following lemma is helpful in this regard.

1 Lemma *For* $\mathrm{j} \in \mathbb{Z}_{\geq 0}$ *there exists a polynomial* $\mathrm{p_j}$ *of degree at most* $2\mathrm{j}$ *such that*

$$\mathrm{f^{(j)}(x)} = \frac{\mathrm{p_j(x)}}{\mathrm{x^{3j}}}\mathrm{e}^{-\frac{1}{x^2}}, \qquad \mathrm{x} \neq 0.$$

Proof We prove this by induction on j. Clearly the lemma holds for $j = 0$ by taking $p_0(x) = 1$. Now suppose the lemma holds for $j \in \{0, 1, \dots, k\}$. Thus

$$f^{(k)}(x) = \frac{p_k(x)}{x^{3k}}\mathrm{e}^{-\frac{1}{x^2}}$$

for a polynomial p_k of degree at most $2k$. Then we compute

$$f^{(k+1)}(x) = \frac{x^3 p_k'(x) - 3kx^2 p_k(x) - 2p_k(x)}{x^{3(k+1)}}\mathrm{e}^{-\frac{1}{x^2}}.$$

Using the rules for differentiation of polynomials, one easily checks that

$$x \mapsto x^3 p_k'(x) - 3kx^2 p_k(x) - 2p_k(x)$$

is a polynomial whose degree is at most $2(k + 1)$. ▼

From the lemma we infer the infinite differentiability of f on $\mathbb{R} \setminus \{0\}$. We now need to consider the derivatives at 0. For this we employ another lemma.

2 Lemma $\lim_{x\to 0} \frac{e^{-\frac{1}{x^2}}}{x^k} = 0$ *for all* $k \in \mathbb{Z}_{\geq 0}$.

Proof We note that

$$\lim_{x\downarrow 0} \frac{e^{-\frac{1}{x^2}}}{x^k} = \lim_{y\to\infty} \frac{y^k}{e^{y^2}}, \qquad \lim_{x\uparrow 0} \frac{e^{-\frac{1}{x^2}}}{x^k} = \lim_{y\to -\infty} \frac{y^k}{e^{y^2}}.$$

We have

$$e^{y^2} = \sum_{j=0}^{\infty} \frac{y^{2j}}{j!}$$

In particular, $e^{y^2} \geq \frac{y^{2k}}{k!}$, and so

$$\left|\frac{y^k}{e^{y^2}}\right| \leq \left|\frac{k!}{y^k}\right|,$$

and so

$$\lim_{x\to 0} \frac{e^{-\frac{1}{x^2}}}{x^k} = 0,$$

as desired. ▼

Now, letting $p_k(x) = \sum_{j=0}^{2k} a_j x^j$, we may directly compute

$$\lim_{x\to 0} f^{(k)}(x) = \lim_{x\to 0} \sum_{j=0}^{2k} a_j x^j \frac{e^{-\frac{1}{x^2}}}{x^{3k}} = \sum_{j=0}^{2k} a_j \lim_{x\to 0} \frac{e^{-\frac{1}{x^2}}}{x^{3k-j}} = 0.$$

Thus we arrive at the conclusion that f is infinitely differentiable on $\mathbb{R}$, and that f and all of its derivatives are zero at $x = 0$. Thus the Taylor series is indeed zero. This is clearly a convergent power series; it converges everywhere to the zero function. However, $f(x) \neq 0$ except when $x = 0$. Thus the Taylor series about 0 for f, while convergent everywhere, converges to f only at $x = 0$. This is, therefore, an example of a function that is infinitely differentiable at a point, but is not equal to its Taylor series at $x = 0$. This function may seem rather useless, but in actuality it is quite an important one. For example, we used it in the construction for the proof of Theorem 1.1.4. It is also used in the construction of partitions of unity which are so important in smooth differential geometry, and whose absence in holomorphic and real analytic differential geometry makes the latter subject so subtle. •

Another way to think of the preceding example is that it tells us that the map $f \mapsto \alpha_f(x_0)$ from $C^\infty(\mathcal{U})$ to $\mathbb{R}[[\xi]]$, while surjective, is not injective.

1.1.5 Convergent power series

In the preceding section we saw that smoothness in the $\mathbb{R}$-valued case was not enough to ensure that the Taylor series of a function has any useful correspondence to the values of the function. This leads us to naturally consider convergent power series. Throughout this section we let $\boldsymbol{\xi} = \{\xi_1, \dots, \xi_n\}$ so $\mathbb{F}[[\boldsymbol{\xi}]]$ denotes the $\mathbb{F}$-formal power series in these indeterminates.

Let us turn to formal power series that converge, and give some of their properties. Recalling notation from Section 1.1.2, we state the following.

1.1.6 Definition (Convergent formal power series) Let $\boldsymbol{\xi} = \{\xi_1, \dots, \xi_n\}$. A formal power series $\alpha \in \mathbb{F}[[\boldsymbol{\xi}]]$ ***converges*** at $\boldsymbol{x} \in \mathbb{F}^n$ if there exists a bijection $\phi\colon \mathbb{Z}_{>0} \to \mathbb{Z}^n_{\geq 0}$ such that the series

$$\sum_{j=1}^{\infty} \alpha(\phi(j))\boldsymbol{x}^{\phi(j)}$$

converges. Let us denote by $\mathcal{R}_{\mathrm{conv}}(\alpha)$ the set of points $\boldsymbol{x} \in \mathbb{F}^n$ such that α converges at $\boldsymbol{x}$. We call $\mathcal{R}_{\mathrm{conv}}(\alpha)$ the ***region of convergence***. We denote by

$$\hat{\mathbb{F}}[[\boldsymbol{\xi}]] = \{\alpha \in \mathbb{R}[[\boldsymbol{\xi}]] \mid \mathcal{R}_{\mathrm{conv}}(\alpha) \neq \{\boldsymbol{0}\}\}$$

the set of power series converging at some nonzero point. •

1.1.7 Remark (On notions of convergence for multi-indexed sums) Note that the definition of convergence we give is quite weak, as we require convergence for *some* arrangement of the index set $\mathbb{Z}^n_{\geq 0}$. A stronger notion of convergence would be that the series

$$\sum_{j=1}^{\infty} \alpha(\phi(j))\boldsymbol{x}^{\phi(j)}$$

converge for *every* bijection $\phi\colon \mathbb{Z}_{>0} \to \mathbb{Z}^n_{\geq 0}$. This, it turns out, is equivalent to absolute convergence of the series, i.e., that

$$\sum_{I \in \mathbb{Z}^n_{\geq 0}} |\alpha(I)||\boldsymbol{x}^I| < \infty.$$

This is essentially explained by Roman [2005] (see Theorem 13.24) and Rudin [1976] (see Theorem 3.55).[1] We shall take an understanding of this for granted. •

Let us now show that convergence as in the definition above at any nontrivial point (i.e., a nonzero point) leads to a strong form of convergence at a "large" subset of other points. To be precise about this, for $\boldsymbol{x} \in \mathbb{F}^n$ let us denote

$$\mathcal{C}(\boldsymbol{x}) = \{(c_1 x_1, \dots, c_n x_n) \in \mathbb{F}^n \mid c_1, \dots, c_n \in \mathsf{D}^1(1, 0)\}.$$

Thus $\mathcal{C}(\boldsymbol{x})$ is the smallest open polydisk centred at the origin whose closure contains $\boldsymbol{x}$.

[1] Also see the interesting paper of Dvoretzky and Rogers [1950] in this regard, where it is shown that the equivalence of absolute and unconditional convergence only holds in finite dimensions.

1.1.8 Theorem (Uniform and absolute convergence of formal power series) *Let* $\alpha \in \mathbb{F}[[\boldsymbol{\xi}]]$ *and suppose that* α *converges at* $\mathbf{x}_0 \in \mathbb{F}^n$*. Then* α *converges uniformly and absolutely on every compact subset of* $\mathcal{C}(\mathbf{x}_0)$*.*

Proof Let $K \subseteq \mathcal{C}(\boldsymbol{x}_0)$ be compact. The proposition holds trivially if $K = \{\mathbf{0}\}$, so we suppose this is not the case. Let $\lambda \in (0,1)$ be such that $|x_j| \leq \lambda |x_{0j}|$ for $\boldsymbol{x} \in K$ and $j \in \{1,\dots,n\}$. Let $\phi\colon \mathbb{Z}_{>0} \to \mathbb{Z}^n_{\geq 0}$ be a bijection such that

$$\sum_{j=1}^{\infty} \alpha(\phi(j))\boldsymbol{x}_0^{\phi(j)}$$

converges. This implies, in particular, that the sequence $(\alpha(\phi(j))\boldsymbol{x}_0^{\phi(j)})_{j\in\mathbb{Z}_{>0}}$ is bounded. Thus there exists $M \in \mathbb{R}_{>0}$ such that $|\alpha(I)||\boldsymbol{x}_0^I| \leq M$ for every $I \in \mathbb{Z}^n_{\geq 0}$. Then $|\alpha(I)||\boldsymbol{x}^I| \leq M\lambda^{|I|}$ for every $\boldsymbol{x} \in K$. In order to complete the proof we use the following lemma.

1 Lemma *For* $\mathrm{x} \in (-1,1)$, $\sum_{\mathrm{j}=0}^{\infty} \frac{(\mathrm{m}+\mathrm{j})!}{\mathrm{j}!}\mathrm{x}^{\mathrm{j}} = \frac{\mathrm{d}^{\mathrm{m}}}{\mathrm{dx}^{\mathrm{m}}}\Big(\frac{\mathrm{x}^{\mathrm{m}}}{1-\mathrm{x}}\Big)$.

Proof Let $a \in (0,1)$ and recall that

$$\frac{1}{1-a} = \sum_{j=0}^{\infty} a^j \quad\Longrightarrow\quad \frac{a^m}{1-a} = \sum_{j=0}^{\infty} a^{m+j}$$

[Rudin 1976, Theorem 3.26]. Also, by the ratio test, the series

$$\sum_{j=0}^{\infty}(m+j)(m+j-1)\cdots(m+j-k)a^{m+j-k-1} \tag{1.3}$$

converges for $k \in \mathbb{Z}_{\geq 0}$.

Now, for $x \in [-a,a]$, since $|\frac{x^m}{1-x}| < \frac{a^m}{1-a}$, we have

$$\sum_{j=0}^{\infty} x^{m+j} = \frac{x^m}{1-x},$$

with the convergence being uniform and absolute on $[-a,a]$. Thus the series can be differentiated term-by-term to give

$$\frac{\mathrm{d}}{\mathrm{d}x}\Big(\frac{x^m}{1-x}\Big) = \sum_{j=0}^{\infty}(m+j)x^{m+j-1}.$$

Since $|(m+j)x^{m+j-1}| \leq (m+j)a^{m+j-1}$, this differentiated series converges uniformly and absolutely on $[-a,a]$ since the series (1.3) converges. In fact, by the same argument, this differentiation can be made m-times to give

$$\frac{\mathrm{d}^n}{\mathrm{d}x^n}\Big(\frac{x^m}{1-x}\Big) = \sum_{j=0}^{\infty}(m+j)\cdots(m+j-m+1)x^j = \sum_{j=0}^{\infty}\frac{(m+j)!}{j!}x^j,$$

as desired. ▼

Now, continuing with the proof, for $x \in K$ and for $m \in \mathbb{Z}_{\geq 0}$ we have

$$\sum_{\substack{I \in \mathbb{Z}^n_{\geq 0} \\ |I| \leq m}} |\alpha(I) x^I| \leq \sum_{\substack{I \in \mathbb{Z}^n_{\geq 0} \\ |I| \leq m}} |\alpha(I)||x^I| \leq \sum_{\substack{I \in \mathbb{Z}^n_{\geq 0} \\ |I| \leq m}} M\lambda^{|I|} < M \sum_{j=0}^{\infty} \binom{n+j-1}{n-1} \lambda^j$$
$$< M \sum_{j=0}^{\infty} \frac{(n+j-1)!}{(n-1)!} \lambda^j = M \frac{\mathrm{d}^{n-1}}{\mathrm{d}\lambda^{n-1}} \Big(\frac{\lambda^{n-1}}{1-\lambda} \Big),$$

using Lemma 1.1.1 and Lemma 1. Thus the sum

$$\sum_{I \in \mathbb{Z}^n_{\geq 0}} \alpha(I) x^I$$

converges absolutely on K, and uniformly in $x \in K$ since our computation above provides a bound independent of x. ■

The result implies that, if we have convergence (in the weak sense of Definition 1.1.6) for a formal power series at some nonzero point in $\mathbb{F}^n$, we have a strong form of convergence in some neighbourhood of the origin. We now define

$$\mathcal{R}_{\mathrm{abs}}(\alpha) = \bigcup_{r \in \mathbb{R}_{>0}} \Big\{ x \in \mathbb{F}^n \;\Big|\; \sum_{I \in \mathbb{Z}^n_{\geq 0}} |\alpha(I) y^I| < \infty \text{ for all } y \in \mathsf{B}^n(r, x) \Big\},$$

which we call the ***region of absolute convergence***. The following result gives the relationship between the two regions of convergence.

1.1.9 Proposition ($\mathrm{int}(\mathcal{R}_{\mathrm{conv}}(\alpha)) = \mathcal{R}_{\mathrm{abs}}(\alpha)$) *For $\alpha \in \mathbb{F}[[\xi]]$, $\mathrm{int}(\mathcal{R}_{\mathrm{conv}})(\alpha) = \mathcal{R}_{\mathrm{abs}}(\alpha)$.*

Proof Let $x \in \mathrm{int}(\mathcal{R}_{\mathrm{conv}}(\alpha))$. Then, there exists $\lambda > 1$ such that $\lambda x \in \mathcal{R}_{\mathrm{conv}}(\alpha)$. For such a λ, $x \in \mathcal{C}(\lambda x)$. Let $K \subseteq \mathcal{C}(\lambda x)$ and $r \in \mathbb{R}_{>0}$ be such that $\mathsf{B}^n(r, x) \subseteq K$, e.g., take K to be a large enough closed polydisk. By Theorem 1.1.8 it follows that

$$\sum_{I \in \mathbb{Z}^n_{\geq 0}} |\alpha(I) y^I| < \infty$$

for $y \in \mathsf{B}^n(r, x) \subseteq K$, and so $x \in \mathcal{R}_{\mathrm{abs}}(\alpha)$.

If $x \in \mathcal{R}_{\mathrm{abs}}(\alpha)$ then there exists $r \in \mathbb{R}_{>0}$ such that

$$\sum_{I \in \mathbb{Z}^n_{\geq 0}} |\alpha(I) y^I| < \infty$$

for $y \in \mathsf{B}^n(r, x)$. In particular, α converges at every $y \in \mathsf{B}^n(r, x)$ and so $x \in \mathrm{int}(\mathcal{R}_{\mathrm{conv}}(\alpha))$. ■

Examples show that the relationship between the region of convergence and the region of absolute convergence can be quite general, apart from the requirement of the preceding proposition.

1.1.10 Examples (Regions of convergence)

1. Consider the formal power series

$$\alpha = \sum_{j=1}^{\infty} \frac{1}{2^j j^2} z^j$$

in one variable. We compute

$$\lim_{j\to\infty} \left| \frac{a_{j+1}}{a_j} \right| = \lim_{j\to\infty} \left| \frac{2^j j^2}{2^{j+1}(j+1)^2} \right| = \frac{1}{2}.$$

By the ratio test [Rudin 1976, Theorem 3.34] it follows that the radius of convergence of the power series $\sum_{j=1}^{\infty} \frac{z^j}{2^j j^2}$ is 2. When $|z| = 2$ we have

$$\left| \sum_{j=1}^{\infty} \frac{1}{2^j j^2} z^j \right| \le \sum_{j=1}^{\infty} \frac{1}{2^j j^2} |z|^j = \sum_{j=1}^{\infty} \frac{1}{j^2},$$

from which we conclude that the series converges absolutely and so converges. Thus $\mathcal{R}_{\mathrm{abs}}(\alpha) = \mathsf{D}^1(2, 0)$, while $\mathcal{R}_{\mathrm{conv}}(\alpha) = \overline{\mathsf{D}}^1(2, 0)$.

2. Now consider the formal power series

$$\alpha = \sum_{j=1}^{\infty} \frac{1}{2^j j} z^j,$$

again in one variable. We again use the ratio test and the computation

$$\lim_{j\to\infty} \left| \frac{a_{j+1}}{a_j} \right| = \lim_{j\to\infty} \left| \frac{2^j j}{2^{j+1}(j+1)} \right| = \frac{1}{2}$$

to deduce that this power series has radius of convergence 2. When $|z| = 2$ let us write $z = 2\mathrm{e}^{\mathrm{i}\theta}$ so the power series at these points becomes

$$\sum_{j=1}^{\infty} \frac{\mathrm{e}^{\mathrm{i}j\theta}}{j} = \sum_{j=1}^{\infty} \frac{\cos(j\theta)}{j} + \mathrm{i} \sum_{j=1}^{\infty} \frac{\sin(j\theta)}{j}.$$

One can look up (I used Mathematica®)

$$\sum_{j=1}^{\infty} \frac{\cos(j\theta)}{j} = -\frac{1}{2}(\log(1 - \mathrm{e}^{\mathrm{i}\theta}) + \log(1 - \mathrm{e}^{-\mathrm{i}\theta})) \tag{1.4}$$

and

$$\sum_{j=1}^{\infty} \frac{\sin(j\theta)}{j} = \frac{\mathrm{i}}{2}(\log(1 - \mathrm{e}^{\mathrm{i}\theta}) - \log(1 - \mathrm{e}^{-\mathrm{i}\theta})). \tag{1.5}$$

From this we conclude that $\mathcal{R}_{\mathrm{abs}}(\alpha) = \mathsf{D}^1(2, 0)$, while

$$\mathcal{R}_{\mathrm{conv}}(\alpha) = \overline{\mathsf{D}}^1(2, 0) \setminus \{2 + \mathrm{i}0\}.$$

3. Now we define the formal power series
$$\alpha = \sum_{k=1}^{\infty} \frac{z^{2k}}{2^k k}$$
in one variable. We have
$$\limsup_{j\to\infty} |a_j|^{1/j} = \limsup_{k\to\infty} \left|\frac{1}{2^k k}\right|^{1/2k} = \frac{1}{\sqrt{2}} \lim_{k\to\infty} \left(\frac{1}{k}\right)^{1/2k} = \frac{1}{\sqrt{2}}.$$
Thus the radius of convergence is $\sqrt{2}$ by the root test. When $|z| = \sqrt{2}$ we write $z = \sqrt{2}e^{i\theta}$ and so the power series becomes
$$\sum_{k=1}^{\infty} \frac{e^{2i\theta}}{k}$$
at these points. By replacing "θ" with "2θ" in the equalities (1.4) and (1.5) we deduce that $\mathcal{R}_{\mathrm{abs}} = \mathsf{D}^1(\sqrt{2}, 0)$ while
$$\mathcal{R}_{\mathrm{conv}} = \overline{\mathsf{D}}^1(\sqrt{2}, 0) \setminus \{\sqrt{2} + i0, -\sqrt{2} + i0\}.$$

If we consider the preceding examples as $\mathbb{R}$-power series, we see that the region of convergence can be an open interval, a closed interval, or an interval that is neither open nor closed. •

This result has the following corollary that will be useful for us.

1.1.11 Corollary (Property of coefficients for convergent power series) *If* $\alpha \in \hat{\mathbb{F}}[[\xi]]$ *and if* $\mathbf{x} \in \mathcal{R}_{\mathrm{abs}}(\alpha)$ *then there exists* $C, \epsilon \in \mathbb{R}_{>0}$ *such that*
$$|\alpha(I)| \le \frac{C}{(|x_1| + \epsilon)^{i_1} \cdots (|x_n| + \epsilon)^{i_n}}$$
for every $I \in \mathbb{Z}^n_{\ge 0}$.

Proof Note that, if $x \in \mathcal{R}_{\mathrm{abs}}(\alpha)$, then
$$(|x_1|, \dots, |x_n|) \in \mathcal{R}_{\mathrm{abs}}(\alpha)$$
by definition of the region of absolute convergence and by Theorem 1.1.8. Now, by Proposition 1.1.9 there exists $\epsilon \in \mathbb{R}_{>0}$ such that
$$(|x_1| + \epsilon, \dots, |x_n| + \epsilon) \in \mathcal{R}_{\mathrm{conv}}(\alpha).$$
Thus there exists a bijection $\phi\colon \mathbb{Z}_{>0} \to \mathbb{Z}_{\ge 0}$ such that
$$\sum_{j=1}^{\infty} \alpha(\phi(j))(|x_1| + \epsilon)^{\phi(j)_1} \cdots (|x_n| + \epsilon)^{\phi(j)_n}$$
converges. Therefore, the terms in this series must be bounded. Thus there exists $C \in \mathbb{R}_{>0}$ such that
$$\alpha(I)(|x_1| + \epsilon)^{i_1} \cdots (|x_n| + \epsilon)^{i_n} < C$$
for every $I \in \mathbb{Z}^n_{\ge 0}$. ■

Using this property of the coefficients of a convergent power series, one can deduce the following result.

1.1.12 Corollary (Convergent power series converge to infinitely $\mathbb{F}$-differentiable functions) *If* $\alpha \in \hat{\mathbb{F}}[[\xi]]$ *then the series*

$$\sum_{\mathrm{I}\in\mathbb{Z}^{\mathrm{n}}_{\geq 0}} \alpha(\mathrm{I})\mathbf{x}^{\mathrm{I}}$$

converges in $\mathcal{R}_{\mathrm{abs}}$ *to an infinitely* $\mathbb{F}$*-differentiable function whose* $\mathbb{F}$*-derivatives are obtained by differentiating the series term-by-term.*

Proof By induction it suffices to show that any partial $\mathbb{F}$-derivative of f is defined on $\mathcal{R}_{\mathrm{abs}}$ by a convergent power series. Consider a term $\alpha(I)\boldsymbol{x}^I$ in the power series for $I \in \mathbb{Z}^n_{\geq 0}$. For $j \in \mathbb{Z}_{>0}$ we have

$$\frac{\partial}{\partial x_j}\alpha(I)\boldsymbol{x}^I = \begin{cases} 0, & i_j = 0, \\ i_j\alpha(I)\boldsymbol{x}^{I-\boldsymbol{e}_j}, & i_j \geq 1. \end{cases}$$

Thus, when differentiating the terms in the power series with respect to x_j, the only nonzero contribution will come from terms corresponding to multi-indices of the form $I + \boldsymbol{e}_j$. In this case,

$$\frac{\partial}{\partial x_j}\alpha(I+\boldsymbol{e}_j)\boldsymbol{x}^{I+\boldsymbol{e}_j} = (i_j+1)\alpha(I+\boldsymbol{e}_j)\boldsymbol{x}^I.$$

Therefore, the power series whose terms are the partial $\mathbb{F}$-derivatives of those for the given power series with respect to x_j is

$$\sum_{I\in\mathbb{Z}^n_{\geq 0}} (i_j+1)\alpha(I+\boldsymbol{e}_j)\boldsymbol{x}^I.$$

Now let $\boldsymbol{x} \in \mathcal{R}_{\mathrm{abs}}$ and, according to Corollary 1.1.11, let $C, \epsilon \in \mathbb{R}_{>0}$ be such that

$$|\alpha(I)| \leq \frac{C}{(|x_1|+\epsilon)^{i_1}\cdots(|x_n|+\epsilon)^{i_n}}, \qquad I \in \mathbb{Z}^n_{\geq 0}.$$

Let $\boldsymbol{y} \in \mathcal{R}_{\mathrm{abs}}$ be such that $\boldsymbol{y} \in \mathsf{D}^n(\frac{\epsilon}{2}, \boldsymbol{x})$. Note that

$$|y_j| \leq |x_j| + |y_j - x_j| < |x_j| + \tfrac{\epsilon}{2}.$$

Also let

$$\lambda = \max\left\{\frac{|x_1|+\frac{\epsilon}{2}}{|x_1|+\epsilon}, \dots, \frac{|x_n|+\frac{\epsilon}{2}}{|x_n|+\epsilon}\right\} \in (0,1).$$

Then, we compute

$$\begin{aligned}\sum_{I\in\mathbb{Z}^n_{\geq 0}} |i_j+1|\,|\alpha(I+\boldsymbol{e}_j)|\,|\boldsymbol{y}^I| &\leq \sum_{I\in\mathbb{Z}^n_{\geq 0}} C|i_j+1|\left(\frac{|x_1|+\frac{\epsilon}{2}}{|x_1|+\epsilon}\right)^{i_1}\cdots\left(\frac{|x_n|+\frac{\epsilon}{2}}{|x_n|+\epsilon}\right)^{i_n} \\ &\leq \sum_{m=0}^{\infty}\sum_{\substack{I\in\mathbb{Z}^n_{\geq 0}\\ |I|=m}} C|i_j+1|\lambda^m \leq \sum_{m=0}^{\infty} C(m+1)\binom{n-m-1}{n-1}\lambda^m,\end{aligned}$$

using Lemma 1.1.1. The ratio test shows that this last series converges. Thus the power series whose terms are the partial $\mathbb{F}$-derivatives of those for the given power series with respect to x_j converges uniformly and absolutely in a neighbourhood of $\boldsymbol{x}$. Thus

$$\frac{\partial}{\partial x_j}\Big(\sum_{I\in\mathbb{Z}^n_{\geq 0}} \alpha(I)\boldsymbol{x}^I\Big) = \sum_{I\in\mathbb{Z}^n_{\geq 0}} (i_j+1)\alpha(I+\boldsymbol{e}_j)\boldsymbol{x}^I,$$

which gives the corollary, after an induction as we indicated at the beginning of the proof. ■

In Theorem 1.1.16 below we shall show that, in fact, convergent power series are real analytic on $\mathcal{R}_{\mathrm{abs}}$.

1.1.6 Holomorphic and real analytic functions

Throughout this section we let $\boldsymbol{\xi} = \{\xi_1, \dots, \xi_n\}$ so $\mathbb{F}[[\boldsymbol{\xi}]]$ denotes the $\mathbb{F}$-formal power series in these indeterminates.

Now understanding some basic facts about convergent power series, we are in a position to use this knowledge to define what we mean by a real analytic function, and give some properties of such functions.

1.1.13 Definition (Holomorphic function, real analytic function) Let $\mathcal{U} \subseteq \mathbb{F}^n$ be open. A function $f\colon \mathcal{U} \to \mathbb{F}$ is

(i) (if $\mathbb{F} = \mathbb{R}$) ***real analytic*** or of ***class*** $\mathbf{C}^{\omega}$ on $\mathcal{U}$, or

(ii) (if $\mathbb{F} = \mathbb{C}$) ***holomorphic*** or of ***class*** $\mathbf{C}^{\mathbf{hol}}$ on $\mathcal{U}$

if, for every $\boldsymbol{x}_0 \in \mathcal{U}$, there exists $\alpha_{\boldsymbol{x}_0} \in \hat{\mathbb{F}}[[\boldsymbol{\xi}]]$ and $r \in \mathbb{R}_{>0}$ such that

$$f(\boldsymbol{x}) = \sum_{I\in\mathbb{Z}^n_{\geq 0}} \alpha_{\boldsymbol{x}_0}(I)(\boldsymbol{x}-\boldsymbol{x}_0)^I = \sum_{I\in\mathbb{Z}^n_{\geq 0}} \alpha_{\boldsymbol{x}_0}(I)(x_1 - x_{01})^{i_1}\cdots(x_n - x_{0n})^{i_n}$$

for all $\boldsymbol{x} \in \mathsf{B}^n(r, \boldsymbol{x}_0)$. The set of real analytic functions on $\mathcal{U}$ is denoted by $\mathrm{C}^{\omega}(\mathcal{U})$.

A map $\boldsymbol{f}\colon \mathcal{U} \to \mathbb{F}^m$ is ***real analytic*** (resp. ***holomorphic***) or of ***class*** $\mathbf{C}^{\omega}$ (resp. ***class*** $\mathbf{C}^{\mathbf{hol}}$) on $\mathcal{U}$ if its components $f_1, \dots, f_m\colon \mathcal{U} \to \mathbb{F}$ are real analytic (resp. holomorphic). The set of real analytic (resp. holomorphic) $\mathbb{F}^m$-valued maps on $\mathcal{U}$ is denoted by $\mathrm{C}^{\omega}(\mathcal{U};\mathbb{R}^m)$ (resp. $\mathrm{C}^{\mathrm{hol}}(\mathcal{U};\mathbb{C}^m)$). •

1.1.14 Notation ("Real analytic" or "analytic") We shall very frequently, especially outside the confines of this chapter, write "analytic" in place of "real analytic." This is not problematic since we use the term "holomorphic" and not the term "analytic" when referring to functions of a complex variable. •

We can now show that a real analytic (resp. holomorphic) function is infinitely $\mathbb{F}$-differentiable with real analytic (resp. holomorphic) derivatives, and that the power series coefficients $\alpha_{\boldsymbol{x}_0}(I)$, $I \in \mathbb{Z}^n_{\geq 0}$, are actually the Taylor series coefficients for f at $\boldsymbol{x}_0$.

1.1.15 Theorem (Holomorphic or real analytic functions have holomorphic or real analytic partial derivatives of all orders) *If $\mathcal{U} \subseteq \mathbb{F}^n$ is open and if $f\colon \mathcal{U} \to \mathbb{F}$ is holomorphic or real analytic, then all partial $\mathbb{F}$-derivatives of f are holomorphic or real analytic, respectively. Moreover, if $\mathbf{x}_0 \in \mathcal{U}$, and if $\alpha \in \hat{\mathbb{F}}[[\boldsymbol{\xi}]]$ and $r \in \mathbb{R}_{>0}$ are such that*

$$f(\mathbf{x}) = \sum_{I \in \mathbb{Z}^n_{\geq 0}} \alpha(I)(\mathbf{x} - \mathbf{x}_0)^I \tag{1.6}$$

for all $\mathbf{x} \in \mathsf{B}^n(r, \mathbf{x}_0)$, then $\alpha = \alpha_f(\mathbf{x}_0)$.

Proof We begin with a lemma.

1 Lemma *If $\mathcal{U} \subseteq \mathbb{F}^n$ is open and if $f\colon \mathcal{U} \to \mathbb{F}$ is holomorphic or real analytic, then f is $\mathbb{F}$-differentiable and its partial $\mathbb{F}$-derivatives are holomorphic or real analytic functions, respectively.*

Proof Let $\boldsymbol{x}_0 \in \mathcal{U}$ and let $r \in \mathbb{R}_{>0}$ and $\alpha \in \hat{\mathbb{F}}[[\boldsymbol{\xi}]]$ be such that

$$f(\boldsymbol{x}) = \sum_{I \in \mathbb{Z}^n_{\geq 0}} \alpha(I)(\boldsymbol{x} - \boldsymbol{x}_0)^I \tag{1.7}$$

for all $\boldsymbol{x} \in \mathsf{B}^n(r, \boldsymbol{x}_0)$. As we showed in the proof of Corollary 1.1.12, the power series whose terms are the partial derivatives of those for the power series for f with respect x_j is

$$\sum_{I \in \mathbb{Z}^n_{\geq 0}} (i_j + 1)\alpha(I + \boldsymbol{e}_j)(\boldsymbol{x} - \boldsymbol{x}_0)^I.$$

Now let $\epsilon \in \mathbb{R}_{>0}$ be such that

$$\boldsymbol{x}_\epsilon \triangleq (x_{01} + \epsilon, \dots, x_{0n} + \epsilon) \in \mathsf{B}^n(r, \boldsymbol{x}_0).$$

Since the series (1.7) converges at $\boldsymbol{x}_\epsilon$, the terms in the series (1.7) must be bounded. Thus there exists $C \in \mathbb{R}_{>0}$ such that, for all $I \in \mathbb{Z}^n_{\geq 0}$,

$$|\alpha(I)(\boldsymbol{x}_\epsilon - \boldsymbol{x}_0)^I| = |\alpha(I)|\epsilon^{|I|} \leq C.$$

Let $\boldsymbol{x} \in \mathsf{B}^n(r, \boldsymbol{x}_0)$ be such that $|x_j - x_{0j}| < \lambda\epsilon$ for some $\lambda \in (0, 1)$. We then estimate

$$\begin{aligned}
\sum_{I \in \mathbb{Z}^n_{\geq 0}} |(i_j + 1)\alpha(I + \boldsymbol{e}_j)(\boldsymbol{x} - \boldsymbol{x}_0)^I| &= \sum_{I \in \mathbb{Z}^n_{\geq 0}} (i_j + 1)|\alpha(I + \boldsymbol{e}_j)||(\boldsymbol{x} - \boldsymbol{x}_0)^I| \\
&\leq C \sum_{I \in \mathbb{Z}^n_{\geq 0}} (i_j + 1)\frac{|(\boldsymbol{x} - \boldsymbol{x}_0)^I|}{\epsilon^{|I|+1}} \leq \frac{C}{\epsilon} \sum_{k=0}^{\infty} \sum_{\substack{I \in \mathbb{Z}^n_{\geq 0} \\ |I| = k}} (i_j + 1)\lambda^{|I|} \\
&\leq \frac{C}{\epsilon} \sum_{k=0}^{\infty} (k + 1)\binom{n + k - 1}{n - 1}\lambda^k,
\end{aligned}$$

where we have used Lemma 1.1.1. The ratio test can be used to show that this last series converges. Since this holds for every $\boldsymbol{x}$ for which $|x_j - x_{0j}| < \lambda\epsilon$ for $\lambda \in (0, 1)$, it follows that there is a neighbourhood of $\boldsymbol{x}_0$ for which the series

$$\sum_{I \in \mathbb{Z}^n_{\geq 0}} \frac{\partial}{\partial x_j} \alpha(I)(\boldsymbol{x} - \boldsymbol{x}_0)^I$$

converges absolutely and uniformly. This means that $\frac{\partial f}{\partial x_j}$ is represented by a convergent power series in a neighbourhood of $\boldsymbol{x}_0$. Since $\boldsymbol{x}_0 \in \mathcal{U}$ is arbitrary, it follows that $\frac{\partial f}{\partial x_j}$ is holomorphic or real analytic in $\mathcal{U}$. ▼

Now, the only part of the statement in the theorem that does not follow immediately from a repeated application of the lemma is the final assertion. This conclusion is proved as follows. If we evaluate (1.6) at $\boldsymbol{x} = \boldsymbol{x}_0$ we see that $\alpha(\boldsymbol{0}) = f(\boldsymbol{x}_0)$. In the proof of the lemma above we showed that

$$\frac{\partial f}{\partial x_j}(\boldsymbol{x}) = \sum_{I \in \mathbb{Z}^n_{\geq 0}} (i_j + 1)\alpha(I + \boldsymbol{e}_j)(\boldsymbol{x} - \boldsymbol{x}_0)^I$$

in a neighbourhood of $\boldsymbol{x}_0$. If we evaluate this at $\boldsymbol{x} = \boldsymbol{x}_0$ we see that $\alpha(\boldsymbol{e}_j) = \frac{\partial f}{\partial x_j}(\boldsymbol{x}_0)$. We can then inductively apply this argument to higher-order $\mathbb{F}$-derivatives to derive the formula

$$\alpha(i_1\boldsymbol{e}_1 + \cdots + i_n\boldsymbol{e}_n) = \frac{1}{i_1! \cdots i_n!}\boldsymbol{D}^{i_1+\cdots+i_n} f(\boldsymbol{x}_0) = \frac{1}{I!}\boldsymbol{D}^I f(\boldsymbol{x}_0),$$

which gives $\alpha(I) = \alpha_f(\boldsymbol{x}_0)(I)$, as desired. ■

By definition, holomorphic and real analytic functions are represented by convergent power series—in fact their Taylor series by Theorem 1.1.15—in a neighbourhood of any point. Conversely, any convergent power series defines a real analytic function on its domain of convergence.

1.1.16 Theorem (Convergent power series define holomorphic or real analytic functions) *If $\alpha \in \hat{\mathbb{F}}[[\xi]]$ then the function $f_\alpha \colon \mathcal{R}_{\mathrm{abs}}(\alpha) \to \mathbb{F}$ defined by*

$$f_\alpha(\boldsymbol{x}) = \sum_{I \in \mathbb{Z}^n_{\geq 0}} \alpha(I)\boldsymbol{x}^I \tag{1.8}$$

is holomorphic or real analytic.

Proof By Corollary 1.1.12 we know that f_α is infinitely $\mathbb{F}$-differentiable and its $\mathbb{F}$-derivatives can be gotten by term-by-term differentiation of the series for f_α. Let $\boldsymbol{x}_0 \in \mathcal{R}_{\mathrm{abs}}(\alpha)$. By an induction on the argument of Corollary 1.1.12, if $J = (j_1, \dots, j_n) \in \mathbb{Z}^n_{\geq 0}$ then

$$\boldsymbol{D}^J f_\alpha(\boldsymbol{x}_0) = \sum_{I \in \mathbb{Z}^n_{\geq 0}} (i_1 + j_1)\cdots(i_1 + 1)\cdots(i_n + j_n)\cdots(i_n + 1)\alpha(I + J)\boldsymbol{x}_0^I.$$

Note that

$$\frac{1}{J!}\boldsymbol{D}^J f_\alpha(\boldsymbol{x}_0) = \sum_{I \in \mathbb{Z}^n_{\geq 0}} \binom{i_1 + j_1}{j_1}\cdots\binom{i_n + j_n}{j_n}\alpha(I + J)\boldsymbol{x}_0^I.$$

We must show that

$$f_\alpha(\boldsymbol{x}) = \sum_{J \in \mathbb{Z}^n_{\geq 0}} \frac{1}{J!}\boldsymbol{D}^J f_\alpha(\boldsymbol{x}_0)(\boldsymbol{x} - \boldsymbol{x}_0)^J$$

for $\boldsymbol{x}$ in some neighbourhood of $\boldsymbol{x}_0$. To this end, the following lemma will be useful.

1 Lemma *If* $\mathrm{a}, \mathrm{b} \in \mathbb{R}_{>0}$ *satisfy* $\mathrm{a} + \mathrm{b} < 1$*, then*

$$\sum_{\mathrm{J} \in \mathbb{Z}^{\mathrm{n}}_{\geq 0}} \sum_{\mathrm{I} \in \mathbb{Z}^{\mathrm{n}}_{\geq 0}} \binom{\mathrm{i}_1 + \mathrm{j}_1}{\mathrm{j}_1} \cdots \binom{\mathrm{i}_\mathrm{n} + \mathrm{j}_\mathrm{n}}{\mathrm{j}_\mathrm{n}} \mathrm{a}^{|\mathrm{I}|} \mathrm{b}^{|\mathrm{J}|} = \Big(\frac{1}{1 - \mathrm{a} - \mathrm{b}}\Big)^{\mathrm{n}}.$$

Proof Recall that for $m \in \mathbb{Z}_{\geq 0}$ we have

$$\sum_{j=0}^{m} \binom{m}{j} a^{m-j} b^j = (a+b)^m,$$

as is easily shown by induction. Therefore,

$$\sum_{m=0}^{\infty} \sum_{j=0}^{m} \binom{m}{j} a^{m-j} b^j = \frac{1}{1-a-b},$$

cf. the proof of Lemma 1 from the proof of Theorem 1.1.8. Note that the sets

$$\{(i,j) \in \mathbb{Z}^2 \mid i + j = m\}, \quad \{(m,j) \in \mathbb{Z}^2 \mid j \leq m\}$$

are in one-to-one correspondence. Using this fact we have

$$\begin{aligned}
\sum_{J \in \mathbb{Z}^n_{\geq 0}} \sum_{I \in \mathbb{Z}^n_{\geq 0}} \binom{i_1 + j_1}{j_1} \cdots \binom{i_n + j_n}{j_n} a^{|I|} b^{|J|} &= \sum_{J \in \mathbb{Z}^n_{\geq 0}} \sum_{I \in \mathbb{Z}^n_{\geq 0}} \binom{i_1 + j_1}{j_1} a^{i_1} b^{j_1} \cdots \binom{i_n + j_n}{j_n} a^{i_n} b^{j_n} \\
&= \Big(\sum_{i_1=0}^{\infty} \sum_{j_1=0}^{m_1} \binom{m_1}{j_1} a^{m_1 - j_1} b^{j_1}\Big) \cdots \Big(\sum_{i_n=0}^{\infty} \sum_{j_n=0}^{m_n} \binom{m_n}{j_n} a^{m_n - j_n} b^{j_n}\Big) \\
&= \Big(\frac{1}{1-a-b}\Big)^m,
\end{aligned}$$

as desired. ▼

Let $\boldsymbol{z} \in \mathcal{R}_{\mathrm{abs}}(\alpha)$ be such that none of the components of $\boldsymbol{z}$ are zero and such that $\left|\frac{x_{0j}}{z_j}\right| < 1$ for $j \in \{1, \dots, n\}$. This is possible by openness of $\mathcal{R}_{\mathrm{abs}}(\alpha)$. Denote

$$\lambda = \max\Big\{\frac{x_{01}}{z_1}, \dots, \frac{x_{0n}}{z_n}\Big\}.$$

Let $\rho \in \mathbb{R}_{>0}$ be such that $\lambda + \rho < 1$. If $\boldsymbol{y} \in \mathcal{R}_{\mathrm{abs}}(\alpha)$ satisfies $|y_j - x_{0j}| < \rho |\boldsymbol{z}_j|$ for $j \in \{1, \dots, n\}$. With these definitions we now compute

$$\begin{aligned}
\sum_{J \in \mathbb{Z}^n_{\geq 0}} \frac{1}{J!} |\boldsymbol{D}^J f_\alpha(\boldsymbol{x}_0)| |(\boldsymbol{x} - \boldsymbol{x}_0)^J| &= \sum_{J \in \mathbb{Z}^n_{\geq 0}} \sum_{I \in \mathbb{Z}^n_{\geq 0}} \binom{i_1 + j_1}{j_1} \cdots \binom{i_n + j_n}{j_n} |\alpha(I+J)| |\boldsymbol{x}_0^I| |(\boldsymbol{x} - \boldsymbol{x}_0)^J| \\
&\leq \sum_{J \in \mathbb{Z}^n_{\geq 0}} \sum_{I \in \mathbb{Z}^n_{\geq 0}} \binom{i_1 + j_1}{j_1} \cdots \binom{i_n + j_n}{j_n} \frac{C}{|\boldsymbol{z}^{I+J}|} |\boldsymbol{x}_0^I| \rho^{|J|} |\boldsymbol{z}^J| \\
&= C \sum_{J \in \mathbb{Z}^n_{\geq 0}} \sum_{I \in \mathbb{Z}^n_{\geq 0}} \binom{i_1 + j_1}{j_1} \cdots \binom{i_n + j_n}{j_n} \lambda^{|I|} \rho^{|J|} = C \Big(\frac{1}{1 - \lambda - \rho}\Big)^n,
\end{aligned}$$

using the lemma above. This shows that the Taylor series for f_α at $\boldsymbol{x}_0$ converges absolutely and uniformly in a neighbourhood of $\boldsymbol{x}_0$. It remains to show that it converges to f_α. We do this separately for the holomorphic and real analytic cases.

First, for the holomorphic case, as in the proof of Corollary 1.1.24, we have

$$\boldsymbol{D}^I f_\alpha(\boldsymbol{z}) = I!\Big(\frac{1}{2\pi\mathrm{i}}\Big)^n \int_{\mathrm{bd}\,\mathsf{D}^n(\hat{r},\boldsymbol{z}_0)} \frac{f_\alpha(\boldsymbol{\zeta})\mathrm{d}\zeta_1\cdots\mathrm{d}\zeta_n}{(\zeta_1-z_1)^{I_1+1}\cdots(\zeta_n-z_n)^{I_n+1}},$$

where $r\in\mathbb{R}_{>0}$ is chosen sufficiently small that $\overline{\mathsf{D}}^n(\hat{r},\boldsymbol{z}_0)\subseteq\mathcal{U}$. Thus we have, using Fubini's Theorem and the Dominated Convergence Theorem,

$$\begin{aligned}
\sum_{I\in\mathbb{Z}^n_{\geq 0}} \frac{1}{I!}\boldsymbol{D}^I f_\alpha(\boldsymbol{z}_0)(\boldsymbol{z}-\boldsymbol{z}_0)^I &= \sum_{I\in\mathbb{Z}^n_{\geq 0}} (\boldsymbol{z}-\boldsymbol{z}_0)^I\Big(\frac{1}{2\pi\mathrm{i}}\Big)^n \int_{\mathrm{bd}\,\mathsf{D}^n(\hat{r},\boldsymbol{z}_0)} \frac{f_\alpha(\boldsymbol{\zeta})\mathrm{d}\zeta_1\cdots\mathrm{d}\zeta_n}{(\zeta_1-z_{01})^{I_1+1}\cdots(\zeta_n-z_{0n})^{I_n+1}}\\
&= \Big(\frac{1}{2\pi\mathrm{i}}\Big)^n \int_{\mathrm{bd}\,\mathsf{D}^n(\hat{r},\boldsymbol{z}_0)} \frac{f_\alpha(\boldsymbol{\zeta})}{(\zeta_1-z_{01})\cdots(\zeta_n-z_{0n})} \sum_{I\in\mathbb{Z}^n_{\geq 0}} \Big(\frac{z_1-z_{01}}{\zeta_1-z_{01}}\Big)^{I_1}\cdots\Big(\frac{z_n-z_{0n}}{\zeta_n-z_{0n}}\Big)^{I_n}\mathrm{d}\zeta_1\cdots\mathrm{d}\zeta_n\\
&= \Big(\frac{1}{2\pi\mathrm{i}}\Big)^n \int_{\mathrm{bd}\,\mathsf{D}^n(\hat{r},\boldsymbol{z}_0)} \frac{f_\alpha(\boldsymbol{\zeta})}{(\zeta_1-z_{01})\cdots(\zeta_n-z_{0n})} \frac{1}{1-\frac{z_1-z_{01}}{\zeta_1-z_{01}}}\cdots\frac{1}{1-\frac{z_n-z_{0n}}{\zeta_n-z_{0n}}}\mathrm{d}\zeta_1\cdots\mathrm{d}\zeta_n\\
&= \Big(\frac{1}{2\pi\mathrm{i}}\Big)^n \int_{\mathrm{bd}\,\mathsf{D}^n(\hat{r},\boldsymbol{z}_0)} \frac{f_\alpha(\boldsymbol{\zeta})}{(\zeta_1-z_1)\cdots(\zeta_n-z_n)}\mathrm{d}\zeta_1\cdots\mathrm{d}\zeta_n = f_\alpha(\boldsymbol{z})
\end{aligned}$$

for $\boldsymbol{z}\in\mathsf{D}^n(\hat{r},\boldsymbol{z}_0)$.

Now we consider the real analytic case, and show that the Taylor series for f_α converges to f_α at $\boldsymbol{x}_0$. Let $\boldsymbol{x}$ be a point in the neighbourhood of $\boldsymbol{x}_0$ where the Taylor series of f_α at $\boldsymbol{x}_0$ converges. Let $k\in\mathbb{Z}_{>0}$. By Taylor's Theorem [Abraham, Marsden, and Ratiu 1988, Theorem 2.4.15] there exists

$$\boldsymbol{z}\in\{(1-t)\boldsymbol{x}_0+t\boldsymbol{x}\mid\ t\in[0,1]\}$$

such that

$$f_\alpha(\boldsymbol{x}) = \sum_{\substack{J\in\mathbb{Z}^n_{\geq 0}\\ |J|\leq k}} \frac{1}{J!}\boldsymbol{D}^J f_\alpha(\boldsymbol{x}_0)(\boldsymbol{x}-\boldsymbol{x}_0)^J + \sum_{\substack{J\in\mathbb{Z}^n_{\geq 0}\\ |J|=k+1}} \frac{1}{J!}\boldsymbol{D}^J f_\alpha(\boldsymbol{z})(\boldsymbol{x}-\boldsymbol{x}_0)^J.$$

By Corollary 1.1.12 we have

$$\frac{1}{J!}\boldsymbol{D}^J f_\alpha(\boldsymbol{z}) = \sum_{I\in\mathbb{Z}^n_{\geq 0}} \binom{i_1+j_1}{j_1}\cdots\binom{i_n+j_n}{j_n}\alpha(I+J)\boldsymbol{z}^I.$$

Therefore,

$$\begin{aligned}
\Big|f_\alpha(\boldsymbol{x}) - \sum_{\substack{J\in\mathbb{Z}^n_{\geq 0}\\ |J|\leq k}} \frac{1}{J!}\boldsymbol{D}^J f_\alpha(\boldsymbol{x}_0)(\boldsymbol{x}-\boldsymbol{x}_0)^J\Big| &\leq \sum_{\substack{J\in\mathbb{Z}^n_{\geq 0}\\ |J|=k+1}} \frac{1}{J!}|\boldsymbol{D}^J f_\alpha(\boldsymbol{z})||(\boldsymbol{x}-\boldsymbol{x}_0)^J|\\
&= \sum_{\substack{J\in\mathbb{Z}^n_{\geq 0}\\ |J|=k+1}} \sum_{I\in\mathbb{Z}^n_{\geq 0}} \binom{i_1+j_1}{j_1}\cdots\binom{i_n+j_n}{j_n}|\alpha(I+J)||\boldsymbol{z}^I||(\boldsymbol{x}-\boldsymbol{x}_0)^J|.
\end{aligned}$$

Just as we did above when we showed that the Taylor series for f_α at $\boldsymbol{x}_0$ converges absolutely, we can show that the series

$$\sum_{J\in\mathbb{Z}^n_{\geq 0}}\sum_{I\in\mathbb{Z}^n_{\geq 0}}\binom{i_1+j_1}{j_1}\cdots\binom{i_n+j_n}{j_n}|\alpha(I+J)||\boldsymbol{z}^I||(\boldsymbol{x}-\boldsymbol{x}_0)^J|$$

converges. Therefore,

$$\lim_{k\to\infty}\sum_{\substack{J\in\mathbb{Z}^n_{\geq 0}\\|J|=k+1}}\sum_{I\in\mathbb{Z}^n_{\geq 0}}\binom{i_1+j_1}{j_1}\cdots\binom{i_n+j_n}{j_n}|\alpha(I+J)||\boldsymbol{z}^I||(\boldsymbol{x}-\boldsymbol{x}_0)^J|=0,$$

and so

$$\lim_{k\to\infty}\Big|f_\alpha(\boldsymbol{x})-\sum_{\substack{J\in\mathbb{Z}^n_{\geq 0}\\|J|\leq k}}\frac{1}{J!}\boldsymbol{D}^J f_\alpha(\boldsymbol{x}_0)(\boldsymbol{x}-\boldsymbol{x}_0)^J\Big|=0,$$

showing that the Taylor series for f_α at $\boldsymbol{x}_0$ converges to f_α in a neighbourhood of $\boldsymbol{x}_0$. ■

As a final result, let us characterise real analytic functions by providing an exact description of their derivatives.

1.1.17 Theorem (Derivatives of holomorphic or real analytic functions) *If $\mathcal{U}\subseteq\mathbb{F}^n$ is open and if $f\colon\mathcal{U}\to\mathbb{F}$ is infinitely $\mathbb{F}$-differentiable, then the following statements are equivalent:*

(i) f is holomorphic or real analytic;

(ii) for each $\mathbf{x}_0\in\mathcal{U}$ there exists a neighbourhood $\mathcal{V}\subseteq\mathcal{U}$ of $\mathbf{x}_0$ and $C,r\in\mathbb{R}_{>0}$ such that

$$|\mathbf{D}^I f(\mathbf{x})|\leq CI!r^{-|I|}$$

for all $\mathbf{x}\in\mathcal{V}$ and $I\in\mathbb{Z}^n_{\geq 0}$.

Proof We consider the holomorphic and real analytic cases separately.

First, in the holomorphic case, if f is holomorphic then it follows from Corollary 1.1.24 that condition (ii) holds. Conversely, suppose that for each $\boldsymbol{x}_0\in\mathcal{U}$ there exists a neighbourhood $\mathcal{V}\subseteq\mathcal{U}$ of $\boldsymbol{x}_0$ and $C,r\in\mathbb{R}_{>0}$ such that

$$|\boldsymbol{D}^I f(\boldsymbol{x})|\leq CI!r^{-|I|}$$

for all $\boldsymbol{x}\in\mathcal{V}$ and $I\in\mathbb{Z}^n_{\geq 0}$. Then, for $\boldsymbol{x}_0\in\mathcal{U}$, let $C,r,\rho\in\mathbb{R}_{>0}$ be such that $\rho<r$ and

$$|\boldsymbol{D}^I f(\boldsymbol{x})|\leq CI!r^{-|I|},\qquad I\in\mathbb{Z}^n_{\geq 0},\ \boldsymbol{x}\in\mathsf{B}^n(\rho,\boldsymbol{x}_0).$$

Then, for $\boldsymbol{x}\in\mathsf{B}^n(\rho,\boldsymbol{x}_0)$ we have

$$\begin{aligned}\sum_{I\in\mathbb{Z}^n_{\geq 0}}\frac{1}{I!}|\boldsymbol{D}^I f(\boldsymbol{x}_0)||(\boldsymbol{x}-\boldsymbol{x}_0)^I|&\leq\sum_{I\in\mathbb{Z}^n_{\geq 0}}\Big(\frac{\rho}{r}\Big)^{|I|}\leq\sum_{k=0}^{\infty}\sum_{\substack{I\in\mathbb{Z}^n_{\geq 0}\\|I|=k}}\Big(\frac{\rho}{r}\Big)^{|I|}\\&\leq\sum_{k=0}^{\infty}\binom{n-k-1}{k-1}\Big(\frac{\rho}{r}\Big)^k,\end{aligned}$$

using Lemma 1.1.1. By the ratio test, the last series converges, giving absolute convergence of the Taylor series of f at x_0. We must also show that the series converges to f. This can be done in the holomorphic case just as in the proof of Theorem 1.1.16.

Now we consider the real analytic case. We will use the following lemmata.

1 **Lemma** *Let* $\mathrm{J} \in \mathbb{Z}^{\mathrm{n}}_{\geq 0}$ *and let* $\mathbf{x} \in \mathbb{R}^{\mathrm{n}}$ *satisfy* $|\mathrm{x}_{\mathrm{k}}| < 1$, $\mathrm{k} \in \{1, \dots, \mathrm{n}\}$. *Then*

$$\sum_{\mathrm{I} \in \mathbb{Z}^{\mathrm{n}}_{\geq 0}} \mathrm{J}! \binom{\mathrm{i}_1 + \mathrm{j}_1}{\mathrm{j}_1} \cdots \binom{\mathrm{i}_\mathrm{n} + \mathrm{j}_\mathrm{n}}{\mathrm{j}_\mathrm{n}} |\mathbf{x}^{\mathrm{I}}| = \frac{\partial^{|\mathrm{J}|}}{\partial \mathbf{x}^{\mathrm{J}}} \Big(\prod_{\mathrm{k}=1}^{\mathrm{n}} \frac{\mathrm{x}_\mathrm{k}^{\mathrm{j}_\mathrm{k}}}{1 - \mathrm{x}_\mathrm{k}} \Big).$$

Proof By Lemma 1 from the proof of Theorem 1.1.8 we have

$$\sum_{i_k=0}^{\infty} j_k! \binom{i_k + j_k}{j_k} x_k^{i_k} = \sum_{i_k=0}^{\infty} \frac{(i_k + j_k)!}{i_k!} x^{i_k} = \frac{\mathrm{d}^{j_k}}{\mathrm{d} x_k^{j_k}} \Big(\frac{x_k^{j_k}}{1 - x_k} \Big), \qquad k \in \{1, \dots, n\}.$$

Therefore,

$$\begin{aligned} \sum_{I \in \mathbb{Z}^n_{\geq 0}} J! \binom{i_1 + j_1}{j_1} \cdots \binom{i_n + j_n}{j_n} |\boldsymbol{x}^I| &= \sum_{I \in \mathbb{Z}^n_{\geq 0}} j_1! \binom{i_1 + j_1}{i_1} x_1^{i_1} \cdots j_n! \binom{i_n + j_n}{i_n} x_n^{i_n} \\ &= \Big(\sum_{i_1=0}^{\infty} j_1! \binom{i_1 + j_1}{i_1} x_1^{i_1} \Big) \cdots \Big(\sum_{i_n=0}^{\infty} j_n! \binom{i_n + j_n}{i_n} x_n^{i_n} \Big) \\ &= \Big(\frac{\mathrm{d}^{j_1}}{\mathrm{d} x_1^{j_1}} \Big(\frac{x_1^{j_1}}{1 - x_1} \Big) \Big) \cdots \Big(\frac{\mathrm{d}^{j_n}}{\mathrm{d} x_n^{j_n}} \Big(\frac{x_n^{j_n}}{1 - x_n} \Big) \Big) \\ &= \frac{\partial^{|J|}}{\partial \boldsymbol{x}^J} \Big(\prod_{k=1}^{n} \frac{x_k^{j_k}}{1 - x_k} \Big), \end{aligned}$$

as desired. ▼

2 **Lemma** *For each* $\mathrm{R} \in (0, 1)$ *there exists* $\mathrm{A}, \lambda \in \mathbb{R}_{>0}$ *such that, for each* $\mathrm{m} \in \mathbb{Z}_{\geq 0}$,

$$\sup \Big\{ \frac{\mathrm{d}^{\mathrm{m}}}{\mathrm{d} \mathrm{x}^{\mathrm{m}}} \Big(\frac{\mathrm{x}^{\mathrm{m}}}{1 - \mathrm{x}} \Big) \Bigm| \mathrm{x} \in [-\mathrm{R}, \mathrm{R}] \Big\} \leq \mathrm{A} \mathrm{m}! \lambda^{-\mathrm{m}}.$$

Proof We first claim that

$$\sum_{j=0}^{\infty} \binom{m + j}{j} x^j = \frac{1}{(1 - x)^{m+1}} \tag{1.9}$$

for $x \in (-1, 1)$ and $m \in \mathbb{Z}_{\geq 0}$. Indeed, by [Rudin 1976, Theorem 3.26] we have

$$\sum_{j=0}^{\infty} x^j = \frac{1}{1 - x},$$

and convergence is uniform and absolute on $[-R, R]$ for $R \in (0, 1)$. Differentiation m-times of both sides with respect to x then gives (1.9).

By Lemma 1 from the proof of Theorem 1.1.8 we have

$$\frac{\mathrm{d}^m}{\mathrm{d}x^m}\Big(\frac{x^m}{1-x}\Big)=\sum_{j=0}^{\infty}\frac{(m+j)!}{j!}x^j$$

for $x\in(-1,1)$. If $x\in[-R,R]$ then

$$\begin{aligned}\Big(\frac{\mathrm{d}^m}{\mathrm{d}x^m}\Big(\frac{x^m}{1-x}\Big)\Big)\frac{(1-R)^m}{m!}&=(1-R)^m\sum_{j=0}^{\infty}\frac{(m+j)!}{m!j!}x^j=(1-R)^m\sum_{j=0}^{\infty}\binom{m+j}{j}x^j\\&=\frac{(1-R)^m}{(1-x)^{m+1}}=\Big(\frac{1-R}{1-x}\Big)^m\frac{1}{1-x}\le\frac{1}{1-R}.\end{aligned}$$

That is to say,

$$\frac{\mathrm{d}^m}{\mathrm{d}x^m}\Big(\frac{x^m}{1-x}\Big)\le\frac{1}{1-R}m!(1-R)^{-m},$$

and so the lemma follows with $A=\frac{1}{1-R}$ and $\lambda=1-R$. ▼

Now, for $\boldsymbol{x}$ in a neighbourhood $\mathcal{V}$ of $\boldsymbol{x}_0$ we have

$$f(\boldsymbol{x})=\sum_{I\in\mathbb{Z}_{\ge0}^n}\frac{1}{I!}\boldsymbol{D}^I f(\boldsymbol{x}_0)(\boldsymbol{x}-\boldsymbol{x}_0)^I.$$

Let us abbreviate $\alpha(I)=\frac{1}{I!}\boldsymbol{D}^I f(\boldsymbol{x}_0)$. By Corollary 1.1.11 there exists $C',\sigma\in\mathbb{R}_{>0}$ such that

$$|\alpha(I)|\le C'\sigma^{-|I|},\qquad I\in\mathbb{Z}_{\ge0}^n.$$

By Corollary 1.1.12 and following the computations from the proof of Theorem 1.1.16, we can write

$$\frac{1}{J!}\boldsymbol{D}^J f(\boldsymbol{x})=\sum_{I\in\mathbb{Z}_{\ge0}^n}\binom{i_1+j_1}{j_1}\cdots\binom{i_n+j_n}{j_n}\alpha(I+J)(\boldsymbol{x}-\boldsymbol{x}_0)^I \tag{1.10}$$

for $J\in\mathbb{Z}_{\ge0}^n$ and for $\boldsymbol{x}$ in a neighbourhood of $\boldsymbol{x}_0$. Therefore, there exists $\rho\in(0,\sigma)$ sufficiently small that, if $\boldsymbol{x}\in\mathbb{R}^n$ satisfies $|x_j-x_{0j}|<\rho$, $j\in\{1,\dots,n\}$, (1.10) holds. Let C_J be the partial derivative

$$\frac{\partial^{|J|}}{\partial\boldsymbol{x}^J}\Big(\prod_{k=1}^{n}\frac{x_k^{j_k}}{1-x_k}\Big)$$

evaluated at $\boldsymbol{x}=(\frac{\rho}{\sigma},\dots,\frac{\rho}{\sigma})$. Let $R\in(0,1)$ satisfy $R>\frac{\rho}{\sigma}$. By the second lemma above there exists $A,\lambda\in\mathbb{R}_{>0}$ such that, for each $k\in\{1,\dots,n\}$ and each $x_k\in[-R,R]$, we have

$$\frac{\mathrm{d}^{j_k}}{\mathrm{d}x_k^{j_k}}\Big(\frac{x_k^{j_k}}{1-x_k}\Big)\le Aj_k!\lambda^{-j_k}$$

It follows that

$$\frac{\partial^{|J|}}{\partial\boldsymbol{x}^J}\Big(\prod_{k=1}^{n}\frac{x_k^{j_k}}{1-x_k}\Big)\le A^nJ!\lambda^{-|J|}$$

whenever $\boldsymbol{x} = (x_1, \dots, x_n)$ satisfies $|x_j| < R$, $j \in \{1, \dots, n\}$. In particular, $C_J \le A^n J! \lambda^{-|J|}$.

Then, for such $\boldsymbol{x}$ such that $|x_j - x_{0j}| < \rho$, $j \in \{1, \dots, n\}$, we have

$$\begin{aligned}
|\boldsymbol{D}^J f(\boldsymbol{x})| &\le \sum_{I \in \mathbb{Z}^n_{\ge 0}} J! \binom{i_1 + j_1}{j_1} \cdots \binom{i_n + j_n}{j_n} |\alpha(I + J)| |(\boldsymbol{x} - \boldsymbol{x}_0)^I| \\
&\le \sum_{I \in \mathbb{Z}^n_{\ge 0}} J! \binom{i_1 + j_1}{j_1} \cdots \binom{i_n + j_n}{j_n} C' \sigma^{-|J|} \Big(\frac{\rho}{\sigma}\Big)^{|I|} \\
&\le C' \sigma^{-|J|} C_J \le C' A^n J! (\lambda + \sigma)^{-|J|},
\end{aligned}$$

using the lemmata above. Thus the second condition in the statement of the theorem holds with $C = C' A^n$ and $r = \lambda + \sigma$.

For the converse part of the proof, we can show as in the holomorphic case that the Taylor series for f converges absolutely in a neighbourhood of $\boldsymbol{x}_0$. We must show that we have convergence to f. Let $k \in \mathbb{Z}_{>0}$, let $\boldsymbol{x} \in \mathsf{B}^n(\rho, \boldsymbol{x}_0)$, and recall from Taylor's Theorem [Abraham, Marsden, and Ratiu 1988, Theorem 2.4.15] that there exists

$$\boldsymbol{z} \in \{(1 - t)\boldsymbol{x}_0 + t\boldsymbol{x} \mid t \in [0, 1]\}$$

such that

$$f(\boldsymbol{x}) = \sum_{\substack{I \in \mathbb{Z}^n_{\ge 0} \\ |I| \le k}} \frac{1}{I!} \boldsymbol{D}^I f(\boldsymbol{x}_0)(\boldsymbol{x} - \boldsymbol{x}_0)^I + \sum_{\substack{I \in \mathbb{Z}^n_{\ge 0} \\ |I| = k+1}} \frac{1}{I!} \boldsymbol{D}^I f(\boldsymbol{z})(\boldsymbol{x} - \boldsymbol{x}_0)^I.$$

Thus

$$\begin{aligned}
\Big| f(\boldsymbol{x}) - \sum_{\substack{I \in \mathbb{Z}^n_{\ge 0} \\ |I| \le k}} \frac{1}{I!} \boldsymbol{D}^I f(\boldsymbol{x}_0)(\boldsymbol{x} - \boldsymbol{x}_0)^I \Big| &\le \sum_{\substack{I \in \mathbb{Z}^n_{\ge 0} \\ |I| = k+1}} \frac{1}{I!} |\boldsymbol{D}^I f(\boldsymbol{z})| |(\boldsymbol{x} - \boldsymbol{x}_0)^I| \\
&\le \sum_{\substack{I \in \mathbb{Z}^n_{\ge 0} \\ |I| = k+1}} \Big(\frac{\rho}{r}\Big)^{|I|} = \binom{n - k}{k} \Big(\frac{\rho}{r}\Big)^{k+1}.
\end{aligned}$$

As we saw above, the series

$$\sum_{k=0}^{\infty} \binom{n - k - 1}{k - 1} \Big(\frac{\rho}{r}\Big)^k$$

converges, and so

$$\lim_{k \to \infty} \binom{n - k}{k} \Big(\frac{\rho}{r}\Big)^{k+1} = 0,$$

giving

$$\lim_{k \to \infty} \Big| f(\boldsymbol{x}) - \sum_{\substack{I \in \mathbb{Z}^n_{\ge 0} \\ |I| \le k}} \frac{1}{I!} \boldsymbol{D}^I f(\boldsymbol{x}_0)(\boldsymbol{x} - \boldsymbol{x}_0)^I \Big| = 0,$$

and so f is equal to its Taylor series about $\boldsymbol{x}_0$ in a neighbourhood of $\boldsymbol{x}_0$. ■

Note that while holomorphic and real analytic functions satisfy the same estimate for derivatives as in the theorem, the nature of the constant C in the estimate is different in the holomorphic and real analytic case. Indeed, from Corollary 1.1.24 below we can take

$$C = \sup\{\|f(z)\| \mid z \in \mathsf{D}^n(\hat{r}, z_0)\}.$$

Thus the derivative bounds are expressed in terms of the values of the function. This is not the case for real analytic functions. This will have consequences, for example, when we place suitable topologies on spaces of holomorphic objects and real analytic objects. Such topologies are typically much more easy to understand in the holomorphic case.

The following result is one that is often useful.

1.1.18 Theorem (Identity Theorem in $\mathbb{F}^n$) *If $\mathcal{U} \subseteq \mathbb{F}^n$ is a connected open set, if $\mathcal{V} \subseteq \mathcal{U}$ is a nonempty open set and if $f, g\colon \mathcal{U} \to \mathbb{F}$ are holomorphic or real analytic and satisfy $f|\mathcal{V} = g|\mathcal{V}$, then $f = g$.*

Proof It suffices to show that if $f(x) = 0$ for $x \in \mathcal{V}$ then f vanishes on $\mathcal{U}$. Let

$$\mathcal{O} = \operatorname{int}(\{x \in \mathcal{U} \mid f(x) = 0\}).$$

Since $\mathcal{O}$ is obviously open, since $\mathcal{U}$ is connected it suffices to show that it is also closed to show that $\mathcal{O} = \mathcal{U}$. Thus let $x_0 \in \mathrm{cl}_{\mathcal{U}}(\mathcal{O})$. Note that for all $r \in \mathbb{R}_{>0}$ sufficiently small that $\mathsf{B}^n(r, x_0) \subseteq \mathcal{U}$ we have $\mathsf{B}^n(\frac{r}{2}, x_0) \cap \mathcal{O} \neq \emptyset$. Thus let $x \in \mathsf{B}^n(\frac{r}{2}, x_0)) \cap \mathcal{O}$ and note that $x_0 \in \mathsf{B}^n(\frac{r}{2}, x) \cap \mathcal{O} \subseteq \mathcal{U}$. Now additionally require that r be sufficiently small that the Taylor series for f at x converges in $\mathsf{B}^n(\frac{r}{2}, x)$. Since $x \in \mathcal{O}$, this means that this Taylor series is zero in a neighbourhood of x and the Taylor coefficients must all vanish. Consequently, f vanishes on $\mathsf{B}^n(\frac{r}{2}, x)$ and, in particular, in a neighbourhood of x_0. Thus $x_0 \in \mathcal{O}$ and so $\mathcal{O}$ is closed. ■

In the one-dimensional case, a stronger conclusion can be drawn.

1.1.19 Proposition (A strong Identity Theorem in one dimension) *If $\mathcal{U} \subseteq \mathbb{F}$ is a connected open set, if $S \subseteq \mathcal{U}$ is a set with an accumulation point, and if $f\colon \mathcal{U} \to \mathbb{F}$ is holomorphic or real analytic function for which $f(x) = 0$ for every $x \in S$, then $f(x) = 0$ for every $x \in \mathcal{U}$.*

Proof Let $x_0 \in \mathcal{U}$ be an accumulation point for S. Then there exists a sequence of distinct points $(x_j)_{j\in\mathbb{Z}_{>0}}$ in S converging to x_0. Suppose, without loss of generality (by translating $\mathcal{U}$ if necessary), that $x_0 = 0$ for simplicity. Note that the sequence $(\frac{x_j}{|x_j|})_{j\in\mathbb{Z}_{>0}}$ in $\overline{\mathsf{D}}^1(1, 0)$ must contain a convergent subsequence by the Bolzano–Weierstrass Theorem [Abraham, Marsden, and Ratiu 1988, Theorem 1.5.4] and compactness of $\overline{\mathsf{D}}^1(1, 0)$. Thus we can suppose, by passing to a subsequence if necessary, that all points in the sequence have real and imaginary parts of the same sign. Let us also suppose, without loss of generality by passing to a subsequence if necessary, that the sequence $(|y_j|)_{j\in\mathbb{Z}_{>0}}$ is monotonically decreasing.

We claim that, for every $r \in \mathbb{Z}_{>0}$, (1) $\boldsymbol{D}^r(f \circ \phi^{-1})(0) = 0$ and (2) there exists a sequence $(y_{r,j})_{j\in\mathbb{Z}_{>0}}$ in $\mathsf{B}^1(R, 0)$ converging to 0 such that $\boldsymbol{D}^r(f \circ \phi^{-1})(y_{r,j}) = 0$. We prove this by induction on r. For $r = 0$, by continuity of f we have

$$f \circ \phi^{-1}(0) = \lim_{j\to\infty} f \circ \phi^{-1}(y_j) = 0,$$

and so our claim holds with $y_{0,j} = y_j$, $j \in \mathbb{Z}_{>0}$. Now suppose that, for $r \in \{1, \dots, k\}$, we have (1) $\boldsymbol{D}^r(f\phi^{-1})(0) = 0$ and (2) there exists a sequence $(y_{r,j})_{j\in\mathbb{Z}_{>0}}$ in $\mathsf{B}^n(R, 0)$ converging to 0 such that $\boldsymbol{D}^r(f \circ \phi^{-1})(y_{r,j}) = 0$. Let $j \in \mathbb{Z}_{>0}$. By the Mean Value Theorem [Abraham, Marsden, and Ratiu 1988, Proposition 2.4.8], there exists

$$y_{k+1,j} \in \{(1-t)y_{k,j} + ty_{k,j+1} \mid t \in [0,1]\}$$

such that $\boldsymbol{D}^{k+1}(f \circ \phi^{-1})(y_{k+1,j}) = 0$. Let us show that $\lim_{j\to\infty} y_{k+1,j} = 0$. Let $t_j \in [0,1]$ be such that

$$y_{k+1,j} = (1-t_j)y_{k,j} + t_j y_{k,j+1}.$$

Then, since

$$|y_{k+1,j}| \le (1-t_j)|y_{k,j}| + t_j|y_{k,j+1}|,$$

we indeed have $\lim_{j\to\infty} y_{k+1,j} = 0$. Continuity of $\boldsymbol{D}^{k+1}(f \circ \phi^{-1})$ ensures that

$$\boldsymbol{D}^{k+1}(f \circ \phi^{-1})(0) = \lim_{j\to\infty} \boldsymbol{D}^{k+1}(f \circ \phi^{-1})(y_{k+1,j}) = 0,$$

giving our claim by induction.

Since $f \circ \phi^{-1}$ is analytic and all of its derivatives vanish at 0, there is a neighbourhood $\mathcal{U}'_0$ of 0 in $\mathsf{B}^n(R, 0)$ such that $f \circ \phi^{-1}$ vanishes identically on $\mathcal{V}'$. The result now follows from the preceding theorem. ■

1.1.7 Some particular properties of holomorphic functions

In this section we provide a few important distinction between the cases $\mathbb{F} = \mathbb{C}$ and $\mathbb{F} = \mathbb{R}$. The first crucial, and perhaps surprising, difference concerns so-called separate holomorphicity.

1.1.20 Definition (Separately holomorphic) Let $\mathcal{U} \subseteq \mathbb{C}^n$ be open and, for $j \in \{1, \dots, n\}$, define the open set

$$\mathcal{U}_j(\hat{z}_1, \dots, \hat{z}_{j-1}, \hat{z}_{j+1}, \hat{z}_n) = \{z \in \mathbb{C} \mid (\hat{z}_1, \dots, \hat{z}_{j-1}, z, z_{j+1}, \dots, \hat{z}_n) \in \mathcal{U}\},$$

provided that this set is nonempty. A function $f: \mathcal{U} \to \mathbb{C}$ is ***separately holomorphic*** if the function

$$z_j \mapsto f(\hat{z}_1, \dots, \hat{z}_{j-1}, z_j, \hat{z}_{j+1}, \dots, \hat{z}_n)$$

on $\mathcal{U}_j(\hat{z}_1, \dots, \hat{z}_{j-1}, \hat{z}_{j+1}, \hat{z}_n)$ is holomorphic (in the usual single variable sense) for each $j \in \{1, \dots, n\}$ and each $\hat{z}_1, \dots, \hat{z}_{j-1}, \hat{z}_{j+1}, \dots, \hat{z}_n \in \mathbb{C}$ for which $\mathcal{U}_j(\hat{z}_1, \dots, \hat{z}_{j-1}, \hat{z}_{j+1}, \hat{z}_n)$ is nonempty. •

The main theorem is that separate holomorphicity implies holomorphicity (which for now we know as $\mathbb{C}$-differentiability), with no additional hypotheses [Hartogs 1906].

1.1.21 Theorem (Hartogs' Theorem) *If $\mathcal{U} \subseteq \mathbb{C}^n$ is open and if $f\colon \mathcal{U} \to \mathbb{C}$ is a separately holomorphic, then f is holomorphic on $\mathcal{U}$.*

Proof See [Krantz 1992, §2.4]. ■

It is not the case that a separately real analytic function is real analytic, and a counterexample is provided by the function $f\colon \mathbb{R}^2 \to \mathbb{R}$ given by

$$f(x_1, x_2) = \begin{cases} \frac{x_1 x_2}{x_1^2 + x_2^2}, & (x_1, x_2) \neq (0, 0), \\ 0, & (x_1, x_2) = (0, 0). \end{cases}$$

A discussion of this may be found in [Krantz and Parks 2002, §4.3].

A nice consequence of Hartogs' Theorem is that it permits the use of single variable methods in some several variable cases. One example of this is the extension to several variables of the Cauchy–Riemann equations. We define

$$\frac{\partial}{\partial z_j} = \frac{1}{2}\Big(\frac{\partial}{\partial x_j} - \mathrm{i}\frac{\partial}{\partial y_j}\Big), \quad \frac{\partial}{\partial \bar{z}_j} = \frac{1}{2}\Big(\frac{\partial}{\partial x_j} + \mathrm{i}\frac{\partial}{\partial y_j}\Big), \qquad j \in \{1, \dots, n\}, \tag{1.11}$$

and so, if $f\colon \mathcal{U} \to \mathbb{C}$ is differentiable in the real variable sense, we can define $\frac{\partial}{\partial z} f$ and $\frac{\partial}{\partial \bar{z}_j} f$. Moreover, we have the following result.

1.1.22 Corollary (Cauchy–Riemann equations) *If $\mathcal{U} \subseteq \mathbb{C}^n$ is open and if $f\colon \mathcal{U} \to \mathbb{C}$ is continuously differentiable in the real variable sense, then f is holomorphic if and only if $\frac{\partial}{\partial \bar{z}_j} f(\mathbf{z}) = 0$ for every $\mathbf{z} \in \mathcal{U}$ and every $j \in \{1, \dots, n\}$.*

If we decompose z and f into their real and imaginary parts as $\boldsymbol{z} = \boldsymbol{x} + \mathrm{i}\boldsymbol{y}$ and $f(z) = g(z) + \mathrm{i}h(z)$, then this reproduces the familiar ***Cauchy–Riemann equations***:

$$\frac{\partial g}{\partial x_j} = \frac{\partial h}{\partial y_j}, \quad \frac{\partial g}{\partial y_j} = -\frac{\partial h}{\partial x_j}, \qquad j \in \{1, \dots, n\}.$$

Another theorem very clearly particular to the complex case is the following.

1.1.23 Theorem ($\mathbb{C}$-differentiable functions are infinitely $\mathbb{C}$-differentiable) *If $\mathcal{U} \subseteq \mathbb{C}^n$ is open and if $f\colon \mathcal{U} \to \mathbb{C}$ is $\mathbb{C}$-differentiable, then f is infinitely $\mathbb{C}$-differentiable.*

Proof For $z_0 \in \mathcal{U}$ we let $\boldsymbol{r} \in \mathbb{R}^n_{>0}$ be sufficiently small that $\overline{\mathsf{D}}^n(\boldsymbol{r}, z_0) \subseteq \mathcal{U}$. Then a repeated application of the Cauchy integral formula in one-dimension and an application of Fubini's Theorem gives

$$\begin{aligned} f(z) &= \Big(\frac{1}{2\pi\mathrm{i}}\Big)^n \int_{|\zeta_1 - z_{01}| = r_1} \frac{\mathrm{d}\zeta_1}{\zeta_1 - z_1} \cdots \int_{|\zeta_n - z_{0n}| = r_n} \frac{f(\zeta)\mathrm{d}\zeta_n}{\zeta_n - z_n} \\ &= \Big(\frac{1}{2\pi\mathrm{i}}\Big)^n \int_{\mathrm{bd}\,\mathsf{D}^n(\boldsymbol{r}, z_0)} \frac{f(\zeta)\mathrm{d}\zeta_1 \cdots \mathrm{d}\zeta_n}{(\zeta_1 - z_1) \cdots (\zeta_n - z_n)} \end{aligned} \tag{1.12}$$

for every $z \in \mathsf{D}^n(r, z_0)$. By the Dominated Convergence Theorem we can then differentiate under the integral sign to get

$$D^I f(z) = \frac{I!}{(2\pi \mathrm{i})^n} \int_{\mathrm{bd}\,\mathsf{D}^n(r,z_0)} \frac{f(\zeta)\mathrm{d}\zeta_1 \cdots \mathrm{d}\zeta_n}{(\zeta_1 - z_1)^{I_1+1} \cdots (\zeta_n - z_n)^{I_n+1}}$$

for all $z \in \mathsf{D}^n(r, z_0)$. Thus we conclude that $\mathbb{C}$-differentiable functions are infinitely $\mathbb{C}$-differentiable. ■

The following corollary, also particular only to the complex case, is useful.

1.1.24 Corollary (Derivative estimates for $\mathbb{C}$-differentiable functions) *If $\mathcal{U} \subseteq \mathbb{C}^n$ is open and if $f: \mathcal{U} \to \mathbb{C}$ is holomorphic, then*

$$\|\mathbf{D}^I f(\mathbf{z}_0)\| \leq \frac{I!}{r_1^{I_1} \cdots r_n^{I_n}} \sup\{|f(\zeta)| \mid \zeta \in \mathsf{D}^n(\mathbf{r}, \mathbf{z}_0)\},$$

for any $\mathbf{z}_0 \in \mathcal{U}$, and where $\mathbf{r} \in \mathbb{R}^n_{>0}$ is such that $\mathsf{D}^n(\mathbf{r}, \mathbf{z}_0) \subseteq \mathcal{U}$.

Proof Denoting

$$M = \sup\{|f(\zeta)| \mid \zeta \in \mathsf{D}^n(r, z_0)\}$$

and using the formula (1.12) with $z = z_0$, along with the fact that $\zeta_j = z_{0j} + r_j \mathrm{e}^{\mathrm{i}\theta}$, $\theta \in [0, 2\pi]$, on the contours along which integration is performed, we have

$$\begin{aligned} \|D^I f(z_0)\| &\leq \frac{I!}{(2\pi)^n} \frac{M}{r_1^{I_1+1} \cdots r_n^{I_n+1}} \int_{|\zeta_1 - z_{01}| = r_1} \mathrm{d}\zeta_1 \cdots \int_{|\zeta_n - z_{0n}| = r_n} \mathrm{d}\zeta_n \\ &= \frac{I!}{(2\pi)^n} \frac{M}{r_1^{I_1+1} \cdots r_n^{I_n+1}} (2\pi r_1) \cdots (2\pi r_n), \end{aligned}$$

which is the desired estimate. ■

As a result of the discussion in this section and in Section 1.1.7, we have the following result.

1.1.25 Theorem (Characterisation of holomorphic functions) *If $\mathcal{U} \subseteq \mathbb{C}^n$ is open and if $f: \mathcal{U} \to \mathbb{C}^m$, then the following statements are equivalent:*

(i) *f is holomorphic;*

(ii) *f is separately holomorphic;*

(iii) *f is $\mathbb{C}$-differentiable;*

(iv) *f is continuously differentiable in the real variable sense and satisfies the Cauchy–Riemann equations.*

As a result of this, we can now dispense with some of the cumbersome terminology we have been using. For example, we do not need the notion of "$\mathbb{C}$-class C^k." Thus, we will dispense with this, and revert to the standard notion of simply saying "holomorphic" in the complex case (since this covers everything) and "class C^k" for $k \in \mathbb{Z}_{>0} \cup \{\infty, \omega\}$ in the real case.

Another useful property of holomorphic functions mirroring what holds in the single variable case is the Maximum Modulus Principle.

1.1.26 Theorem (Maximum Modulus Principle in $\mathbb{C}^n$) *If $\mathcal{U} \subseteq \mathbb{C}^n$ is a connected open set, if $f \in C^{\mathrm{hol}}(\mathcal{U})$, and if there exists $\mathbf{z}_0 \in \mathcal{U}$ such that $|f(\mathbf{z})| \leq |f(\mathbf{z}_0)|$ for every $\mathbf{z} \in \mathcal{U}$, then f is constant on $\mathcal{U}$.*

Proof Let $r \in \mathbb{R}_{>0}$ be such that $\overline{\mathsf{D}}^n(\hat{r}, z_0) \subseteq \mathcal{U}$ and calculate, using the variable substitution $\zeta_j = z_{0j} + re^{i\theta_j}$, $j \in \{1, \dots, n\}$,

$$\begin{aligned} f(z_0) &= \frac{1}{(2\pi i)^n} \int_{\mathrm{bd}(\overline{\mathsf{D}}^n(\hat{r},z_0))} \frac{f(\zeta)}{(\zeta_1 - z_{01}) \cdots (\zeta_n - z_{0n})} \mathrm{d}\zeta_1 \cdots \mathrm{d}\zeta_n \\ &= \frac{1}{(2\pi)^n} \int_{\mathrm{bd}(\overline{\mathsf{D}}^n(\hat{r},z_0))} f(\boldsymbol{z_0} + (re^{i\theta_1}, \dots, re^{i\theta_n}))\mathrm{d}\theta_1 \cdots \mathrm{d}\theta_n. \end{aligned}$$

Therefore,

$$\begin{aligned} |f(z_0)| &\leq \frac{1}{(2\pi)^n} \int_{\mathrm{bd}(\overline{\mathsf{D}}^n(\hat{r},z_0))} |f(\boldsymbol{z_0} + (re^{i\theta_1}, \dots, re^{i\theta_n}))|\mathrm{d}\theta_1 \cdots \mathrm{d}\theta_n \\ &\leq \frac{1}{(2\pi)^n} \int_{\mathrm{bd}(\overline{\mathsf{D}}^n(\hat{r},z_0))} |f(z_0)|\mathrm{d}\theta_1 \cdots \mathrm{d}\theta_n = |f(z_0)|, \end{aligned}$$

using the fact that z_0 is a maximum for f. Thus we must have

$$\int_{\mathrm{bd}(\overline{\mathsf{D}}^n(\hat{r},z_0))} (|f(\boldsymbol{z_0} + (re^{i\theta_1}, \dots, re^{i\theta_n}))| - |f(x_0)|)\mathrm{d}\theta_1 \cdots \mathrm{d}\theta_n = 0,$$

implying that

$$|f(\boldsymbol{z_0} + (re^{i\theta_1}, \dots, re^{i\theta_n}))| - |f(x_0)| = 0$$

for all $\theta_1, \dots, \theta_n \in [0, 2\pi)$ since the integrand is continuous and nonnegative. Thus f has constant modulus in $\overline{\mathsf{D}}^n(\hat{r}, z_0)$. Let us write

$$f(z) = e^{i\theta(z)}|f(z_0)|$$

for a differentiable function $\theta \colon \overline{\mathsf{D}}^n(\hat{r}, z_0) \to \mathbb{R}$. We then have

$$0 = \frac{\partial f}{\partial \bar{z}_j}(z) = i f(z) \frac{\partial \theta}{\partial \bar{z}_j}(z)$$

for all $j \in \{1, \dots, n\}$ and $z \in \mathsf{D}^n(\hat{r}, z_0)$. Thus θ is a real-valued holomorphic function on $\mathsf{D}^n(\hat{r}, z_0)$. Thus, using the Cauchy–Riemann equations,

$$\frac{\partial \theta}{\partial x_j}(z) = -i \frac{\partial \theta}{\partial y_j}(z)$$

for all $j \in \{1, \dots, n\}$ and $z \in \mathsf{D}^n(\hat{r}, z_0)$. However, since both partial derivatives are real functions, they must both be zero, and so θ must be constant. Thus $f(z) = f(z_0)$ for all $z \in \mathsf{D}^n(\hat{r}, z_0)$, and so f is constant by the Identity Theorem, Theorem 1.1.18. ■

1.2 Holomorphic and real analytic multivariable calculus

Now that we know what holomorphic and real analytic functions are, and what are some of their properties, we can turn to the calculus of holomorphic and real analytic functions. We describe here the bare basics of this theory, enough so that we can do holomorphic and real analytic differential geometry.

1.2.1 Holomorphicity, real analyticity, and operations on functions

Let us verify that holomorphicity and real analyticity is respected by the standard ring operations for $\mathbb{F}$-valued functions on a set. The proofs we give here are real variable in nature, meaning that they are direct proofs using Taylor series. It is usual, when working only with holomorphic things, to give proofs using complex variable methods.

1.2.1 Proposition (Holomorphicity, real analyticity, and algebraic operations) *If $\mathcal{U} \subseteq \mathbb{F}^n$ is open and if $f, g\colon \mathcal{U} \to \mathbb{F}$ are holomorphic (resp. real analytic) then $f+g$ and fg are holomorphic (resp. real analytic). If, moreover, $g(\mathbf{x}) \neq 0$ for all $\mathbf{x} \in \mathcal{U}$, then $\frac{f}{g}$ is holomorphic (resp. real analytic).*

Proof Let us first prove that $f+g$ and fg are holomorphic (resp. real analytic). Let $x_0 \in \mathcal{U}$ and let $r \in \mathbb{R}_{>0}$ be such that, for $x \in \mathsf{B}^n(r, x_0)$, we have

$$f(x) = \sum_{I \in \mathbb{Z}^n_{\geq 0}} \alpha_f(x_0)(I)(x - x_0)^I, \quad g(x) = \sum_{I \in \mathbb{Z}^n_{\geq 0}} \alpha_g(x_0)(I)(x - x_0)^I,$$

with the convergence being uniform and absolute on $\mathsf{B}^n(r, x_0)$. Absolute convergence implies that for any bijection $\phi\colon \mathbb{Z}_{\geq 0} \to \mathbb{Z}^n_{\geq 0}$ we have

$$f(x) = \sum_{j=0}^{\infty} \alpha_f(x_0)(\phi(j))(x - x_0)^{\phi(j)}, \quad g(x) = \sum_{j=0}^{\infty} \alpha_g(x_0)(\phi(j))(x - x_0)^{\phi(j)}$$

The standard results on sums and products [Rudin 1976, Theorem 3.4] of series now apply (noting that convergence is absolute) to show that

$$f(x) + g(x) = \sum_{j=0}^{\infty} (\alpha_f(x_0)(\phi(j)) + \alpha_g(x_0)(\phi(j)))(x - x_0)^{\phi(j)},$$

$$f(x)g(x) = \sum_{k=0}^{\infty} \sum_{j=0}^{k} (\alpha_f(x_0)(\phi(j))\alpha_g(x_0)(\phi(k-j)))(x - x_0)^{\phi(j)+\phi(k-j)}$$

for all $x \in \mathsf{B}^n(r, x_0)$, with convergence being absolute in both series. Absolute convergence then implies that we can "de-rearrange" the series to get

$$f(x) + g(x) = \sum_{I \in \mathbb{Z}^n_{\geq 0}} (\alpha_f(x_0)(I) + \alpha_g(x_0(I))(x - x_0)^I,$$

$$f(x)g(x) = \sum_{I \in \mathbb{Z}^n_{\geq 0}} \sum_{\substack{I_1, I_2 \in \mathbb{Z}^n_{\geq 0} \\ I_1 + I_2 = I}} \alpha_f(x_0)(I_1)\alpha_g(x_0)(I_2)(x - x_0)^I$$

for $x \in \mathsf{B}^n(r, x_0)$. Thus the power series $\alpha_f(x_0) + \alpha_g(x_0)$ and $\alpha_f(x_0)\alpha_g(x_0)$ converge in a neighbourhood of x_0 to $f + g$ and fg, respectively. In particular, $f + g$ and fg are holomorphic (resp. real analytic).

To show that $\frac{f}{g}$ is holomorphic (resp. real analytic) if g is nonzero on $\mathcal{U}$, we show that $\frac{1}{g}$ is holomorphic (resp. real analytic); that $\frac{f}{g}$ is holomorphic (resp. real analytic) then follows by our conclusion above for multiplication of holomorphic (resp. real analytic) functions. Let $x_0 \in \mathcal{U}$ and let $r \in \mathbb{R}_{>0}$ be such that

$$g(x) = \sum_{I \in \mathbb{Z}^n_{\geq 0}} \alpha_g(\mathbf{0})(x - x_0)^I \tag{1.13}$$

for $x \in \mathsf{B}^n(r, x_0)$, convergence being absolute and uniform in $\mathsf{B}^n(r, x_0)$. Let us abbreviate $\alpha = \alpha_g(x_0)$. Since g is nowhere zero on $\mathcal{U}$ it follows that $\alpha(\mathbf{0}) \neq 0$ and so, by Proposition 1.1.3, α is a unit in $\mathbb{F}[[\xi]]$ with inverse defined by

$$\alpha^{-1}(I) = \frac{1}{\alpha(\mathbf{0})} \sum_{k=0}^{\infty} \Big(1 - \frac{\alpha(I)}{\alpha(\mathbf{0})}\Big)^k.$$

We will show that this is a convergent power series. Let $\epsilon \in \mathbb{R}_{>0}$ be such that

$$x_\epsilon \triangleq (x_{01} + \epsilon, \dots, x_{0n} + \epsilon) \in \mathsf{B}^n(r, x_0).$$

Since the series (1.13) converges at x_ϵ, the terms in the series must be bounded. Thus there exists $C' \in \mathbb{R}_{>0}$ such that, for all $I \in \mathbb{Z}^n_{\geq 0}$,

$$|\alpha(I)(x_\epsilon - x_0)^I| = |\alpha(I)|\epsilon^{|I|} \leq C'.$$

Therefore, for $I \in \mathbb{Z}^n_{\geq 0}$,

$$\Big|1 - \frac{\alpha(I)}{\alpha(\mathbf{0})}\Big| \leq 1 + \frac{C'}{\alpha(\mathbf{0})}.$$

Let

$$C = \max\Big\{1, 1 + \frac{C'}{\alpha(\mathbf{0})}\Big\}$$

and let $\lambda \in \mathbb{R}_{>0}$ be such that $C\lambda \in (0, 1)$. If $x \in \mathsf{B}^n(r, x_0)$ satisfies $|x_j - x_{0j}| < \lambda$, $j \in \{1, \dots, n\}$, we have

$$\begin{aligned}
\sum_{I \in \mathbb{Z}_{\geq 0}} \sum_{k=0}^{\infty} \Big|1 - \frac{\alpha(I)}{\alpha(\mathbf{0})}\Big|^k |x - x_0|^I &= \sum_{I \in \mathbb{Z}_{\geq 0}} \sum_{k=0}^{|I|} \Big|1 - \frac{\alpha(I)}{\alpha(\mathbf{0})}\Big|^k |x - x_0|^I \\
&\leq \sum_{m=0}^{\infty} \sum_{\substack{I \in \mathbb{Z}_{\geq 0} \\ |I| = m}} \sum_{k=0}^{m} C^k \lambda^m \leq \sum_{m=0}^{\infty} \sum_{\substack{I \in \mathbb{Z}_{\geq 0} \\ |I| = m}} \sum_{k=0}^{m} (C\lambda)^m \\
&= \sum_{m=0}^{\infty} \sum_{\substack{I \in \mathbb{Z}_{\geq 0} \\ |I| = m}} (m+1)(C\lambda)^m = \sum_{m=0}^{\infty} \binom{n - m - 1}{n - 1} (m+1)(C\lambda)^m,
\end{aligned}$$

using Lemma 1.1.1 and the fact, from Lemma 1 in the proof of Proposition 1.1.3, that $(1 - \frac{\alpha(I)}{\alpha(\mathbf{0})})^k(I) = 0$ whenever $|I| \in \{0, 1, \dots, k\}$.. The last series can be shown to converge by the ratio test, and this shows that the series

$$\frac{1}{\alpha(\mathbf{0})} \sum_{I \in \mathbb{Z}_{\geq 0}} \sum_{k=0}^{\infty} \Big(1 - \frac{\alpha(I)}{\alpha(\mathbf{0})}\Big)^k (\boldsymbol{x} - \boldsymbol{x}_0)^I$$

converges in a neighbourhood of $\boldsymbol{x}_0$. ■

Of course, the addition part of the preceding result also applies to $\mathbb{R}^m$-valued real analytic maps. That is, if $\boldsymbol{f}, \boldsymbol{g} \colon \mathcal{U} \to \mathbb{R}^m$ are holomorphic (resp. real analytic), then $\boldsymbol{f} + \boldsymbol{g}$ is holomorphic (resp. real analytic).

Next we consider compositions.

1.2.2 Proposition (Compositions of holomorphic or real analytic maps are holomorphic or real analytic) *Let $\mathcal{U} \subseteq \mathbb{F}^n$ and $\mathcal{V} \subseteq \mathbb{F}^m$ be open, and let $\mathbf{f} \colon \mathcal{U} \to \mathcal{V}$ and $\mathbf{g} \colon \mathcal{V} \to \mathbb{R}^p$ be holomorphic (resp. real analytic). Then $\mathbf{g} \circ \mathbf{f} \colon \mathcal{U} \to \mathbb{F}^p$ is holomorphic (resp. real analytic).*

Proof It suffices to consider the case where $p = 1$, and so we use g rather than $\boldsymbol{g}$. We denote the components of $\boldsymbol{f}$ by $f_1, \dots, f_m \colon \mathcal{U} \to \mathbb{F}$. Let $\boldsymbol{x}_0 \in \mathcal{U}$ and let $\boldsymbol{y}_0 = \boldsymbol{f}(\boldsymbol{x}_0) \in \mathcal{V}$. For $\boldsymbol{x}$ in a neighbourhood $\mathcal{U}' \subseteq \mathcal{U}$ of $\boldsymbol{x}_0$ and for $k \in \{1, \dots, m\}$ we write

$$f_k(\boldsymbol{x}) = \sum_{I \in \mathbb{Z}^n_{\geq 0}} \alpha_k(I)(\boldsymbol{x} - \boldsymbol{x}_0)^I \tag{1.14}$$

and for $\boldsymbol{y}$ in a neighbourhood $\mathcal{V}' \subseteq \mathcal{V}$ of $\boldsymbol{y}_0$ we write

$$g(\boldsymbol{y}) = \sum_{J \in \mathbb{Z}^m_{\geq 0}} \beta(J)(\boldsymbol{y} - \boldsymbol{y}_0)^J. \tag{1.15}$$

Following Theorem 1.1.8, let $\bar{\boldsymbol{x}} \in \mathcal{U}'$ be such that the series (1.14) converges absolutely at $\boldsymbol{x} = \bar{\boldsymbol{x}}$ for each $k \in \{1, \dots, m\}$ and such that $\bar{x}_j - x_{0j} \in \mathbb{R}_{>0}$, $j \in \{1, \dots, n\}$. In like fashion, let $\bar{\boldsymbol{y}} \in \mathcal{V}'$ be such that (1.15) converges absolutely at $\boldsymbol{y} = \bar{\boldsymbol{y}}$ and such that $\bar{y}_k - y_{0k} \in \mathbb{R}_{>0}$, $k \in \{1, \dots, m\}$. Thus we have $A, B \in \mathbb{R}_{>0}$ such that

$$\sum_{I \in \mathbb{Z}^n_{\geq 0}} |\alpha_k(I)|(\bar{\boldsymbol{x}} - \boldsymbol{x}_0)^I < A,\ k \in \{1, \dots, m\}, \quad \sum_{J \in \mathbb{Z}^m_{\geq 0}} |\beta(J)|(\bar{\boldsymbol{y}} - \boldsymbol{y}_0)^J < B.$$

Let

$$r = \min\Big\{1, \frac{\bar{y}_1 - y_{01}}{A}, \dots, \frac{\bar{y}_m - y_{0m}}{A}\Big\}.$$

If $\boldsymbol{x} \in \mathcal{U}'$ and $\rho \in (0, 1)$ satisfies $|x_j| < \rho(\bar{x}_j - x_{0j})$, $j \in \{1, \dots, n\}$, then, for $k \in \{1, \dots, m\}$,

$$\sum_{\substack{I \in \mathbb{Z}^n_{\geq 0} \\ |I| \geq 1}} |\alpha_k(I)||\boldsymbol{x} - \boldsymbol{x}_0|^I \leq \sum_{\substack{I \in \mathbb{Z}^n_{\geq 0} \\ |I| \geq 1}} |\alpha_k(I)|\rho^{|I|}(\bar{\boldsymbol{x}} - \boldsymbol{x}_0)^I = \sum_{\substack{I \in \mathbb{Z}^n_{\geq 0} \\ |I| \geq 1}} |\alpha_k(I)|\rho^{|I|}(\bar{\boldsymbol{x}} - \boldsymbol{x}_0)^I \leq \rho A \leq (\bar{y}_k - y_{0k})$$

since $\rho \leq 1$. Therefore, by (1.15),

$$\sum_{J \in \mathbb{Z}^m_{\geq 0}} |\beta(J)| \Bigg(\sum_{\substack{I_1 \in \mathbb{Z}^n_{\geq 0} \\ |I_1| \geq 1}} |\alpha_1(I_1)||\boldsymbol{x} - \boldsymbol{x}_0|^{I_1} \Bigg)^{j_1} \cdots \Bigg(\sum_{\substack{I_n \in \mathbb{Z}^n_{\geq 0} \\ |I_n| \geq 1}} |\alpha_n(I_n)||\boldsymbol{x} - \boldsymbol{x}_0|^{I_n} \Bigg)^{j_k} \leq B.$$

It follows that

$$\sum_{J\in\mathbb{Z}^m_{\geq 0}} \beta(J)\Bigg(\sum_{\substack{I_1\in\mathbb{Z}^n_{\geq 0}\\ |I_1|\geq 1}} \alpha_1(I_1)(\boldsymbol{x}-\boldsymbol{x}_0)^{I_1}\Bigg)^{j_1}\cdots\Bigg(\sum_{\substack{I_n\in\mathbb{Z}^n_{\geq 0}\\ |I_n|\geq 1}} \alpha_n(I_n)(\boldsymbol{x}-\boldsymbol{x}_0)^{I_n}\Bigg)^{j_k}$$

converges absolutely for $\boldsymbol{x}\in\mathcal{U}'$ satisfying $|x_j|<\rho(\bar{x}_j-x_{0j})$, $j\in\{1,\dots,n\}$. Note, however, that since $\alpha_k(\boldsymbol{0})=y_{0k}$, $k\in\{1,\dots,m\}$, this means that the series

$$\sum_{J\in\mathbb{Z}^m_{\geq 0}} \beta(J)\Bigg(\Big(\sum_{I_1\in\mathbb{Z}^n_{\geq 0}} \alpha_1(I_1)(\boldsymbol{x}-\boldsymbol{x}_0)^{I_1}\Big)-y_{01}\Bigg)^{j_1}\cdots\Bigg(\Big(\sum_{I_n\in\mathbb{Z}^n_{\geq 0}} \alpha_n(I_n)(\boldsymbol{x}-\boldsymbol{x}_0)^{I_n}\Big)-y_{0k}\Bigg)^{j_k}$$

converges absolutely for $\boldsymbol{x}\in\mathcal{U}'$ satisfying $|x_j|<\rho(\bar{x}_j-x_{0j})$, $j\in\{1,\dots,n\}$. This last series, however, is precisely $\boldsymbol{g}\circ\boldsymbol{f}(\boldsymbol{x})$. This series is also a power series after a rearrangement, and any rearrangement will not affect convergence, cf. Remark 1.1.7. Thus $\boldsymbol{g}\circ\boldsymbol{f}$ is expressed as a convergent power series in a neighbourhood of $\boldsymbol{x}_0$. ■

1.2.2 The holomorphic and real analytic Inverse Function Theorem

The Inverse Function Theorem lies at the heart of many of the constructions in differential geometry. Thus it is essential for us to have at our disposal the holomorphic and real analytic versions of the Inverse Function Theorem.

We make the following obvious definition.

1.2.3 Definition (Holomorphic or real analytic diffeomorphism) If $\mathcal{U},\mathcal{V}\subseteq\mathbb{F}^n$ are open sets, a map $\boldsymbol{f}\colon\mathcal{U}\to\mathcal{V}$ is a ***holomorphic*** (resp. ***real analytic***) ***diffeomorphism*** if it (i) is a bijection, (ii) is holomorphic (resp. real analytic), and (iii) has a holomorphic (resp. real analytic) inverse. •

In the proof we shall use the so-called "method of majorants" used by Cauchy in the proof of what we now call the Cauchy–Kowaleski Theorem.

1.2.4 Definition (Majorisation of formal power series) Given two formal $\mathbb{F}$-power series

$$\sum_{I\in\mathbb{Z}^n_{\geq 0}} a_I \xi^I, \quad \sum_{I\in\mathbb{Z}^n_{\geq 0}} b_I \xi^I,$$

the former is ***majorised*** by the latter if $|a_I| < |b_I|$ for every $I \in \mathbb{Z}^n_{\geq 0}$. •

By Theorem 1.1.17 it follows that if a formal power series is majorised by a convergent power series, the majorised power series also converges, and so converges to a holomorphic or real analytic function by Theorem 1.1.16. This is a popular means of proving that a given power series gives a holomorphic or real analytic function. The following lemma provides a useful result for these sorts of computations.

1.2.5 Lemma (A useful majorisation) *Let $\mathcal{U} \subseteq \mathbb{F}^n$ be an open set, let $f\colon \mathcal{U} \to \mathbb{F}$ have the property that $|\boldsymbol{D}^I f(\boldsymbol{x})| \leq CI!r^{-|I|}$, and define*

$$\phi_{C,r}(\boldsymbol{x}) = \frac{Cr}{r-(x_1+\cdots+x_n)}$$

for $\boldsymbol{x}$ in a sufficiently small neighbourhood of $\boldsymbol{0}$. Then the Taylor series for f at $\boldsymbol{x}$ is majorised by the Taylor series for $\phi_{C,r}$ at $\boldsymbol{0}$.

Proof Let $\chi\colon \mathbb{F}^n \to \mathbb{F}$ and $\psi_{C,r}\colon (-1,1) \to \mathbb{F}$ be defined by

$$\chi(\boldsymbol{x}) = x_1+\cdots+x_n, \quad \psi_{C,r}(y) = \frac{Cr}{r-y},$$

so that $\phi_{C,r} = \psi_{C,r}\circ\chi$. A straightforward induction on the order of the derivatives gives

$$\boldsymbol{D}^I\phi_{C,r} = CI!r^{-|I|},$$

and the result follows immediately from this. ■

We now have the following theorem.

1.2.6 Theorem (Holomorphic or real analytic Inverse Function Theorem) *Let $\mathcal{U} \subseteq \mathbb{F}^n$ and let $\boldsymbol{f}\colon \mathcal{U} \to \mathbb{F}^n$ be holomorphic (resp. real analytic). If the matrix*

$$\boldsymbol{D}\boldsymbol{f}(\boldsymbol{x}_0) = \begin{bmatrix} \frac{\partial f_1}{\partial x_1}(\boldsymbol{x}_0) & \cdots & \frac{\partial f_1}{\partial x_n}(\boldsymbol{x}_0) \\ \vdots & \ddots & \vdots \\ \frac{\partial f_n}{\partial x_1}(\boldsymbol{x}_0) & \cdots & \frac{\partial f_n}{\partial x_n}(\boldsymbol{x}_0) \end{bmatrix}$$

is invertible for $\boldsymbol{x}_0 \in \mathcal{U}$, then there exists a neighbourhood $\mathcal{U}'$ of $\boldsymbol{x}_0$ such that $\boldsymbol{f}|\mathcal{U}'\colon \mathcal{U}' \to \boldsymbol{f}(\mathcal{U}')$ is a holomorphic (resp. real analytic) diffeomorphism.

Proof Let us begin by making some simplifying assumptions, all of which can be made without loss of generality, which can be easily shown. We first suppose that $\boldsymbol{x}_0 = \boldsymbol{f}(\boldsymbol{x}_0) = \boldsymbol{0}$. By a linear change of coordinates we also suppose that $\boldsymbol{D}\boldsymbol{f}(\boldsymbol{0}) = \boldsymbol{I}_n$.

Note that holomorphic mappings are smooth as real variable mappings. This is because the $\mathbb{C}$-partial derivatives contain in their expression the $\mathbb{R}$-partial derivatives. Therefore, we can use the smooth Inverse Function Theorem [Abraham, Marsden, and Ratiu 1988, Theorem 2.5.2] to assert the existence of a neighbourhood $\mathcal{U}'$ of $\boldsymbol{0}$ such that $\boldsymbol{f}\colon \mathcal{U}' \to \mathcal{V}' = \boldsymbol{f}(\mathcal{U}')$ is a smooth diffeomorphism. We denote by $\boldsymbol{g}\colon \mathcal{V}' \to \mathcal{U}'$ the inverse of $\boldsymbol{f}$.

Let us write

$$\boldsymbol{f}(\boldsymbol{x}) = \sum_{I\in\mathbb{Z}^n_{\geq 0}} \boldsymbol{a}(I)\boldsymbol{x}^I$$

for $\boldsymbol{x}$ in the neighbourhood $\mathcal{U}'$ of $\boldsymbol{0}$ (maybe by shrinking $\mathcal{U}'$). Let us suppose, by Theorem 1.1.17, that

$$|a_j(I)| \leq Cr^{-|I|}, \qquad I \in \mathbb{Z}^n_{\geq 0},\ j \in \{1,\dots,n\}, \tag{1.16}$$

for some appropriate $C, r \in \mathbb{R}_{>0}$. Let us consider the map $\boldsymbol{F}\colon \mathcal{U}' \times \mathcal{V}' \to \mathbb{R}^n$ given by

$$\boldsymbol{F}(\boldsymbol{x},\boldsymbol{y}) = \boldsymbol{f}(\boldsymbol{x}) - \boldsymbol{y}.$$

Note that

$$\boldsymbol{F}(\boldsymbol{g}(\boldsymbol{y}),\boldsymbol{y}) = \boldsymbol{f}\circ\boldsymbol{g}(\boldsymbol{y}) - \boldsymbol{y} = \boldsymbol{0}, \qquad \boldsymbol{y} \in \mathcal{V}'.$$

Given our assumptions on $\boldsymbol{f}$ we have

$$\boldsymbol{F}(\boldsymbol{x},\boldsymbol{y}) = \boldsymbol{x} + \sum_{\substack{I\in\mathbb{Z}^n_{\geq 0}\\ |I|\geq 2}} \boldsymbol{a}(I)\boldsymbol{x}^I - \boldsymbol{y},$$

and let us define

$$\boldsymbol{G}(\boldsymbol{x},\boldsymbol{y}) = \sum_{\substack{I\in\mathbb{Z}^n_{\geq 0}\\ |I|\geq 2}} \boldsymbol{b}(I)\boldsymbol{x}^I + \boldsymbol{y},$$

where $\boldsymbol{b}(I) = -\boldsymbol{a}(I)$, so that $\boldsymbol{g}(\boldsymbol{y}) = \boldsymbol{G}(\boldsymbol{g}(\boldsymbol{y}),\boldsymbol{y})$.

Let us write the Taylor series for $\boldsymbol{g}$ as

$$\sum_{I\in\mathbb{Z}_{\geq 0}} \boldsymbol{c}(I)\boldsymbol{y}^I,$$

understanding that this is a formal series as we do not know yet that it converges. The equality $\boldsymbol{g}(\boldsymbol{y}) = \boldsymbol{G}(\boldsymbol{g}(\boldsymbol{y}),\boldsymbol{y})$, at the level of formal power series, reads

$$\sum_{I\in\mathbb{Z}_{\geq 0}} \boldsymbol{c}(I)\boldsymbol{y}^I = \sum_{\substack{I\in\mathbb{Z}^n_{\geq 0}\\ |I|\geq 2}} \boldsymbol{b}(I)\Bigg(\sum_{I_1\in\mathbb{Z}^n_{\geq 0}} c_1(I_1)\boldsymbol{y}^{I_1}\Bigg)^{i_1}\cdots\Bigg(\sum_{I_n\in\mathbb{Z}^n_{\geq 0}} c_n(I_n)\boldsymbol{y}^{I_n}\Bigg)^{i_n} + \boldsymbol{y}. \tag{1.17}$$

Since $\boldsymbol{g}(\boldsymbol{0}) = \boldsymbol{0}$, $\boldsymbol{c}(\boldsymbol{0}) = \boldsymbol{0}$. Thus (1.17) correctly determines $\boldsymbol{c}(\boldsymbol{0})$. Since the Jacobian of $\boldsymbol{g}$ at $\boldsymbol{0}$ is $\boldsymbol{I}_n$ by the smooth Inverse Function Theorem, $\boldsymbol{c}(\boldsymbol{e}_j) = \boldsymbol{e}_j$ for $j \in \{1,\dots,n\}$. Thus (1.17) determines $\boldsymbol{c}(I)$ for $|I| = 1$. Let $I \in \mathbb{Z}^n_{\geq 0}$ be such that $|I| = m \geq 2$. Then $\boldsymbol{c}(I)$ will be a linear combination of terms of the form

$$\boldsymbol{b}(J)c_1(J_{1,1})\cdots c_1(J_{1,j_1})\cdots c_n(J_{n,1})\cdots c_n(J_{n,j_n}), \tag{1.18}$$

where, for each $k \in \{1,\dots,n\}$, the sum of the kth components of the multi-indices $J_{1,1},\dots,J_{1,j_1},\dots,J_{n,1},\dots,J_{n,j_n}$ is i_k. We claim that, if (1.18) is nonzero, then

$$|J_{1,1}|,\dots,|J_{1,j_1}|,\dots,|J_{n,1}|,\dots,|J_{n,j_n}| < |I|.$$

Suppose that $|J_{l,s}| \geq |I|$ for some $l \in \{1,\dots,n\}$ and some $s \in \{1,\dots,j_l\}$. Since the jth component of any of these multi-indices cannot exceed i_k, this implies that $J_{l,s} = I$. But this implies that the other multi-indices are zero. However, since $\boldsymbol{c}(\boldsymbol{0}) = \boldsymbol{0}$ this implies that the expression (1.18) is zero. Thus one gets a recursive formula for determining the Taylor coefficients $\boldsymbol{c}(I)$ in terms of the Taylor coefficients $\boldsymbol{b}(J)$ and Taylor coefficients $\boldsymbol{c}(J)$ for multi-indices J of lower order than I. Moreover, an examination of (1.17) shows that the coefficients of the linear combination of the terms (1.18) are positive. We shall use this fact later.

Next we find a holomorphic (resp. real analytic) mapping whose Taylor series majorises that for $\boldsymbol{G}$. Recalling (1.16) and [Rudin 1976, Theorem 3.26], note that

$$\phi_j(\boldsymbol{x}) = \sum_{\substack{I\in\mathbb{Z}^n_{\geq 0}\\ |I|\geq 2}} Cr^{-|I|}\boldsymbol{x}^I = \frac{Cr}{r-(x_1+\cdots+x_n)} - C - \frac{C}{r}(x_1+\cdots+x_n)$$

majorises

$$\sum_{\substack{I\in\mathbb{Z}^n_{\geq 0}\\ |I|\geq 2}} b_j(I)\boldsymbol{x}^I$$

for each $j \in \{1,\dots,n\}$. By Lemma 1.2.5, if we define

$$\boldsymbol{H}(\boldsymbol{x},\boldsymbol{y}) = \boldsymbol{\phi}(\boldsymbol{x}) + \boldsymbol{y},$$

then the Taylor series for H_j majorises that for G_j for $j \in \{1,\dots,n\}$. Now define

$$h_j(\boldsymbol{y}) = \frac{r^2 + 2n(n+r)y_j - (2Cn+r)\sum\boldsymbol{y} + r\sqrt{r^2 - 2r\sum\boldsymbol{y} - (4Cn - \sum\boldsymbol{y})(\sum\boldsymbol{y})}}{2n(Cn+r)},$$

where $\sum\boldsymbol{y} = y_1 + \cdots + y_n$. A tedious computation then verifies that $\boldsymbol{h}(\boldsymbol{y}) = \boldsymbol{H}(\boldsymbol{h}(\boldsymbol{y}),\boldsymbol{y})$. It turns out that the Taylor series for $\boldsymbol{h}$ majorises that for $\boldsymbol{g}$.

1 **Lemma** *With the above notation, the Taylor series for* $\mathrm{h_j}$ *at* $\boldsymbol{0}$ *majorises that for* $\mathrm{g_j}$ *at* $\boldsymbol{0}$ *for each* $\mathrm{j} \in \{1,\dots,\mathrm{n}\}$.

Proof We have

$$\boldsymbol{g}(\boldsymbol{y}) = \boldsymbol{G}(\boldsymbol{g}(\boldsymbol{y}),\boldsymbol{y}), \quad \boldsymbol{h}(\boldsymbol{y}) = \boldsymbol{H}(\boldsymbol{h}(\boldsymbol{y}),\boldsymbol{y}).$$

Let us write the Taylor series for $\boldsymbol{h}$ as

$$\sum_{I\in\mathbb{Z}^n_{\geq 0}} \boldsymbol{d}(I)\boldsymbol{y}^I.$$

Then we have

$$\sum_{I\in\mathbb{Z}^n_{\geq 0}} \boldsymbol{d}(I)\boldsymbol{y}^I = \sum_{\substack{I\in\mathbb{Z}^n_{\geq 0}\\ |I|\geq 2}} \frac{\boldsymbol{D}^I\boldsymbol{\phi}(\boldsymbol{0})}{I!}\Big(\sum_{I_1\in\mathbb{Z}^n_{\geq 0}} d_1(I_1)\boldsymbol{y}^{I_1}\Big)^{i_1}\cdots\Big(\sum_{I_n\in\mathbb{Z}^n_{\geq 0}} d_n(I_n)\boldsymbol{y}^{I_n}\Big)^{i_n} + \boldsymbol{y}.$$

As we saw above, this means that $\boldsymbol{d}(I)$ is a positive linear combination of terms of the form

$$\boldsymbol{D}^J\boldsymbol{\phi}(\boldsymbol{0})d_1(J_{1,1})\cdots d_1(J_{1,j_1})\cdots d_n(J_{n,1})\cdots d_n(J_{n,j_n}),$$

where

$$|J_{1,1}|,\dots,|J_{1,j_1}|,\dots,|J_{n,1}|,\dots,|J_{n,j_n}| < |I|.$$

Moreover, this is the same positive linear combination as for the Taylor coefficients for $\boldsymbol{g}$. Since

$$|b_j(I)| < \frac{1}{I!}\boldsymbol{D}^I\phi_j(\boldsymbol{0}), \qquad I\in\mathbb{Z}^N_{\geq 0},\ j\in\{1,\dots,n\},$$

we get the desired conclusion by an inductive argument. ▼

Since $\boldsymbol{h}$ is obviously real analytic in a neighbourhood of $\boldsymbol{0}$, we conclude that $\boldsymbol{g}$ is also analytic in a neighbourhood of $\boldsymbol{0}$. ■

1.2.3 Some consequences of the Inverse Function Theorem

Having the holomorphic and real analytic Inverse Function Theorems at our disposal, we are now in a position to perform some of the more or less standard constructions that follow from it. Gratifyingly, these follow from the Inverse Function Theorem just as they do in the differentiable case.

We first state the holomorphic and real analytic Implicit Function Theorem.

1.2.7 Theorem (Holomorphic or real analytic Implicit Function Theorem) *Let $\mathrm{m}, \mathrm{n} \in \mathbb{Z}_{>0}$, let $\mathcal{U} \times \mathcal{V} \subseteq \mathbb{F}^{\mathrm{n}} \times \mathbb{F}^{\mathrm{m}}$ be open, and let $\mathbf{f}\colon \mathcal{U} \times \mathcal{V} \to \mathbb{F}^{\mathrm{m}}$ be holomorphic (resp. real analytic). Denote a point in $\mathbb{F}^{\mathrm{n}} \times \mathbb{F}^{\mathrm{m}}$ by $(\mathbf{x}, \mathbf{y})$. If the matrix*

$$\mathbf{D}_2\mathbf{f}(\mathbf{x}_0) = \begin{bmatrix} \frac{\partial \mathrm{f}_1}{\partial \mathrm{y}_1}(\mathbf{x}_0, \mathbf{y}_0) & \cdots & \frac{\partial \mathrm{f}_1}{\partial \mathrm{y}_{\mathrm{m}}}(\mathbf{x}_0, \mathbf{y}_0) \\ \vdots & \ddots & \vdots \\ \frac{\partial \mathrm{f}_{\mathrm{m}}}{\partial \mathrm{y}_1}(\mathbf{x}_0, \mathbf{y}_0) & \cdots & \frac{\partial \mathrm{f}_{\mathrm{m}}}{\partial \mathrm{y}_{\mathrm{m}}}(\mathbf{x}_0, \mathbf{y}_0) \end{bmatrix}$$

is invertible for $(\mathbf{x}_0, \mathbf{y}_0) \in \mathcal{U} \times \mathcal{V}$, then there exist

(i) neighbourhoods $\mathcal{U}'$ of $\mathbf{x}_0$ and $\mathcal{W}'$ of $\mathbf{f}(\mathbf{x}_0, \mathbf{y}_0)$, respectively, and

(ii) a holomorphic (resp. real analytic) map $\mathbf{g}\colon \mathcal{U}' \times \mathcal{W}' \to \mathcal{V}$

such that $\mathbf{f}(\mathbf{x}, \mathbf{g}(\mathbf{x}, \mathbf{z})) = \mathbf{z}$ for all $(\mathbf{x}, \mathbf{z}) \in \mathcal{U}' \times \mathcal{W}'$.

Proof Let us define $\boldsymbol{h}\colon \mathcal{U} \times \mathcal{V} \to \mathbb{F}^n \times \mathbb{F}^m$ by $\boldsymbol{h}(\boldsymbol{x}, \boldsymbol{y}) = (\boldsymbol{x}, \boldsymbol{f}(\boldsymbol{x}, \boldsymbol{y}))$. The Jacobian matrix of partial derivatives of $\boldsymbol{h}$ is

$$\boldsymbol{Dh}(\boldsymbol{x}_0, \boldsymbol{y}_0) = \left[\begin{array}{ccc|ccc} 1 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 1 & 0 & \cdots & 0 \\ \hline \frac{\partial f_1}{\partial x_1}(\boldsymbol{x}_0, \boldsymbol{y}_0) & \cdots & \frac{\partial f_1}{\partial x_n}(\boldsymbol{x}_0, \boldsymbol{y}_0) & \frac{\partial f_1}{\partial y_1}(\boldsymbol{x}_0, \boldsymbol{y}_0) & \cdots & \frac{\partial f_1}{\partial y_m}(\boldsymbol{x}_0, \boldsymbol{y}_0) \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \frac{\partial f_m}{\partial x_1}(\boldsymbol{x}_0, \boldsymbol{y}_0) & \cdots & \frac{\partial f_m}{\partial x_n}(\boldsymbol{x}_0, \boldsymbol{y}_0) & \frac{\partial f_m}{\partial y_1}(\boldsymbol{x}_0, \boldsymbol{y}_0) & \cdots & \frac{\partial f_m}{\partial y_m}(\boldsymbol{x}_0, \boldsymbol{y}_0) \end{array}\right].$$

By hypothesis, the lower right block is invertible. Since the upper left block is invertible and the upper right block is zero, it follows that the matrix is invertible. Therefore, by the Inverse Function Theorem there exists a neighbourhood $\mathcal{U}' \times \mathcal{V}'$ of $(\boldsymbol{x}_0, \boldsymbol{y}_0)$ such that $\boldsymbol{h}|\mathcal{U}' \times \mathcal{V}'$ is a holomorphic (resp. real analytic) diffeomorphism. Given the form of $\boldsymbol{h}$, $\boldsymbol{h}(\mathcal{U}' \times \mathcal{V}) = \mathcal{U}' \times \mathcal{W}'$ and the inverse of $\boldsymbol{h}|\mathcal{U}' \times \mathcal{V}'$ has the form

$$(\boldsymbol{x}, \boldsymbol{z}) \mapsto (\boldsymbol{x}, \boldsymbol{g}(\boldsymbol{x}, \boldsymbol{z})) \tag{1.19}$$

for some holomorphic (resp. real analytic) $\boldsymbol{g}\colon \mathcal{U}' \times \mathcal{W}' \to \mathcal{V}' \subseteq \mathcal{V}$. One can easily verify that $\boldsymbol{g}(\boldsymbol{x}, \boldsymbol{z}) = \boldsymbol{z}$ for all $(\boldsymbol{x}, \boldsymbol{z}) \in \mathcal{U}' \times \mathcal{W}'$ by virtue of the fact that the map (1.19) is the inverse of $\boldsymbol{h}|\mathcal{U}' \times \mathcal{V}'$. ■

The next result gives a local normal form for holomorphic (resp. real analytic) maps whose derivative is surjective at a point.

1.2.8 Theorem (Holomorphic or real analytic local submersion theorem) *Let $\mathcal{U} \subseteq \mathbb{F}^n$ be open and let $\mathbf{f}\colon \mathcal{U} \to \mathbb{F}^m$ be holomorphic (resp. real analytic). If the matrix*

$$\mathbf{D}\mathrm{f}(\mathbf{x}_0) = \begin{bmatrix} \frac{\partial \mathrm{f}_1}{\partial \mathrm{x}_1}(\mathbf{x}_0) & \cdots & \frac{\partial \mathrm{f}_1}{\partial \mathrm{x}_n}(\mathbf{x}_0) \\ \vdots & \ddots & \vdots \\ \frac{\partial \mathrm{f}_m}{\partial \mathrm{x}_1}(\mathbf{x}_0) & \cdots & \frac{\partial \mathrm{f}_m}{\partial \mathrm{x}_m}(\mathbf{x}_0) \end{bmatrix}$$

has rank m for $\mathbf{x}_0 \in \mathcal{U}$, then there exists

(i) a neighbourhood $\mathcal{U}_1 \subseteq \mathcal{U}$ of $\mathbf{x}_0$,

(ii) an open set $\mathcal{U}_2 \subseteq \mathbb{F}^m \times \mathbb{F}^{n-m}$, and

(iii) a holomorphic (resp. real analytic) diffeomorphism $\Phi\colon \mathcal{U}_2 \to \mathcal{U}_1$

such that $\mathbf{f} \circ \Phi(\mathbf{y}, \mathbf{z}) = \mathbf{y}$ for all $(\mathbf{y}, \mathbf{z}) \in \mathcal{U}_2$.

Proof Let $\mathsf{U} \subseteq \mathbb{F}^n$ be a complement to $\ker(\boldsymbol{D}\boldsymbol{f}(\boldsymbol{x}_0))$ so that $\boldsymbol{D}\boldsymbol{f}(\boldsymbol{x}_0)|\mathsf{U}$ is an isomorphism onto $\mathbb{F}^m$ [Roman 2005, Theorem 3.5]. Note that $\mathbb{F}^n \simeq \mathsf{U} \oplus \ker(\boldsymbol{D}\boldsymbol{f}(\boldsymbol{x}_0))$. Choose a basis $(\boldsymbol{\alpha}_1, \dots, \boldsymbol{\alpha}_m, \boldsymbol{\beta}_1, \dots, \boldsymbol{\beta}_{n-m})$ for $\mathbb{F}^n$ for which $(\boldsymbol{\alpha}_1, \dots \boldsymbol{\alpha}_m)$ is a basis for U and $(\boldsymbol{\beta}_1, \dots, \boldsymbol{\beta}_{n-m})$ is a basis for $\ker(\boldsymbol{D}\boldsymbol{f}(\boldsymbol{x}_0))$. Define an isomorphism $\psi\colon \mathbb{F}^m \oplus \mathbb{F}^{n-m} \to \mathbb{F}^n$ by

$$\psi(\boldsymbol{u} \oplus \boldsymbol{v}) = u_1\boldsymbol{\alpha}_1 + \dots + u_m\boldsymbol{\alpha}_m + v_1\boldsymbol{\beta}_1 + \dots + v_{n-m}\boldsymbol{\beta}_{n-m}.$$

Then define $\hat{\boldsymbol{f}}\colon \psi^{-1}(\mathcal{U}) \to \mathbb{F}^m$ by $\hat{\boldsymbol{f}} = \boldsymbol{f} \circ \psi^{-1}$ so that $\boldsymbol{D}\hat{\boldsymbol{f}}(\boldsymbol{x}_0) = \boldsymbol{D}\boldsymbol{f}(\boldsymbol{x}_0) \circ \psi^{-1}$. Now define $\hat{\boldsymbol{g}}\colon \psi^{-1}(\mathcal{U}) \to \mathbb{F}^m \times \mathbb{F}^{n-m}$ by

$$\hat{\boldsymbol{g}}(\boldsymbol{y}, \boldsymbol{z}) = (\hat{\boldsymbol{f}}(\boldsymbol{y} \oplus \boldsymbol{z}), \boldsymbol{z}).$$

Let $(\boldsymbol{y}_0, \boldsymbol{z}_0) = \psi^{-1}(\boldsymbol{x}_0)$. Note that the Jacobian matrix of partial derivatives of $\hat{\boldsymbol{g}}$ is

$$\boldsymbol{D}\hat{\boldsymbol{g}}(\boldsymbol{y}_0, \boldsymbol{z}_0) = \left[\begin{array}{ccc|ccc} \frac{\partial \hat{f}_1}{\partial y_1}(\boldsymbol{y}_0, \boldsymbol{z}_0) & \cdots & \frac{\partial \hat{f}_1}{\partial y_m}(\boldsymbol{y}_0, \boldsymbol{z}_0) & \frac{\partial \hat{f}_1}{\partial z_1}(\boldsymbol{y}_0, \boldsymbol{z}_0) & \cdots & \frac{\partial \hat{f}_1}{\partial z_{n-m}}(\boldsymbol{y}_0, \boldsymbol{z}_0) \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \frac{\partial \hat{f}_m}{\partial y_1}(\boldsymbol{y}_0, \boldsymbol{z}_0) & \cdots & \frac{\partial \hat{f}_m}{\partial y_m}(\boldsymbol{y}_0, \boldsymbol{z}_0) & \frac{\partial \hat{f}_m}{\partial z_1}(\boldsymbol{y}_0, \boldsymbol{z}_0) & \cdots & \frac{\partial \hat{f}_m}{\partial z_{n-m}}(\boldsymbol{y}_0, \boldsymbol{z}_0) \\ \hline 0 & \cdots & 0 & 1 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & 0 & \cdots & 1 \end{array}\right].$$

We claim that the upper left block of the Jacobian matrix, i.e., $\boldsymbol{D}_1\hat{\boldsymbol{f}}(\boldsymbol{y}_0, \boldsymbol{z}_0)$, is invertible. This follows since ψ is an isomorphism from $\mathbb{F}^m \oplus \{\boldsymbol{0}\}$ onto U and since U is defined so that $\boldsymbol{D}\boldsymbol{f}(\boldsymbol{x}_0)|\mathsf{U}$ is an isomorphism. It then follows that $\boldsymbol{D}\hat{\boldsymbol{g}}(\boldsymbol{y}_0, \boldsymbol{z}_0)$ is an isomorphism. By the Inverse Function Theorem there exists a neighbourhood $\mathcal{U}'$ of $(\boldsymbol{y}_0, \boldsymbol{z}_0)$ such that $\hat{\boldsymbol{g}}|\mathcal{U}'$ is a holomorphic (resp. real analytic) diffeomorphism onto its image. Denote $\mathcal{U}_2 = \hat{\boldsymbol{g}}(\mathcal{U}')$ and denote by $\Psi\colon \mathcal{U}_2 \to \mathcal{U}'$ the inverse of $\hat{\boldsymbol{g}}|\mathcal{U}'$. Let us write $\Psi(\boldsymbol{y}, \boldsymbol{z}) = (\Psi_1(\boldsymbol{y}, \boldsymbol{z}), \Psi_2(\boldsymbol{y}, \boldsymbol{z}))$. Note that

$$\hat{\boldsymbol{g}} \circ \Psi(\boldsymbol{y}, \boldsymbol{z}) = \hat{\boldsymbol{g}}(\Psi_1(\boldsymbol{y}, \boldsymbol{z}), \Psi_2(\boldsymbol{y}, \boldsymbol{z})) = (\hat{\boldsymbol{f}}(\Psi_1(\boldsymbol{y}, \boldsymbol{z}), \Psi_2(\boldsymbol{y}, \boldsymbol{z})), \Psi_2(\boldsymbol{z})) = (\boldsymbol{y}, \boldsymbol{z})$$

for all $(\boldsymbol{y}, \boldsymbol{z}) \in \mathcal{U}_2$. Thus $\Psi_2(\boldsymbol{y}, \boldsymbol{z}) = \boldsymbol{z}$ for all $(\boldsymbol{y}, \boldsymbol{z}) \in \mathcal{U}_2$. Moreover, if we define $\mathcal{U}_1 = \psi(\mathcal{U}')$ and $\Phi\colon \mathcal{U}_2 \to \mathcal{U}_1$ by $\Phi = \psi \circ \Psi$, then we have

$$\boldsymbol{y} = \hat{\boldsymbol{f}}(\Psi_1(\boldsymbol{y}, \boldsymbol{z}), \Psi_2(\boldsymbol{y}, \boldsymbol{z})) = \boldsymbol{f} \circ \psi \circ \Psi(\boldsymbol{y}, \boldsymbol{z}) = \boldsymbol{f} \circ \Phi(\boldsymbol{y}, \boldsymbol{z}),$$

as desired. ■

Next we give a similar result when the derivative is injective at a point.

1.2.9 Theorem (Holomorphic or real analytic local immersion theorem) *Let $\mathcal{U} \subseteq \mathbb{F}^n$ be open and let $\mathbf{f}\colon \mathcal{U} \to \mathbb{F}^m$ be holomorphic (resp. real analytic). If the matrix*

$$\mathbf{D}f(\mathbf{x}_0) = \begin{bmatrix} \frac{\partial f_1}{\partial x_1}(\mathbf{x}_0) & \cdots & \frac{\partial f_1}{\partial x_n}(\mathbf{x}_0) \\ \vdots & \ddots & \vdots \\ \frac{\partial f_m}{\partial x_1}(\mathbf{x}_0) & \cdots & \frac{\partial f_m}{\partial x_m}(\mathbf{x}_0) \end{bmatrix}$$

has rank n for $\mathbf{x}_0 \in \mathcal{U}$, then there exists

(i) a neighbourhood $\mathcal{V}_1$ of $\mathbf{f}(\mathbf{x}_0)$,

(ii) a neighbourhood $\mathcal{V}_2 \subseteq \mathcal{U} \times \mathbb{F}^{m-n}$ of $(\mathbf{x}_0, \mathbf{0})$, and

(iii) a holomorphic (resp. real analytic) diffeomorphism $\Psi\colon \mathcal{V}_1 \to \mathcal{V}_2$

such that $\Psi \circ \mathbf{f}(\mathbf{x}) = (\mathbf{x}, \mathbf{0})$ for all $\mathbf{x} \in \mathcal{U}$ for which $(\mathbf{x}, \mathbf{0}) \in \mathcal{V}_2$.

Proof Since $\operatorname{rank}(\boldsymbol{D}\boldsymbol{f}(x_0)) = n$, $\boldsymbol{D}\boldsymbol{f}(x_0)$ is injective. Let $\mathsf{V} \subseteq \mathbb{F}^m$ be a complement to $\operatorname{image}(\boldsymbol{D}\boldsymbol{f}(x_0))$. Choose a basis $(\boldsymbol{\alpha}_1, \dots, \boldsymbol{\alpha}_n, \boldsymbol{\beta}_1, \dots, \boldsymbol{\beta}_{m-n})$ for $\mathbb{F}^m$ such that $(\boldsymbol{\alpha}_1, \dots, \boldsymbol{\alpha}_n)$ is a basis for $\operatorname{image}(\boldsymbol{D}\boldsymbol{f}(x_0))$ and such that $(\boldsymbol{\beta}_1, \dots, \boldsymbol{\beta}_{m-n})$ is a basis for V. Define an isomorphism $\psi\colon \mathbb{F}^n \oplus \mathbb{F}^{m-n} \to \mathbb{F}^m$ by

$$\psi(\boldsymbol{u} \oplus \boldsymbol{v}) = u_1\boldsymbol{\alpha}_1 + \cdots + u_n\boldsymbol{\alpha}_n + \cdots + v_1\boldsymbol{\beta}_1 + \cdots + v_{m-n}\boldsymbol{\beta}_{m-n}.$$

Then define $\hat{\boldsymbol{f}}\colon \mathcal{U} \to \mathbb{F}^n \times \mathbb{F}^{m-n}$ by $\hat{\boldsymbol{f}} = \psi^{-1} \circ \boldsymbol{f}$ so that $\boldsymbol{D}\hat{\boldsymbol{f}}(x_0) = \psi^{-1} \circ \boldsymbol{D}\boldsymbol{f}(x_0)$. Also define $\hat{\boldsymbol{g}}\colon \mathcal{U} \times \mathbb{F}^{m-n} \to \mathbb{F}^n \times \mathbb{F}^{n-m}$ by

$$\hat{\boldsymbol{g}}(x, \boldsymbol{y}) = \hat{\boldsymbol{f}}(x) + (\boldsymbol{0}, \boldsymbol{y}).$$

Note that the Jacobian matrix of $\hat{\boldsymbol{g}}$ at $(x_0, \boldsymbol{0})$ is

$$\boldsymbol{D}\hat{\boldsymbol{g}}(x_0, \boldsymbol{0}) = \left[\begin{array}{ccc|ccc} \frac{\partial \hat{f}_1}{\partial x_1}(x_0, \boldsymbol{0}) & \cdots & \frac{\partial \hat{f}_1}{\partial x_m}(x_0, \boldsymbol{0}) & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \frac{\partial \hat{f}_m}{\partial x_1}(x_0, \boldsymbol{0}) & \cdots & \frac{\partial \hat{f}_m}{\partial x_m}(x_0, \boldsymbol{0}) & 0 & \cdots & 0 \\ \hline 0 & \cdots & 0 & 1 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & 0 & \cdots & 1 \end{array}\right].$$

The upper left corner of this matrix is invertible since ψ is an isomorphism of $\mathbb{F}^n \oplus \{\boldsymbol{0}\}$ onto $\operatorname{image}(\boldsymbol{D}\boldsymbol{f}(x_0))$ and since $\boldsymbol{D}\boldsymbol{f}(x_0)$ is injective. It follows that $\boldsymbol{D}\hat{\boldsymbol{g}}(x_0)$ is an isomorphism and so there exists a neighbourhood $\mathcal{V}_2$ of $(x_0, \boldsymbol{0})$ such that $\hat{\boldsymbol{g}}|\mathcal{V}_2$ is a diffeomorphism onto its image. Let $\mathcal{V}' = \hat{\boldsymbol{g}}(\mathcal{V}_2)$ and denote by $\Phi\colon \mathcal{V}' \to \mathcal{V}_2$ the inverse of $\hat{\boldsymbol{g}}|\mathcal{V}_2$. Note that

$$\Phi \circ \hat{\boldsymbol{g}}(x, \boldsymbol{y}) = (x, \boldsymbol{y})$$

for every $(x, \boldsymbol{y}) \in \mathcal{V}'$. Therefore,

$$\Phi \circ \hat{\boldsymbol{g}}(x, \boldsymbol{0}) = \Phi \circ \hat{\boldsymbol{f}}(x) = \Phi \circ \psi^{-1} \circ \boldsymbol{f}(x) = x.$$

Thus, if we define $\mathcal{V}_1 = \psi(\mathcal{V}')$ and $\Psi = \Phi \circ \psi^{-1}$, we see that the conclusions of the theorem hold. ■

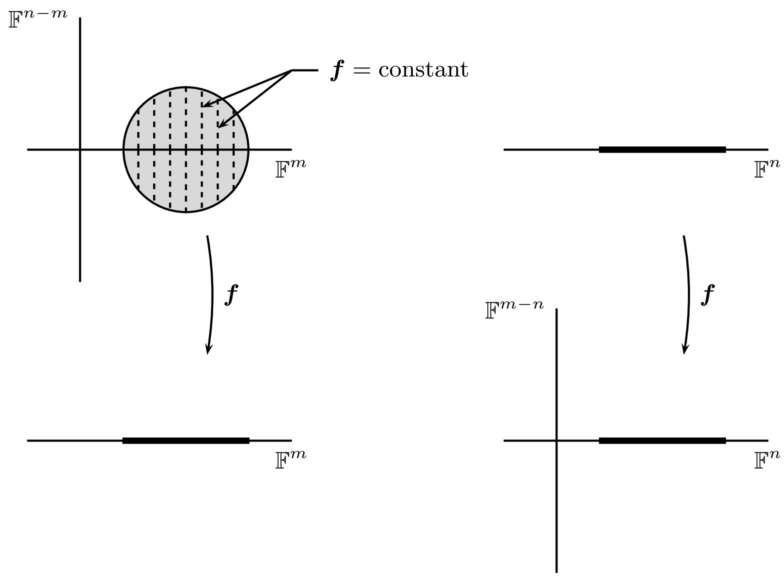

Figure 1.3 The character of local submersions (left) and immersions (right)

In Figure 1.3 we depict the behaviour of maps satisfying the hypotheses of Theorems 1.2.8 and 1.2.9 after the application of the diffeomorphisms from the theorem.

Even when the derivative is neither injective nor surjective, we can still provide a characterisation of the local behaviour of a map.

1.2.10 Theorem (Holomorphic or real analytic local representation theorem) *Let $\mathcal{U} \subseteq \mathbb{F}^{\mathrm{n}}$ be open and let $\mathbf{f}\colon \mathcal{U} \to \mathbb{F}^{\mathrm{m}}$ be holomorphic (resp. real analytic). If $\mathrm{k} = \operatorname{rank}(\mathbf{Df}(\mathbf{x}_0))$ for $\mathbf{x}_0 \in \mathcal{U}$, then there exists*

(i) a neighbourhood $\mathcal{U}_1 \subseteq \mathcal{U}$ of $\mathbf{x}_0$,

(ii) an open set $\mathcal{U}_2 \subseteq \mathbb{F}^{\mathrm{k}} \times \mathbb{F}^{\mathrm{n-k}}$,

(iii) a holomorphic (resp. real analytic) diffeomorphism $\Phi\colon \mathcal{U}_2 \to \mathcal{U}_1$,

(iv) an isomorphism $\psi\colon \mathbb{F}^{\mathrm{m}} \to \mathbb{F}^{\mathrm{k}} \oplus \mathbb{F}^{\mathrm{m-k}}$, and

(v) a holomorphic (resp. real analytic) map $\mathbf{g}\colon \mathcal{U}_2 \to \mathbb{F}^{\mathrm{m-k}}$

such that $\psi \circ \mathbf{f} \circ \Phi(\mathbf{y}, \mathbf{z}) = (\mathbf{y}, \mathbf{g}(\mathbf{y}, \mathbf{z}))$ for all $(\mathbf{y}, \mathbf{z}) \in \mathcal{U}_2$ and such that $\mathbf{Dg}(\Phi^{-1}(\mathbf{x}_0)) = 0$.

Proof Let $(\boldsymbol{\alpha}_1, \dots, \boldsymbol{\alpha}_k, \boldsymbol{\beta}_1, \dots, \boldsymbol{\beta}_{m-k})$ be a basis for $\mathbb{F}^m$ such that $(\boldsymbol{\alpha}_1, \dots, \boldsymbol{\alpha}_k)$ is a basis for $\operatorname{image}(\boldsymbol{D}\boldsymbol{f}(\boldsymbol{x}_0))$ and $(\boldsymbol{\beta}_1, \dots, \boldsymbol{\beta}_{m-k})$ is a basis for a complement V to $\operatorname{image}(\boldsymbol{D}\boldsymbol{f}(\boldsymbol{x}_0))$. In like manner we let $(\boldsymbol{a}_1, \dots, \boldsymbol{a}_k, \boldsymbol{b}_1, \dots, \boldsymbol{b}_{n-k})$ be a basis for $\mathbb{F}^n$ such that $(\boldsymbol{b}_1, \dots, \boldsymbol{b}_{n-k})$ is a basis for $\ker(\boldsymbol{D}\boldsymbol{f}(\boldsymbol{x}_0))$ and $(\boldsymbol{a}_1, \dots, \boldsymbol{a}_k)$ is a basis for a complement U to $\ker(\boldsymbol{D}\boldsymbol{f}(\boldsymbol{x}_0))$. Define isomorphisms $\phi\colon \mathbb{F}^k \oplus \mathbb{F}^{n-k} \to \mathbb{F}^n$ by

$$\phi(\boldsymbol{u} \oplus \boldsymbol{v}) = u_1\boldsymbol{a}_1 + \cdots + u_k\boldsymbol{a}_k + v_1\boldsymbol{b}_1 + \cdots + v_{n-k}\boldsymbol{b}_{n-k}$$

and $\psi\colon \mathbb{F}^m \to \mathbb{F}^k \oplus \mathbb{F}^{m-k}$ by

$$\psi^{-1}(\boldsymbol{r} \oplus \boldsymbol{s}) = r_1\boldsymbol{\alpha}_1 + \cdots + r_k\boldsymbol{\alpha}_k + s_1\boldsymbol{\beta}_1 + \cdots + s_{m-k}\boldsymbol{\beta}_{m-k}.$$

Let us define $\hat{f}\colon \mathcal{U} \to \mathbb{F}^k \times \mathbb{F}^{m-k}$ by $\hat{f} = \psi \circ f$. For $x \in \mathcal{U}$ write $\hat{f}(x) = (\hat{f}_1(x), \hat{f}_2(x))$. Note that $D\hat{f}_1(x_0)$ is injective since ψ is an isomorphism of image$(Df(x_0))$ with $\mathbb{F}^k \oplus \{0\}$. Thus $\hat{f}_1$ satisfies the hypotheses of the local submersion theorem, and so the conclusions of that theorem furnish us with a neighbourhood $\mathcal{U}_1$ of x_0, an open set $U_2 \subseteq \mathbb{F}^k \times \mathbb{F}^{n-k}$, and a holomorphic (resp. real analytic) diffeomorphism $\Phi\colon \mathcal{U}_2 \to \mathcal{U}_1$ such that $\hat{f}_1 \circ \Phi(y, z) = y$ for all $(y, z) \in \mathcal{U}_2$. Let us define $g\colon \mathcal{U}_2 \to \mathbb{F}^{m-k}$ by $g = \hat{f}_2 \circ \Phi$ so that

$$\hat{f} \circ \Phi(y, z) = (y, g(y, z)).$$

Note that $Df_2(x_0) = 0$ since ϕ maps $\{0\} \oplus \mathbb{F}^{n-k}$ to $\ker(Df(x_0))$. Thus $Dg(\Phi^{-1}(x_0)) = 0$. Finally,

$$\psi \circ f \circ \Phi(y, z) = \hat{f} \circ \Phi(y, z) = (y, g(y, z)),$$

giving the theorem. ■

Finally, we consider the case where the derivative is not necessarily injective nor surjective, but is of constant rank.

1.2.11 Theorem (Holomorphic or real analytic local rank theorem) *Let $\mathcal{U} \subseteq \mathbb{F}^{\mathrm{n}}$ be open and let $\mathbf{f}\colon \mathcal{U} \to \mathbb{F}^{\mathrm{m}}$ be holomorphic (resp. real analytic). If the rank of the matrix*

$$\mathbf{Df}(\mathbf{x}) = \begin{bmatrix} \frac{\partial \mathrm{f}_1}{\partial \mathrm{x}_1}(\mathbf{x}) & \cdots & \frac{\partial \mathrm{f}_1}{\partial \mathrm{x}_\mathrm{n}}(\mathbf{x}) \\ \vdots & \ddots & \vdots \\ \frac{\partial \mathrm{f}_\mathrm{m}}{\partial \mathrm{x}_1}(\mathbf{x}) & \cdots & \frac{\partial \mathrm{f}_\mathrm{m}}{\partial \mathrm{x}_\mathrm{n}}(\mathbf{x}) \end{bmatrix}$$

is equal to k in a neighbourhood $\mathcal{U}' \subseteq \mathcal{U}$ of $\mathbf{x}_0$, then there exists

(i) *a neighbourhood $\mathcal{U}_1 \subseteq \mathcal{U}$ of $\mathbf{x}_0$,*
(ii) *an open set $\mathcal{U}_2 \subseteq \mathbb{F}^{\mathrm{k}} \times \mathbb{F}^{\mathrm{n-k}}$,*
(iii) *an open set $\mathcal{V}_1 \subseteq \mathbb{F}^{\mathrm{m}}$,*
(iv) *an open set $\mathcal{V}_2 \subseteq \mathbb{F}^{\mathrm{k}} \times \mathbb{F}^{\mathrm{m-k}}$, and*
(v) *a holomorphic (resp. real analytic) diffeomorphisms $\Phi\colon \mathcal{U}_2 \to \mathcal{U}_1$ and $\Psi\colon \mathcal{V}_1 \to \mathcal{V}_2$*

such that $\Psi \circ \mathbf{f} \circ \Phi(\mathbf{y}, \mathbf{z}) = (\mathbf{y}, \mathbf{0})$ for all $(\mathbf{y}, \mathbf{z}) \in \mathcal{U}_2$.

Proof The local representation theorem furnishes us with a neighbourhood $\mathcal{U}_1 \subseteq \mathcal{U}$ of x_0, an open subset $\mathcal{U}_2 \subseteq \mathbb{F}^k \times \mathbb{F}^{n-k}$, a holomorphic (resp. real analytic) diffeomorphism $\Phi\colon \mathcal{U}_2 \to \mathcal{U}_1$, an isomorphism $\psi\colon \mathbb{F}^m \to \mathbb{F}^k \times \mathbb{F}^{m-k}$, and a holomorphic (resp. real analytic) map $g\colon \mathcal{U}_2 \to \mathbb{F}^{m-k}$ such that $\psi \circ f \circ \Phi(y, z) = (y, g(y, z))$ for all $(y, z) \in \mathcal{U}_2$ and such that $Dg(\Phi^{-1}(x_0)) = 0$. We shall assume that $\mathcal{U}_1 \subseteq \mathcal{U}'$. Let us denote $F(y, z) = (y, g(y, z))$. If $\mathrm{pr}_1\colon \mathbb{F}^k \oplus \mathbb{F}^{m-k} \to \mathbb{F}^k$ is the projection onto the first factor, we have

$$\mathrm{pr}_1 \circ DF(y, z) \cdot (u, v) = u,$$

This means that the map

$$(u, 0) \mapsto DF(y, z) \cdot (u, 0) \in \mathrm{image}(DF(y, z)) \tag{1.20}$$

is injective. By hypothesis image($\boldsymbol{D}\boldsymbol{F}(\boldsymbol{y},\boldsymbol{z})$) has dimension k, and so the map (1.20) is an isomorphism for each $(\boldsymbol{y},\boldsymbol{z}) \in \mathcal{U}_1$. Now let

$$(\boldsymbol{u}, \boldsymbol{D}\boldsymbol{g}(\boldsymbol{y},\boldsymbol{z}) \cdot (\boldsymbol{u},\boldsymbol{v})) \in \operatorname{image}(\boldsymbol{D}\boldsymbol{F}(\boldsymbol{y},\boldsymbol{z})).$$

Since the map (1.20) is an isomorphism there exists $\boldsymbol{u}' \in \mathbb{F}^k$ such that

$$\boldsymbol{D}\boldsymbol{F}(\boldsymbol{y},\boldsymbol{z}) \cdot (\boldsymbol{u}',\boldsymbol{0}) = (\boldsymbol{u}', \boldsymbol{D}\boldsymbol{g}(\boldsymbol{y},\boldsymbol{z}) \cdot (\boldsymbol{u}',\boldsymbol{0})) = (\boldsymbol{u}', \boldsymbol{D}_1\boldsymbol{g}(\boldsymbol{y},\boldsymbol{z}) \cdot \boldsymbol{u}') = (\boldsymbol{u}, \boldsymbol{D}\boldsymbol{g}(\boldsymbol{y},\boldsymbol{z}) \cdot (\boldsymbol{u},\boldsymbol{v})).$$

Thus $\boldsymbol{u}' = \boldsymbol{u}$ and

$$\boldsymbol{D}\boldsymbol{g}(\boldsymbol{y},\boldsymbol{z}) \cdot (\boldsymbol{u},\boldsymbol{v}) = \boldsymbol{D}_1\boldsymbol{g}(\boldsymbol{y},\boldsymbol{z}) \cdot \boldsymbol{u} + \boldsymbol{D}_2\boldsymbol{g}(\boldsymbol{y},\boldsymbol{z}) \cdot \boldsymbol{v} = \boldsymbol{D}_1\boldsymbol{g}(\boldsymbol{y},\boldsymbol{z}) \cdot \boldsymbol{u}$$

for all $\boldsymbol{v} \in \mathbb{F}^{n-k}$. Thus $\boldsymbol{D}_2\boldsymbol{g}(\boldsymbol{y},\boldsymbol{z}) \cdot \boldsymbol{v} = \boldsymbol{0}$ for all $(\boldsymbol{y},\boldsymbol{z}) \in \mathcal{U}_2$ and $\boldsymbol{v} \in \mathbb{F}^{n-k}$. Thus $\boldsymbol{g}$ is not a function of $\boldsymbol{z}$. Since

$$\boldsymbol{D}_2\boldsymbol{F}(\boldsymbol{y},\boldsymbol{z}) \cdot \boldsymbol{v} = (\boldsymbol{0}, \boldsymbol{D}_2\boldsymbol{g}(\boldsymbol{y},\boldsymbol{z}) \cdot \boldsymbol{v}),$$

we conclude that $\boldsymbol{F}$ also does not depend on $\boldsymbol{z}$. Now define $\hat{\boldsymbol{F}}\colon \operatorname{pr}_1(\mathcal{U}_2) \to \mathbb{F}^k \times \mathbb{F}^{m-k}$ by $\hat{\boldsymbol{F}}(\boldsymbol{y}) = \boldsymbol{F}(\boldsymbol{y},\boldsymbol{z}_{\boldsymbol{y}})$ where $\boldsymbol{z}_{\boldsymbol{y}} \in \mathbb{F}^{n-k}$ is such that $(\boldsymbol{y},\boldsymbol{z}_{\boldsymbol{y}}) \in \mathcal{U}_2$. It follows from the fact that the map (1.20) is an isomorphism and that $\boldsymbol{F}$ is independent of $\boldsymbol{z}$ that $\boldsymbol{D}\hat{\boldsymbol{F}}(\boldsymbol{y})$ is injective for every $\boldsymbol{y} \in \operatorname{pr}_1(\boldsymbol{U}_2)$. In particular, this holds at $\operatorname{pr}_1 \circ \Phi^{-1}(\boldsymbol{x}_0)$. We can now apply the local immersion theorem to give a neighbourhood $\mathcal{V}_1'$ of $\hat{\boldsymbol{F}}(\operatorname{pr}_1 \circ \Phi^{-1}(\boldsymbol{x}_0))$, a neighbourhood $\mathcal{V}_2 \subseteq \operatorname{pr}_1(\boldsymbol{U}_2) \times \mathbb{F}^{m-k}$ of $(\boldsymbol{x}_0, \boldsymbol{0})$, and a diffeomorphism $\Psi'\colon \mathcal{V}'_1 \to \mathcal{V}_2$ such that $\Psi' \circ \hat{\boldsymbol{F}}(\boldsymbol{y}) = (\boldsymbol{y},\boldsymbol{0})$ for all $\boldsymbol{y} \in \operatorname{pr}_1(\boldsymbol{U}_2)$ such that $(\boldsymbol{y},\boldsymbol{0}) \in \mathcal{V}_2$. Now let $(\boldsymbol{y},\boldsymbol{z}) \in \mathcal{U}_2$ and note that

$$(\boldsymbol{y},\boldsymbol{0}) = \Psi' \circ \hat{\boldsymbol{F}}(\boldsymbol{y}) = \Psi' \circ \boldsymbol{F}(\boldsymbol{y},\boldsymbol{z}) = \Psi' \circ \psi \circ \boldsymbol{f} \circ \Phi(\boldsymbol{y},\boldsymbol{z}),$$

giving the theorem after taking $\mathcal{V}_1 = \psi^{-1}(\mathcal{V}_1')$ and $\Psi = \Psi' \circ \psi$. ■

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