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Conjectural Applications of Arithmetic Surfaces to Arithmetic

§1. The ABC-conjecture:

Notation: $\overset{\text{cond}}{\mathrm{rad}}(n) = \prod_{p \mid n} p \qquad (n \in \mathbb{Z},\ n \neq 0, \pm 1)$ or $n \in \mathbb{Z}^{\circ}$

ABC-Conjecture (Masser–Oesterlé): $\exists\, c_1, c_2 \in \mathbb{R}_{>0}$ s.th. for any $a, b, c \in \mathbb{Z}^{\circ}$ with $a + b = c$, $(a,b) = 1$ we have:

$$\mathrm{Max}(|a|, |b|, |c|) \leq c_2\, \mathrm{rad}(abc)^{c_1}$$

equiv.: $\quad h_{\mathbb{P}^2}(a:b:c) \leq c_1 \log \mathrm{rad}(abc) + \log c_2$

Rmk. Masser–Oesterlé: $c_1 = 1 + \varepsilon$, $c_2 = C(\varepsilon)$.

Ex. abc-conj $\Rightarrow$ asymp. (gen.) Fermat:

For A, B, C fixed $\quad Ax^n + By^n = Cz^n \quad$ has no non-trivial sol'ns for $n \gg 0$

$$c_3 |ABC| |xyz|^n \leq \mathrm{Max}(|Ax^n|, |By^n|, |Cz^n|) \leq c(|ABCxyz|)^{c_2}$$

$\Rightarrow n$ bounded.

Rmk Wiles $\not\Rightarrow$ ABC or gen. asympt. Fermat.

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§2. The Height Conjecture for elliptic curves

Let E/K be an elliptic curve def^d over a no. fld K.

$\pi : \mathcal{E} \to S = \mathrm{Spec}(\mathcal{O}_K)$ the assoc. arith. surface

Assume: $\mathcal{E}/S$ semi-stable

$$h(E/K) := \deg_{\bar{S}} (\pi_* \omega_{\mathcal{E}/S}) \qquad \text{"Faltings height of } E\text{"}$$

$$\mathrm{cond}(E/K) = \sum_{v \text{ bad}} v \in \mathrm{Div}(S) \qquad (\text{sum of place where } v \text{ had bad red'n})$$

↳ fibre $\mathcal{E}_v$ singular

$\underline{\underline{H_E}}$ - Conjecture. There exist constants c_1, c_2 (depending on K) such that for every s.s. elliptic curve E/K we have

$$h(E/K) \leq c_1 \deg_{\bar{S}} \mathrm{cond}(E/K) + c_2$$

Prop. 1: (Frey) $H_E \Rightarrow ABC$

Pf idea: given $a+b=c$, look at Frey curve

$$E_{a,b} : \; y^2 = x(x-a)(x+b) \qquad \text{semistable } (2^4 | a,\ b \equiv -1 (4)),\ (a,b)=1$$

$$\mathrm{cond}(E_{a,b}) \sim \mathrm{rad}(abc)$$
$$h(E_{a,b}) \sim h(a:b:c).$$

Rmk. $HC_E \Rightarrow$ Spiro conj $\Rightarrow ABC$

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§3. Shafarevich Conjecture

Observation: Conj. H_E (with explicit constants c_1, c_2) yields an effective version of Shafarevich Conjecture:

Shafarevich Conj. (Th.) ($Ш_E$) for ell. curves: For a given set of places S_0 of K, there exist only finitely many ell. curves E/K with good reduction outside S_0

Pf. (Shaf.) ineffective: uses Siegel's th. on integral pts.

H_E: effective (for s.s. curves only) because
$\{E/K : h(E/K) \leq c\}$ is finite + eff. computable
↳ height on moduli space ↑ of Silverman

Note: $\mathrm{cond}(E/K) < c \Rightarrow$ only finitely many S_0 possible.

Variants: $Ш_C(g)$ ($Ш_A(g)$): For a given set S_0 of K and fixed $g \geq 2$, there are only f. many iso. classes of curves C/K (princ. pol. ab. var A/K) with good reduction outside S_0

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<u>Note</u>. 1) $Ш_C(g)$, $Ш_A(g)$ were proved by Faltings
(rmk: $Ш_A(g) \to Ш_C(g)$ by Torelli)

2) $Ш_C(g)$, $g \gg 0$ $\Longrightarrow$ Mordell's Conj.
↑ Parshin (Parshin–Kod. constr.)

<u>Natural</u> ext'ns of H_E:

<u>Height</u> <u>Conj</u>. <u>for</u> <u>curves</u> (ab. var's): $H_C(g)$ ($H_A(g)$): Fix g, K. Then
$\exists\, c_1, c_2$ (depending on g, K) s.th. all semi-stable[*)] C/K (A/K)

$$h(C/K) \le c_1 \deg \operatorname{cond}(C/K) + c_2 \qquad (A)$$

Here: $h(C/K) = \deg_{\bar S} \det \pi_* \omega_{X/S}$; $\operatorname{cond}(C/K)$,
$\quad\;\; \shortparallel$ $\qquad\qquad\qquad\qquad\qquad\qquad$ $\operatorname{cond}(A/K)$ as
$h(J_C/K)$ $\qquad\qquad\qquad\qquad\qquad$ before

<u>Note</u>: $\operatorname{cond}(J_C/K) \le \operatorname{cond}(C/K) \le \delta + O(1)$
↑ of Noether formula

Thus: $H_J(g) \to H_C(g)$

Again, these give eff. versions of Shaf. Conj.

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*) assoc. min. model $X_C \to S$ is semi-stable
(cf. notes of Noether formula)

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§4. The classical BMY inequality

Let X/k be a smooth proj. surface ($k = \bar{k}$, char 0).

X gen'l type $\Rightarrow$ $c_1^2 \le \lambda c_2$ $\quad \lambda = 8$ van de Ven

$\lambda = 4$ Bog.

$\lambda = 3$ Miy.

X arbitrary $\overset{\text{Parshin}}{\Longrightarrow}$ $c_1^2 \le \mathrm{Max}(2c_2, 3c_2)$

Relative version: (Parshin) (M-B) $\quad \pi : X \to B$ semi-stable, $g(X_K) = g \ge 2$ geom. gen. smooth fi.

Then ineq. $c_1^2 \le \lambda c_2 \iff$

$$\boxed{(\omega_{X/B} \cdot \omega_{X/B}) \le \lambda \cdot \delta + (\lambda - 2)c},$$

relative BMY ($\lambda = 3$)

$c = (g-1)(q-1)$, $q = g(B)$.

$\delta = \sum_{v \in B} \delta_v$, $\delta_v = \#\mathrm{sing}(X_v)$

Important refinement:

Consider
$$\begin{array}{ccc} X & \xrightarrow{\sigma} & X' \\ \pi \downarrow & \swarrow \pi' & \\ B & & \end{array}$$

σ : blow-down of all (-2)-rat'l curves ($\simeq \mathbb{P}^1$) in fibre

(X' singular surface!)

$\delta'_v = \#$ sing. pts of X'_v $\quad (\le \delta_v)$; $\quad \delta' = \sum_{v \in S} \delta_v$

Th. 1 (Parshin) If $g \ge 2$, $q \ge 2$ then

$$(\omega_{X/B} \cdot \omega_{X/B}) \le 3\delta' + 4c .$$

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Cor. $(\omega_{X/B}\cdot\omega_{X/B}) \leq g(g-1)\deg_B \mathrm{cond}(X/B) + 4c$

Pf. It is known that $\delta'_v \leq 3g-3$ for each $v \in B$. (cf [MB])

Rmk. As before: $\mathrm{cond}(X/B) = \sum\limits_{X_v \text{ singular}} v$

Note: By the Noether formula one has:

$$\omega^2 = 12\,h(X/B) - \delta$$

where, analogous to before,

$$h(X/B) = \deg f_*\omega_{X/B}$$

is the Faltings (or modular height).

Th.2 $\exists$ $c_1 = c(g)$ s.t. for every s.s. X/B (as above) of genus $g \geq 2$

$c_2 = c(g,q)$

$H_c(g)$ for fields.

$$h(X/B) \leq \overset{\frac{g-1}{2}}{c_1} \deg \mathrm{cond}(X/B) + \overset{(g-1)q-1}{c_2}$$

Pf. [V], p.162

Rmk. Th.2 $\Rightarrow$ Shaf. conj (for fn fields)

with some work

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§5. Arithmetic Analogue of BMY

K no. field. $\exists \lambda \geq 3$

BMY (naive): $g(C) \geq 2$, $\pi : \mathcal{X} \to S = \mathrm{Spec}(\mathcal{O}_K)$ assoc. arith. surface s.s.

$$(\omega_{X/B} \cdot \omega_{X/B}) \leq \lambda \deg_{\bar S} \Delta'(X/S) + (\lambda-2)(2g-2)[K:\mathbb{Q}]\, d(K)$$

where $d(F) = \log |\mathrm{disc}(K/\mathbb{Q})|$

$$\Delta' = \sum_{v \in S} \delta'_v$$

Bost/Mestre/MB (1988): $\exists$ cons. curve of genus 2 / $\mathbb{Q}$ s.th.

$$(\omega_{X/B} \cdot \omega_{X/B}) \leq \lambda \deg_S |\Delta|(X/S) + (\lambda-2)(2g-2) \ldots$$

is false for every $\lambda > 0$.

note: $|\Delta|$, not Δ'.

(Parshin, Vojta)

BMY_P: $\exists$ constants c_1, c_2, c_3 s.th.

depends on g

$$\omega^2_{X/B} \leq c_1(\quad) + c_2(\quad) + c_3[K:\mathbb{Q}]$$

Th. (Parshin): BMY for $g \gg 2 \Rightarrow HC_E$.

Pf. via Parshin-Kodaira construction.

Rmk. (MB) weaker BMY $\Rightarrow$ ABC

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$\approx S_c(g)$

BMY_i: $\exists\, c_1, c_2$ depending on K, g s.th. for all s.s. C/K

$$\omega^2_{X/S} \leq c_1 \deg \operatorname{cond}(X/S) + c_2$$

Note. $BMY_p \Rightarrow S_c(g)$.

$S_J(g)$: ..

$$\omega^2_{X/S} \leq c_1 \deg \operatorname{cond}(J_C) + c_2$$

Implications:

$$\begin{array}{ccc} H_J(g) & \Rightarrow & H_C(g) \\ \Downarrow & & \Downarrow \\ S_J(g) & \Rightarrow & S_c(g) \sim BMY_i . \end{array}$$

Th. (Frey/K) $S_g(2) \Rightarrow H_E$

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Idea of proof: fix test ell. curve E_1, good red'n, int. N.

for each E, construct C_E of genus 2 together w.

$$\begin{tikzcd} & C \arrow[dl] \arrow[dr] & \\ E_1 & & E \end{tikzcd} \quad \deg N$$

$$\mathrm{cond}(E) \approx \mathrm{cond}(J_C) \qquad J_C \sim E_1 \times E$$

Then: $$h(E) \approx h(C) + O(1)$$

aim: find inequality

$$h(E) \leq c_1'(\omega.\omega) + c_2' \deg \mathrm{cond}(E) + c_3'$$

- relate Arakelov geometry of C to that of E_1 and E.
(intersection th., Néron models).