

## Conjectural Applications of Arithmetic Surfaces to Arithmetic

§1. The ABC-conjecture:

Notation:  $\text{rad}(n) = \prod_{p|n} p$   $(n \in \mathbb{Z}, n \neq 0, \pm 1)$   $\ll n \in \mathbb{Z}^*$

ABC-Conjecture (Masser-Oesterlé):  $\exists c_1, c_2 \in \mathbb{R}_{>0}$  s.t. for any

$a, b, c \in \mathbb{Z}^*$  with  $a+b=c, \gcd(a,b)=1$  we have:

$$\max(|a|, |b|, |c|) \leq c_2 \text{rad}(abc)^{c_1}$$

equiv.:  $h_{\text{pt}}(a:b:c) \leq c_1 \log \text{rad}(abc) + \log c_2$

Rmk. Masser-Oesterlé:  $c_1 = 1 + \epsilon, c_2 = c(\epsilon)$ .

Ex.  $abc$ -conj  $\Rightarrow$  asymp. (gen.) Fermat:

For  $A, B, C$  fixed  $Ax^n + By^n = Cz^n$  has no non-trivial sol<sup>n</sup>s for  $n \gg 0$

$$c_1 |ABC| |xyz|^n \leq \max(|Ax^n|, |By^n|, |Cz^n|) \leq c_2 (|ABC| |xyz|)^{c_2}$$

$\Rightarrow n$  bounded.

Rmk. Wiles  $\nRightarrow$  ABC or gen. asymp. Fermat.



## §2. The Height Conjecture for elliptic curves

Let  $E/K$  be an elliptic curve def'd over a no. fld  $K$ .

$\pi: \mathcal{E} \rightarrow S = \text{Spec}(\mathcal{O}_K)$  the assoc. arith. surface

Assume:  $\mathcal{E}$  semi-stable

$$h(E/K) := \deg_S(\pi_* \omega_{\mathcal{E}/S}) \quad \text{"Faltings height of } E \text{"}$$

$$\text{cond}(E/K) = \sum_{\substack{\nu \\ \nu \text{ bad}}} \nu \in \text{Div}(S) \quad \left( \begin{array}{l} \text{sum of place where} \\ \nu \text{ had bad red'n} \end{array} \right)$$

$\hookrightarrow$  fibre  $\mathcal{E}_\nu$  singular

$H_E$  - Conjecture. There exist constants  $c_1, c_2$  (depending on  $K$ ) such that for every s.s. elliptic curve  $E/K$  we have

$$h(E/K) \leq c_1 \deg_S \text{cond}(E/K) + c_2$$

Prop. 1: (Frey)  $H_E \Rightarrow ABC$

Pf idea: given  $a+b=c$ , look at Frey curve

$$E_{a,b}: y^2 = x(x-a)y(x+b)$$

semistable ( $2^4 | a, b \equiv -1(4)$ )  
( $a, b$ ) = 1

$$\text{cond}(E_{a,b}) \sim \text{rad}(abc)$$

$$h(E_{a,b}) \sim h(a:b:c).$$

Rank.  $H_{E_{a,b}} \xrightarrow{\text{conj}} \text{Spind} \rightarrow ABC$



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### §3. Shafarevich Conjecture

Observation: Conj.  $H_E$  (with explicit constants  $c_1, c_2$ ) yields an effective version of Shafarevich Conjecture:

Shafarevich Conj. <sup>Th.</sup>  $(III)_E$  for all-curves: For a given set of places  $S_0$  of  $K$ , there exist only finitely many ell. curves  $E/K$  with good reduction outside  $S_0$

Pf. (Shaf.) ineffective: uses Siegel's th. on integral pts.

$H_E$ : effective (for s.s. curves only) because

$\{E/K: h(E/K) \leq c\}$  is finite + eff. computable

↳ height on moduli space  $\uparrow$  of Silverman

Note:  $\text{cond}(E/K) < c \Rightarrow$  only finitely many  $S_0$  possible.

Variants:  $III_A(g)$  <sup>fixed</sup>  $III_C(g)$ : For a given set  $S_0$  of  $K$  and  $g \geq 2$ ,  
princ. pt. ab. var  $A/K$   
 there are only f. many iso. classes of curves  $C/K$   
 with good reduction outside  $S_0$



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Note. 1)  $H_c(g), H_A(g)$  were proved by Faltings

(with:  $H_A(g) \rightarrow H_c(g)$  by Torelli)

2)  $H_c(g), g \gg 0 \Rightarrow$  Mordell's Conj.  
 $\uparrow$  Parshin (Parshin-Kod. const.)

Natural ext'n of  $H_E$ :

Height Conj. for curves:  $H_c(g)$ : Fix  $g, K$ . Then  
 $\exists c_1, c_2$  (depending on  $g, K$ ) s.t. all semi-stable  $A/K$   
 $h(C/K) \leq c_1 \deg \text{cond}(C/K) + c_2$

Here:  $h(C/K) = \deg_S \det \pi_* \omega_{X/S}$ ;  $\text{cond}(C/K)$ ,  
 $h(J_C/K)$   $\text{cond}(A/K)$  as  
 before

Note:  $\text{cond}(J_C/K) \leq \text{cond}(C/K) \leq \mathcal{O} + O(1)$   
 $\uparrow$  of Noether formula

Thus:  $H_g(g) \rightarrow H_c(g)$

Again, these give eff. versions of Shaf. Conj.

\* ) error. min. model  $X_c \rightarrow S$   $\square$  semi-stable  
 (q. uses of Noether formula)





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Cor.  $(\omega_{X/B} - \omega_{X/B}) \leq g(g-1) \deg_B \text{cond}(X/B) + 4C$

Pf. It is known that  $\delta'_v \leq 3g-3$  for each  $v \in B$ . (cf [M8])

Remark. As before:  $\text{cond}(X/B) = \sum_{X_v \text{ singular}} v$

Note: By the Noether formula one has:

$$\omega^2 = 12h(X/B) - \delta$$

where, analogous to before,

$$h(X/B) = \deg f_* \omega_{X/B}$$

is the Faltings (or modular height).

Th. 2  $\exists c_1 = c(g)$  s.t. for every s.s.  $X/B$  (as above) of gen  $g \geq 2$

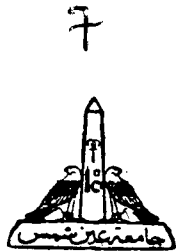
$$h_c(g) \leq c_1 \deg \text{cond}(X/B) + c_2$$

for all fields.  $c_2 = c(g, g) \rightarrow \frac{g-1}{2} \rightarrow (g-1)g-1$

Pf. [V], p. 162

Remark. Th. 2  $\Rightarrow$  Shaf. conj (for  $\mu$ -fields)

$\uparrow$  with some work



### §5. Arithmetic Analogue of BMY

$K$  no. fld.

$\lambda \geq 3$

BMY (naive):  $g(L) \geq 2$ ,  $\pi: X \rightarrow S = \text{Spec}(O_U)$  assoc. with surface s.s.

$$(\omega_{X/B} \otimes \omega_{X/B}^\lambda) \leq \lambda \deg_S \Delta'(X/S) + (\lambda-2)(2g-2)[K:\mathbb{Q}]d(K)$$

where  $d(F) = \log |disc(K/\mathbb{Q})|$

$$\Delta' = \sum_{\sigma \in S} \delta'_\sigma$$

Bost/Kodaira/MB:  $\exists$  cons. curve of genus 2 /  $\mathbb{Q}$  s.t.

$$(\omega_{X/B} \otimes \omega_{X/B}^\lambda) \leq \lambda \deg_S \Delta(X/S) + (\lambda-2)(2g-2) \dots$$

$\circ$  false for every  $\lambda > 0$ .   
 note:  $|\Delta|$ , not  $\Delta'$ .

(Parshin, Vojta)

BMY<sub>P</sub>:  $\exists$  constants  $c_1, c_2, c_3$  s.t.

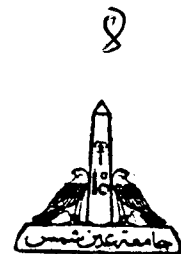
$$\omega_{X/B}^2 \leq c_1 ( ) + c_2 ( ) + c_3 [K:\mathbb{Q}]$$

depends on  $g$

Th. (Parshin): BMY for  $g \gg 2 \Rightarrow HC_E$ .

Pf. via Parshin-Kodaira construction.

Remark. (MB) weaker BMY  $\Rightarrow ABC$ .



$$= S_c(g)$$

BH<sub>1</sub>:  $\exists c_1, c_2$  depending on  $K, g$  s.t. for all s.s.  $C/K$

$$\omega_{X/S}^2 \leq c_1 \deg \text{cond}(X/S) + c_2$$

Note.  $BH_{10} \Rightarrow S_c(g)$ .

$S_g(g)$ : ..

$$\omega_{X/S}^2 \leq c_1 \deg \text{cond}(J_C) + c_2$$

Implications:

$$H_g(g) \Rightarrow H_c(g)$$

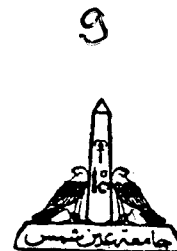
$$\Downarrow$$

$$\Downarrow$$

$$S_g(g) \Rightarrow S_c(g) \sim BH_1.$$

Th. (Freitag)  $S_g(2) \Rightarrow H_E$





Idea of proof: fix test curve  $E_1$ , with  $N$  good red'n.

for each  $E$ , construct  $C_E$  of genus 2 together w.

$$\begin{array}{ccc} & C & \text{deg } N \\ & \downarrow & \\ E_1 & \downarrow & E \\ & \text{f.c.} & \sim E_1 \times E \end{array}$$

Then:  $\text{cond}(E) \approx \text{cond}(f_c)$   
 $h(E) \approx h(C) + O(1)$

aim: find inequality

$$h(E) \leq c'_1(\omega \cdot \omega) + c'_2 \text{deg cond}(E) + c'_3$$

- relate Arakelov geometry of  $C$  to that of  $E_1$  and  $E$ .  
 (intersection th, Neom models).