Modular Diagonal Quotient Surfaces

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Outline

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1. Introduction

Aim: Let $N \geq 3$ and $\varepsilon \in (\mathbb{Z}/N\mathbb{Z})^\times$.

We want to study the properties of a certain (affine) surface $Z_{N,\varepsilon}$ defined over $\mathbb{Q}$, called the modular diagonal quotient surface (MDQS) of type (N, ε).

Analytic Description:

$$(Z_{N,\varepsilon})_{/\mathbb{C}} = \tilde{\Delta}_{N,\varepsilon} \backslash (\mathfrak{H} \times \mathfrak{H}),$$

where $\tilde{\Delta}_{N,\varepsilon} \leq \Gamma(1) \times \Gamma(1)$ is a certain subgroup containing $\Gamma(N) \times \Gamma(N)$.

More precisely, $\tilde{\Delta}_{N,\varepsilon}$ is such that the quotient

$$\begin{aligned} \Delta_{N,\varepsilon} &:= \tilde{\Delta}_{N,\varepsilon}/(\Gamma(N) \times \Gamma(N)) \\ &= \{(g, \alpha_\varepsilon(g)) \in G \times G : g \in G\} \end{aligned}$$

is the twisted diagonal subgroup of the group

$$G = \mathrm{SL}_2(\mathbb{Z}/NZ) = \Gamma(1)/\Gamma(N)$$

with respect to the automorphism

$$\alpha_\varepsilon(g) = Q_\varepsilon g Q_\varepsilon^{-1}, Q_\varepsilon = \left(\begin{smallmatrix} \varepsilon & 0 \\ 0 & 1 \end{smallmatrix}\right) \in \mathrm{GL}_2(\mathbb{Z}/N\mathbb{Z}).$$

Since the analytic properties of these surfaces are analogous to those of the usual Hilbert modular surfaces, they may also be called degenerate Hilbert modular surfaces of discriminant N^2. (Terminology of C.F. Hermann).

Modular Description: The surface $Z_{N,\varepsilon}$ "classifies" isomorphisms (of determinant ε) between mod N Galois representations attached to elliptic curves E/K:

$$\bar{\rho}_{E/K,N} : G_K \to \mathrm{Aut}(E[N]) \simeq \mathrm{GL}_2(\mathbb{Z}/N\mathbb{Z}).$$

This can be used to understand:

- Mazur's Question on isomorphisms between mod N Galois representations attached to elliptic curves.
  $\to$ construction of ∞'ly many such isomorphisms for $N = 11$ (K.-Rizzo)
  $\to$ explanation of the examples of Kraus/Oesterlé, Halberstadt/Kraus for $N = 7$.
- Frey's Conjecture ($\leftrightarrow$ Asymptotic Fermat Conjecture) and Darmon's Conjectures.
  $+$ other related conjectures.

2. The Modular Interpretation

The Functor $\mathcal{Z}_N$: The assignment $K \mapsto \mathcal{Z}_N(K)$, where

$$\mathcal{Z}_N(K) = \{(E, E', \psi)_{/K} : \psi\colon E[N] \xrightarrow{\sim} E'[N]\}/\simeq,$$

naturally extends to a functor $\mathcal{Z}_N : \mathrm{Sch}_{/\mathbb{Q}} \to \mathrm{Sets}$.
We have a natural decomposition (of functors)

$$\mathcal{Z}_N(K) \;=\; \coprod_{\varepsilon\in(\mathbb{Z}/N\mathbb{Z})^\times} \mathcal{Z}_{N,\varepsilon}(K),$$

where

$$\mathcal{Z}_{N,\varepsilon}(K) = \{(E, E', \psi) \in \mathcal{Z}_N(K) : \det(\psi) = \varepsilon\};$$

here, the determinant of ψ is defined by

$$e_N^{E'}(\psi(P), \psi(Q)) = e_N^E(P, Q)^{\det(\psi)}, \forall P, Q \in E[N].$$

Theorem 1: The functors $\mathcal{Z}_N$ and $\mathcal{Z}_{N,\varepsilon}$ are coarsely representable by normal affine surfaces Z_N and $Z_{N,\varepsilon}$ over $\mathbb{Q}$. Moreover, the surfaces $Z_{N,\varepsilon}$ are the connected components of Z_N, and each $Z_{N,\varepsilon}$ is geometrically irreducible; in fact, we have

$$Z_{N,\varepsilon} \otimes \mathbb{C} \simeq (Z_{N,\varepsilon})_{/\mathbb{C}} := \tilde{\Delta}_{N,\varepsilon}\backslash(\mathfrak{H} \times \mathfrak{H}).$$

where $\tilde{\Delta}_{N,\varepsilon} \leq \Gamma(1) \times \Gamma(1)$ is as above.

Theorem 2: For every number field K, the representation map

$$r_{N,K} : \mathcal{Z}_N(K) \to Z_N(K)$$

is surjective, and is injective up to simultaneous twists.

Remarks: 1) Thus, the classification of isomorphisms between the $\bar{\rho}_{E/K,N}$'s is essentially the same as the study of rational points on the surfaces $Z_{N,\varepsilon}$.

2) By considering (as in Deligne/Rapoport) generalized elliptic curves (for which the order of the Néron polygon is divisible by N), we can define a compactification $\bar{\mathcal{Z}}_N$ (resp. $\bar{\mathcal{Z}}_{N,\varepsilon}$) of the above moduli functors. These are then coarsely represented by projective normal surfaces $\bar{Z}_N$ and $\bar{Z}_{N,\varepsilon}$.

3) Let $X(N)_{/\mathbb{Q}}$ denote Shimura's canonical model/$\mathbb{Q}$ of the modular curve $X(N)_{/\mathbb{C}} = \Gamma(N)\backslash\mathfrak{H}^\star$. Then we have natural ($\mathbb{Q}$-rational) morphisms

$$X(N)_{/\mathbb{Q}} \times X(N)_{/\mathbb{Q}} \xrightarrow{\varphi} \bar{Z}_{N,\varepsilon} \xrightarrow{\psi} X(1)_{/\mathbb{Q}} \times X(1)_{/\mathbb{Q}}.$$

Moreover, both are finite covering maps of degree

$$\deg(\varphi) = \deg(\psi) = \tfrac{1}{2}\left|\mathrm{SL}_2(\mathbb{Z}/N\mathbb{Z})\right|.$$

3. CM-Points and Singularities

CM-Points: A CM-point (or "special point") P on Z_N is a point of the form

$$P = P_{E,E',h} := (E, E', h_{|E[N]}),$$

where E has CM and $0 \neq h \in \mathrm{Hom}(E, E') \simeq \mathbb{Z}^2$ ($\Rightarrow (\deg(h), N) = 1$). For fixed elliptic curves E, E' with CM we let

$$Z^{CM}_{N,E,E'} = \{P_{E,E',h} \in Z_N : h \in \mathrm{Hom}(E, E')\}.$$

denote the set of CM-points of type (E, E').

Notes: 1) Each CM-point $P = P_{E,E'h}$ defines a positive definite binary quadratic form q_P of discriminant $N^2\mathrm{disc}(q_{E,E'})$, where $q_{E,E'} = \deg_{|\mathrm{Hom}(E,\,E')}$. Explicitly, q_P is given by $q_P = \deg_{|\mathbb{Z}h+N\mathrm{Hom}(E,E')}$. Thus, the number of CM-points of fixed type is given by (a quotient) of class numbers of binary quadratic forms.

2) Each CM-point $P_{E,E',h}$ is clearly rational over any field over which E, E' and h are defined (and in some cases can be rational over a smaller field). However, in general they will not be rational over $\mathbb{Q}$.

Example: Take $E = E' = E_k$, $k = 0, 1$, where E_k has CM by $\mathfrak{O}_k = \mathbb{Z}[\zeta_{4+2k}]$ (so $\mathfrak{O}_0 = \mathbb{Z}[i], \mathfrak{O}_1 = \mathbb{Z}[\omega], \omega^3 = 1$). Then the map $h \mapsto (E_k, E'_k, h_{E_k[N]})$ defines a bijection

$$(\mathfrak{O}_k/N\mathfrak{O}_k)^\times/(\mathfrak{O}_k)^\times \xrightarrow{\sim} \mathcal{S}^+_{N,k} := Z^{CM}_{N,E_k,E_k}.$$

Thus, if $\mathcal{S}^+_{N,k,\varepsilon} = S^+_{N,k} \cap Z_{N,\varepsilon}$, then one has

$$\mathcal{S}^+_{N,0,\varepsilon} = \tfrac{1}{2}h(-4N^2), \text{ if } (4, N) = 1,$$
$$\mathcal{S}^+_{N,1,\varepsilon} = \tfrac{1}{2}h(-3N^2), \text{ if } (3, N) = 1.$$

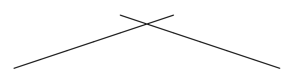
Remark: It is sometimes useful to also consider the twist $P^{(g)} = P_{E,E',h,g} = (E, E', h_{|E[N]} \circ g)$ of a CM-point $P = P_{E,E',h} \in Z^{CM}_{N,E,E'}$ by an element $g \in \mathrm{Aut}(E[N]) \simeq \mathrm{GL}_2(\mathbb{Z}/N\mathbb{Z})$.

In particular, each CM-point $P = P_{E_k,E_k,h} \in \mathcal{S}^+_{N,k}$ gives rise to a unique anti-CM-point $P^- := P^{(w_k)}$, where $w_k = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ (with respect to a basis of $E_k[N]$ of the form $\{P, \sigma_k(P)\}$, where $\mathfrak{O}_k^\times = \langle \sigma_k \rangle$. Thus we have a bijection $\mathcal{S}^+_{N,k} \xrightarrow{\sim} \mathcal{S}^-_{N,k} = \{P^- : P \in S^+_{N,k}\}$.

Theorem 3: The singularities of Z_N are the points $\mathcal{S}_{N,0} \cup \mathcal{S}_{N,1}$, where $\mathcal{S}_{N,k} = S^+_{N,k} \cup S^-_{N,k}$, and all are cyclic quotient singularities (cqs). More precisely,

each point $P \in S_{N,0}$ is a cqs of type $(2,1)$,
each point $P \in S^+_{N,1}$ is a cqs of type $(3,1)$,
each point $P \in S^-_{N,1}$ is a cqs of type $(3,2)$.

Remarks. 1) Thus, Z_N is desingularized by:

- replacing each $P \in S_{N,0}$ by a (-2)-curve E_P,
- replacing each $P \in S^+_{N,1}$ by a (-3)-curve E_P,
- replacing each $P \in S^-_{N,1}$ by a chain E_P consisting of two intersecting (-2)-curves $E_{P,i}$: 

2) Each $Z_{N,\varepsilon}$ is compactified by adding two (smooth) curves isomorphic to $X_1(N)$. These give rise to further cyclic quotient singularities $S_{N,\infty}$ located "at the cusps" which are of type (n,q) with $n|N$ and $(n,q) = 1$. Each $P \in S_\infty$ of type (n,q) is then resolved by a $\mathbb{P}^1$-chain

$$E_P = E_{P,1} \cup \ldots \cup E_{P,r},$$

where r and the self-intersection numbers $E^2_{P,i} = -n_i$ are determined by the (modified) continued fraction expansion of $\frac{n}{q}$.

Example: The Case $(N, \varepsilon) = (11, 1)$.

Here we have

$$|\mathcal{S}^+_{11,0,1}| = |\mathcal{S}^-_{11,0,1}| = 3$$
$$|\mathcal{S}^+_{11,1,1}| = |\mathcal{S}^-_{11,1,1}| = 2$$
$$|S_{11,\infty,1}| = 5.$$

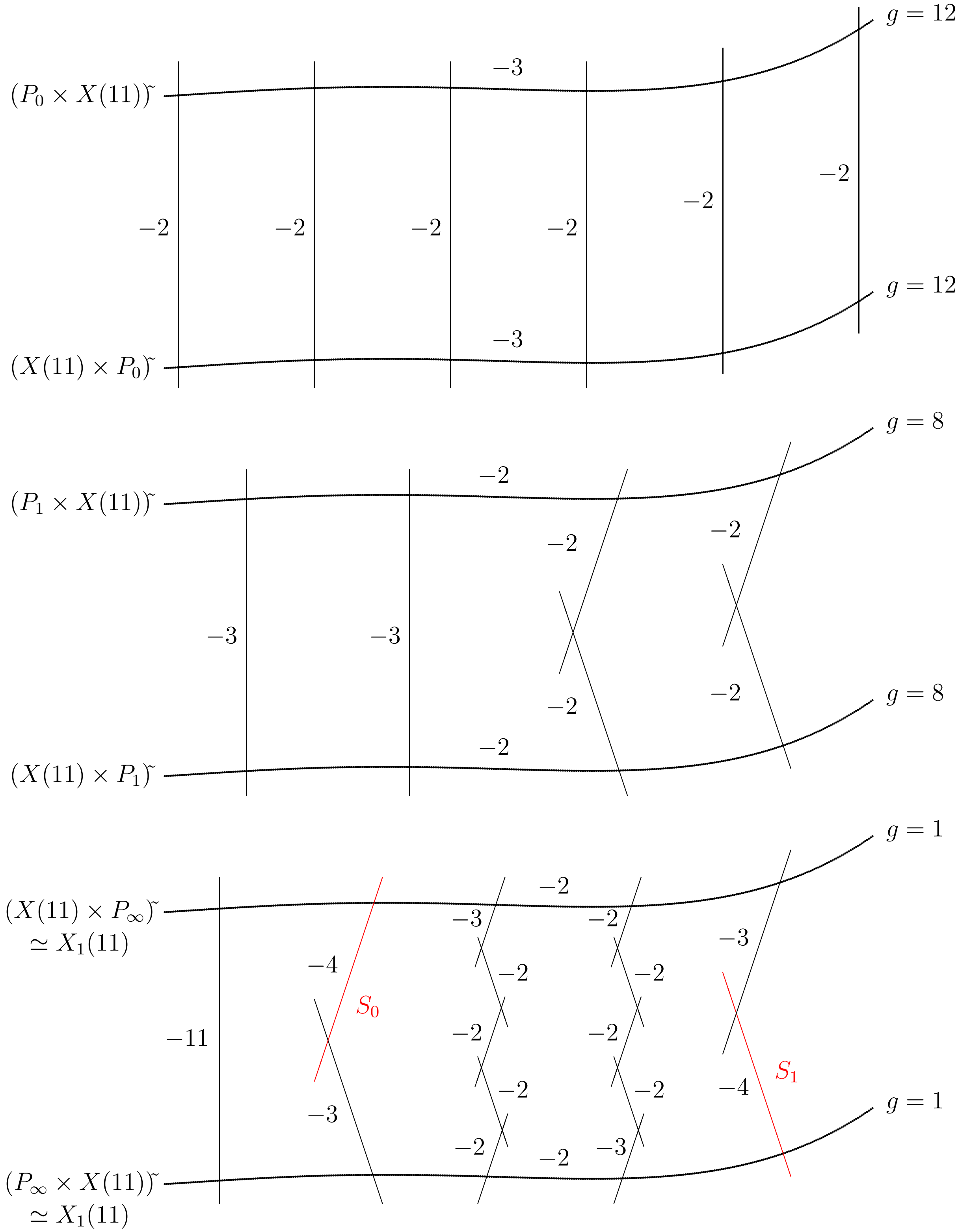
Furthermore, the 5 singularities at the cusps have the following types:

Type	r	Continued Fraction
$(11, 1)$	1	$\frac{11}{1} = [11]$
$(11, 3)$	2	$\frac{11}{3} = [4, 3]$
$(11, 4)$	2	$\frac{11}{4} = [3, 4]$
$(11, 5)$	5	$\frac{11}{5} = [3, 2, 2, 2, 2]$
$(11, 9)$	5	$\frac{11}{9} = [2, 2, 2, 2, 3]$

For example,

$$\frac{11}{3} = 4 - \frac{1}{3}, \qquad \frac{11}{5} = 3 - \cfrac{1}{2 - \cfrac{1}{2 - \cfrac{1}{2 - \frac{1}{2}}}},$$

Basic Curves on the Surface $\tilde{Z}_{11,1}$



4. The Geometry of $Z_{N,\varepsilon}$

Recall: For any smooth projective variety V/K of dimension n, its geometric genus is the dimension

$$p_g(V) = \dim_K \Omega^n_{V/K}$$

of the space $\Omega^n_{V/K}$ of holomorphic n-forms on V.

Theorem 4 (C.F. Hermann; K.-Schanz): The rough classification type of the desingularization $\tilde{Z}_{N,\varepsilon}$ of $\overline{Z}_{N,\varepsilon}$ is completely determined by its geometric genus $p_g = p_g(\tilde{Z}_{N,\varepsilon})$:

$$\begin{array}{lcl} \tilde{Z}_{N,\varepsilon} & \text{is a rational surface} & \text{if } p_g = 0, \\ \tilde{Z}_{N,\varepsilon} & \text{is a blown-up K3 surface} & \text{if } p_g = 1, \\ \tilde{Z}_{N,\varepsilon} & \text{is an honestly elliptic surface} & \text{if } p_g = 2, \\ \tilde{Z}_{N,\varepsilon} & \text{is a surface of general type} & \text{if } p_g \geq 3. \end{array}$$

In particular, its Kodaira dimension is

$$\kappa(\tilde{Z}_{N,\varepsilon}) = \min(p_g - 1, 2).$$

Remark: One has a natural identification

$$(1) \quad \Omega^2_{\tilde{Z}_{N,\varepsilon}/\mathbb{C}} \simeq (\Omega^1_{X(N)/\mathbb{C}} \otimes \Omega^1_{X(N)/\mathbb{C}})^{\Delta_{N,\varepsilon}},$$

and this leads to an explicit formula for $p_g(\tilde{Z}_{N,\varepsilon})$:

$$p_g = \mathbb{G}_{N,\varepsilon} - \mathbb{S}_{N,\varepsilon},$$

where

$$\mathbb{G}_{N,\varepsilon} = \frac{m(N-12)}{144N} - 1 + \tfrac{1}{8}\phi(N) + \tfrac{1}{8}r_0 + \tfrac{1}{6}r_1 + \tfrac{1}{4}r_\infty$$
$$\mathbb{S}_{N,\varepsilon} = \tfrac{1}{18}(|\mathcal{S}^+_{N,1,\varepsilon}| - |S^-_{N,1,\varepsilon}|) + \mathbb{S}_{N,\infty,\varepsilon},$$

in which $m = \frac{1}{2}|\mathrm{SL}_2(\mathbb{Z}/N\mathbb{Z})|$, $r_k = |\mathcal{S}_{N,k,\varepsilon}|$ for $k = 0, 1, \infty$ and $\mathbb{S}_{N,\infty,\varepsilon}$ is a sum of certain Dedekind sums.

In fact, all the Betti and Hodge numbers of $\tilde{Z}_{N,\varepsilon}$ have similar explicit expressions.

However: No explicit formula is known for any of the plurigenera

$$P_n(\tilde{Z}_{N,\varepsilon}) = \dim H^0(\tilde{Z}_{N,\varepsilon}, \omega^{\otimes n}), \text{ for } n \geq 2$$

where $\omega = \omega_{\tilde{Z}_{N,\varepsilon}}$ denotes the canonical sheaf on $\tilde{Z}_{N,\varepsilon}$. (Of course, $P_1 = p_g$ is known by the above formula.)

In particular, one does not know how many exceptional curves on $\tilde{Z}_{N,\varepsilon}$ have to be blown down so as to arrive at the minimal model.

Corollary: $\tilde{Z}_{N,\varepsilon}$ is of general type $\forall \varepsilon \Leftrightarrow N \geq 13$.

Remark: A complete list of all (N, ε) for which the surface $\tilde{Z}_{N,\varepsilon}$ is special (i.e. rational, K3 or elliptic) is given in K.-Schanz. In particular:

The surface $\tilde{Z}_{7,1}$ is rational $\rightarrow$ Halberstadt-Kraus.

The surface $\tilde{Z}_{11,1}$ is elliptic $\rightarrow$ K.-Rizzo.

Note: In a recent paper, D. Carlton introduced the notion of cusp forms (in two variables) of weight (k_1, k_2) on Z_N and uses the isomorphism (1) to establish the isomorphism

$$S_{(2,2)}(Z_N) \simeq \bigoplus_{\varepsilon} \Omega^2_{\tilde{Z}_{N,\varepsilon}/\mathbb{C}}.$$

He further shows that the Hecke Algebra $\mathbb{T}_N \otimes \mathbb{T}_N$ naturally acts on the space $S_{(k_1,k_2)}(Z_N)$ of cusp forms of weight (k_1, k_2) and develops a (weak version of) Atkin-Lehner Theory for such cusp forms.

5. Modular Curves on $Z_{N,\varepsilon}$

Recall: The modular curve $Y_0(n) = X_0(n) \setminus \{\text{cusps}\}$ coarsely represents the functor

$$K \to \mathcal{Y}_0(n)(K) = \{(E, E', f)_{/K} : E \xrightarrow{f} E' \text{ cyclic}, \ \deg(f) = n\}.$$

Thus: if n, k satisfy $(nk, N) = 1$, then the rule

$$(E, E', f) \mapsto (E, E', (kf)_{|E[N]})$$

defines a morphism of functors and of varieties:

$$\tau_{n,k} : \mathcal{Y}_0(n) \to \mathcal{Z}_N \text{ and } \tau_{n,k} : X_0(n) \to \bar{Z}_N.$$

We call the image

$$\bar{T}_{n,k} = \mathrm{Im}(\tau_{n,k}) \subset \bar{Z}_{N,\varepsilon}$$

a modular or Hecke curve; here $\varepsilon \equiv nk^2 (\mathrm{mod}\, N)$.

Remarks: 1) The map $\tau_{n,k} : X_0(n) \to \bar{T}_{n,k}$ is finite and birational. If $\tilde{T}_{n,k}$ denotes the total transform of $\bar{T}_{n,k}$ on $\tilde{Z}_{N,\varepsilon}$, then $\tilde{T}_{n,k}$ has ordinary singularities on $\tilde{Z}_{N,\varepsilon} \setminus \{\text{cuspidal curves}\}$, and singularities of type $x^a = y^b$ "at the cusps".

2) In the finite part, the curves $\tilde{T}_{n,k}$ intersect only at CM-points, and their intersection multiplicity can be computed in terms of representation numbers of binary quadratic forms and/or as sums of class numbers ($\longrightarrow$ Hurwitz).

3) From the above modular description it is clear that the $\tilde{T}_{n,k}$'s are defined over $\mathbb{Q}$.

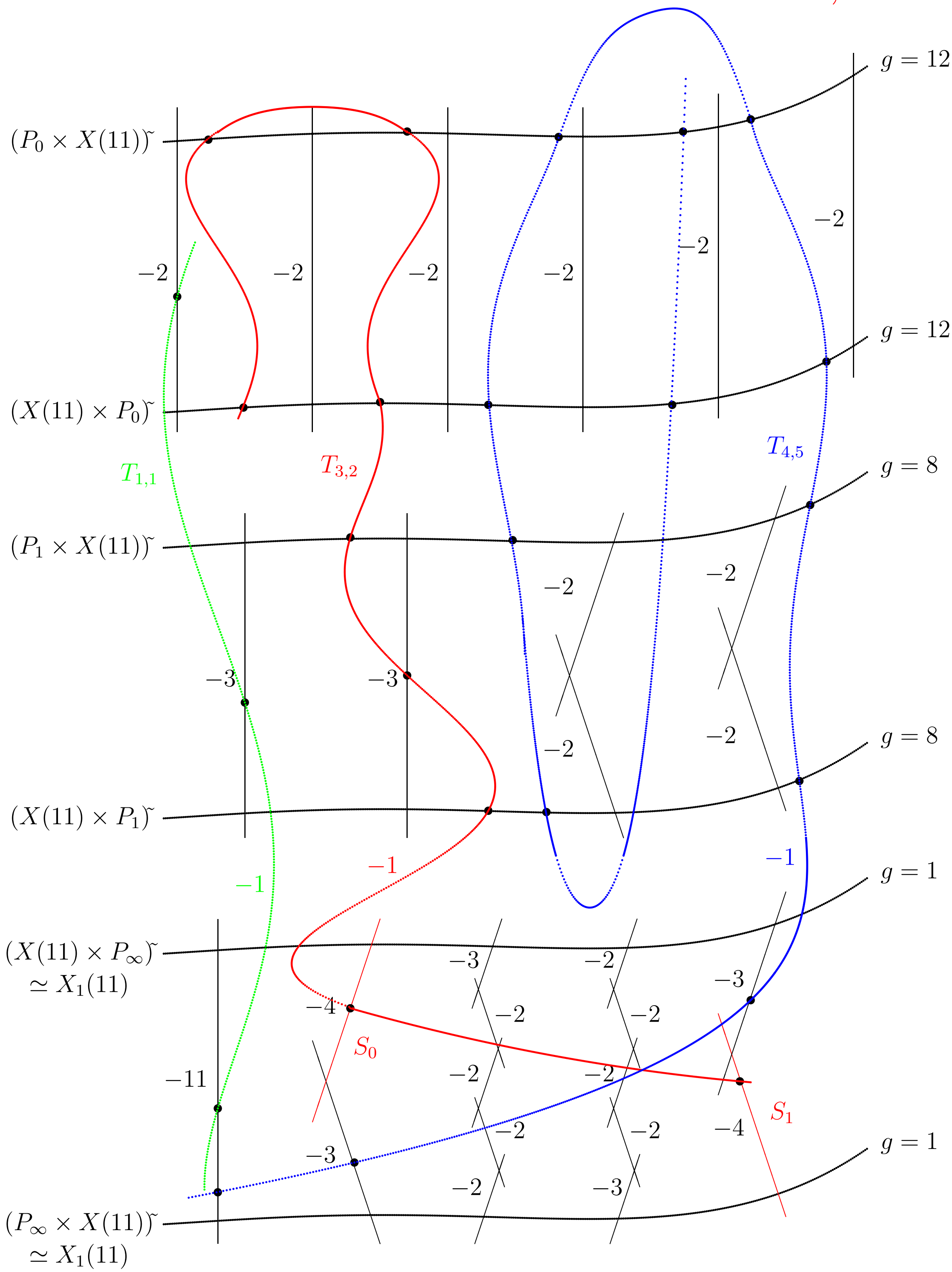
4) As the name suggests, the $\tilde{T}_{n,k}$'s are closely connected to the modular (Hecke) correspondences T_n on $Y(N) := X(N) \times X(N)$ which were already studied by Klein, Gierster, Hurwitz in 1880ff:

$$
\begin{array}{ccccccl}
 & & T_n & & & \rightsquigarrow & T_n \subset Y = X(N) \times X(N) \\
 & {\scriptstyle p_n} \swarrow & & \searrow {\scriptstyle p_n \circ w_n} & & & \\
X(N) & & \downarrow & & X(N) & \rightsquigarrow & T_{n,k} = (\langle k \rangle \times id) T_n \subset Y \\
\downarrow & & X_0(n) & & \downarrow & \Delta_\varepsilon & \downarrow \\
 & \swarrow & & \searrow & & & \\
X(1) & & & & X(1) & & \bar{T}_{n,k} \subset Z = \Delta_\varepsilon \backslash Y
\end{array}
$$

Example: The case $(N, \varepsilon) = (11, 1)$:

(n,k)	$p_a(\tilde{T}_{n,k})$	$g(X_0(n))$	$T^2_{n,k}$	Notes
$(1,1)$	0	0	-1	*
$(3,2)$	0	0	-1	*
$(4,5)$	0	0	-1	*
$(5,3)$	0	0	-2	
$(9,4)$	0	0	-2	
$(12,1)$	0	0	-2	
$(14,2)$	1	1	-2	
$(15,5)$	1	1	-2	
$(16,3)$	0	0	-2	
$(20,4)$	1	1	-2	
$(23,1)$	2	2	0	
$(25,2)$	0	0	-2	
$(26,5)$	2	2	0	
$(27,3)$	1	1	-2	
$(31,4)$	3	2	0	Not smooth
$(34,1)$	5	3	4	Not smooth
$(36,2)$	4	1	2	Not smooth
$(37,5)$	4	2	2	Not smooth

Curves on the Surface $\tilde{Z}_{11,1}$



6. Intersection Numbers of Modular Curves

Fact: Distinct Hecke curves meet only at cusps and at CM-points.

Local Intersection Numbers: Let $\mathcal{R}_q(n)$ be the set of primitive representations of n by q, i.e.,

$$\mathcal{R}_q(n) = \{(x, y) \in \mathbb{Z}^2 : q(x, y) = n, \gcd(x, y) = 1\},$$

and put

$$\begin{aligned} & r_q(n, k, n', k'; N) \\ & \quad = \#\{\vec{v} \times \vec{v}' \in \mathcal{R}_n \times \mathcal{R}_n : k\vec{v} \equiv k'\vec{v}' \pmod{N}\}. \end{aligned}$$

Then the local intersection number (at the CM-points associated to (E, E')) is

$$\begin{aligned} (T_{n,k} \cdot T_{n',k'})_{(E,E')} &:= \sum_{\psi} (T_{n,k} \cdot T_{n',k'})_{(E,E',\psi)} \\ &= \frac{1}{e_q} r_q(n, k, n', k'; N), \end{aligned}$$

where $q = q_{E,E'}$ and $e_q = |\mathrm{Aut}(q)|$.

Thus we have

$$(T_{n,k} \cdot T_{n',k'})_{fin} = \sum_q \frac{c_q}{e_q} r_q(n, k, n', k'; N),$$

If, instead of the irreducible curve $\overline{T}_{n,k}$, we consider the divisor

$$\hat{T}_{n,k} = \sum_{d^2|n} \overline{T}_{n/d^2,k/d},$$

then the intersection numbers become simpler:

Theorem 5: If $n \equiv k^2 \pmod{N}$, then

$$(2) \qquad (\hat{T}_{1,1}.\hat{T}_{n,k}) = H_{N^2,k}(n) + I_{N^2,k}(n),$$

where

$$I_{N^2,k}(n) = \frac{1}{N} \sum_{\substack{d_1 d_2 = n \\ d_1 \equiv d_2 \equiv \pm k\,(N)}} \min(d_1, d_2).$$

$$H_{N^2,k}(n) = \sum_{x \in S_{N^2}(n,k)} H\left(\frac{4n - x^2}{N^2}\right),$$

with

$$S_{N^2}(n,k) = \{x \in S_{N^2}(n) : x \equiv \pm 2k + \tfrac{4n-x^2}{N}\ (2N)\},$$

$$S_{N^2}(n) = \{x \in \mathbb{Z} : x^2 \leq 4n,\ x^2 \equiv 4n\ (N^2)\}.)$$

Furthermore, if $n > 0$, then $H(n)$ denotes the (weighted) number of classes of positive definite binary quadratic forms of discriminant $-n$, i.e.

$$H(n) = \sum_{f^2|n} \hat{h}(-n/f^2),$$

where

$$\hat{h}(\Delta) = \begin{cases} 2|Cl(\mathcal{O}_\Delta)|/|\mathcal{O}_\Delta^\times|, & \text{if } \Delta \equiv 0,\ 1 \pmod 4, \\ 0 & \text{otherwise;} \end{cases}$$

here $\mathcal{O}_\Delta \subset \mathbb{Q}(\sqrt{\Delta})$ denotes the order of discriminant Δ. In addition, we put $H(0) = -\frac{1}{12}$.

Remarks. 1) This formula generalizes a formula of Gierster/Hurwitz (who considered only the case that $N = p$ is prime). It corrects a (trace) formula of Eichler (1967) (which is incorrect when $8|N$).

2) From the above formula one can compute arbitrary intersection numbers between the $\hat{T}_{n,k}$'s and hence between the $T_{n,k}$'s. Indeed,

$$(\hat{T}_{n,k}.\hat{T}_{m,l}) = \sum_{d|(n,m)} (\hat{T}_{1,1}.\hat{T}_{mn/d^2,ln/(kd)}),$$

if $nk^{-2} \equiv ml^{-2} \pmod N$, and is 0 otherwise.

3) The above intersection numbers of curves on $Z_{N,\varepsilon}$ are in general rational numbers (since $Z_{N,\varepsilon}$ is singular). The corresponding intersection numbers on $\tilde{Z}_{N,\varepsilon}$ require a small correction term.

7. Some Geometric Conjectures

Conjecture 1: If $N \geq 23$ is prime, then every curve C on $\bar{Z}_{N,\varepsilon}$ of genus $g(C) \leq 1$ is modular, i.e. $C = \bar{T}_{n,k}$, for some n, k.

Conjecture 2: (Hermann, 1991) If $N \geq 7$, then the minimal model $\tilde{Z}^{min}_{N,\varepsilon}$ of $\tilde{Z}_{N,\varepsilon}$ is obtained by blowing down "known curves".

Remarks. 1) Conjecture 1 $\Rightarrow$ Conjecture 2 (if $N \geq 23$ is prime).

2) Conjecture 2 $\Leftrightarrow$ explicit formula for $P_2(\tilde{Z}_{N,\varepsilon})$.

3) Conjecture 2 is a natural analogue of a Conjecture of Hirzebruch/Van de Ven/Zagier for Hilbert modular surfaces; this latter conjecture was proven by C.F. Hermann in 1987 in many cases. His method also yields:

Theorem 6 (Hermann) If $N \equiv 7 \pmod 8$ is prime and $\varepsilon \equiv -1 \pmod N$, then Conjecture 2 is true.

Theorem 7: Conjecture 2 is true for $N \leq 13$.

8. Applications to Mazur's Question

Question: To what extent is the isogeny class of E/K determined by the isomorphism class of $\bar{\rho}_{E/K,N}$?

Mazur (1978): $\exists$? E and $E'/\mathbb{Q}$ with $E \not\sim E'$ such that $\bar{\rho}_{E/K,N} \simeq \bar{\rho}_{E'/K,N}$ for some $N \geq 7$?

Kraus-Oesterlé (1992): Yes! (for $N = 7$).

Frey + group ($\sim$ 1993): Computer search: lots of examples for $N = 7, 11$.

Halberstadt-Kraus (1997): $\exists$ ∞'ly many examples for $N = 7$.

K.-Rizzo (1999): $\exists$ ∞'ly many families of examples for $N = 11$.

Note: Faltings' Theorem (=Mordell Conjecture) $\Rightarrow$ $\mathbb{S}_{N,E}(K) \stackrel{\text{def}}{=} \{E'/K : \bar{\rho}_{E'/K,N} \simeq \bar{\rho}_{E/K,N}, \}/\simeq$ is finite, for all $N \geq 7$.

Conjecture 3 (Frey, 1988): $\exists$ a constant $M_{E,K}$ s. th. $\mathbb{S}'_{N,E}(K) := \{E' \in \mathbb{S}_{N,E}(K) : E' \not\sim E\} = \phi$, for $N \geq M_{E,K}$.

Theorem (Frey, 1996): For $K = \mathbb{Q}$, Conjecture 3 is equivalent to the Asymptotic Fermat Conjecture:

(AFC) For every $a, b, c \in \mathbb{Z}, abc \neq 0$, the set

$$F_{a,b,c} = \bigcup_{n \geq 4} \{(x_n, y_n, z_n) \in \mathbb{Z}^3 : ax_n^n + by_n^n = cz_n^n, \quad (x_n, y_n, z_n) = 1\}$$

is finite.

Conjecture 4 (Darmon, 1994): $\exists$ constant M_K s. th.

$$\mathbb{S}'_N(K) := \bigcup_{E/K} \mathbb{S}'_{N,E}(K) = \phi, \quad \forall N \geq M_K.$$

Conjecture 5 (Darmon, 1994): $\exists$ constant M s. th.

$$\#(\mathbb{S}'_N(K)/\text{twists}) < \infty, \quad \forall N \geq M.$$

Conjecture 5′: Conjecture 5 is true for $M = 23$.

Remark: Via Lang's Conjecture, Conjecture 5' is essentially equivalent to the (geometric) Conjecture 1. Indeed, if $P = (E, E', \psi) \in Z_N(K)$, then $E \sim E'$ if and only if P lies on (a twist of) some $T_{n,k}$.

On the other hand, for small N's one does not expect that such finiteness statements are true. For example, this is false for $N = 11$ because we have

Theorem 8 (K. - Rizzo) Let Z_{min} denote the minimal model of (the desingularization of) the modular diagonal quotient surface $\bar{Z} = (\bar{Z}_{11,1})_{/\mathbb{Q}}$. Then the canonical map defines an elliptic fibration

$$f_{can} : Z_{min} \to \mathbb{P}^1_{\mathbb{Q}}$$

which has an infinite number of sections

$$S_i : \mathbb{P}^1_{\mathbb{Q}} \to Z_{min}.$$

Notes: 1) The surface Z_{min} is obtained from the minimal desingularization $\tilde{Z}$ of $\bar{Z}$ by blowing down 5 (explicit) curves (cf. K.-Schanz).

2) The sections S_i are constructed as follows. We start with two explicit sections S_0 and S_1 of f_{can} which arise as components of the desingularization curves of the quotient singularities at the cusps of $\bar{Z}$. Since these meet on Z_{min}, S_1 defines a point of infinite order on the associated elliptic curve (taking S_0 as the origin), and so the i-th multiples S_i of S_1 define infinitely many distinct sections of f_{can}.

Corollary. There exist infinitely many one-parameter families of (isomorphism classes of) pairs (E, E') of non-isogenous elliptic curves E, E' defined over $\mathbb{Q}$ whose associated Galois representations mod 11 are symplectically isomorphic:

$$\bar{\rho}_{E/\mathbb{Q},11} \simeq \bar{\rho}_{E'/\mathbb{Q},11}.$$

Note: By a one-parameter family of such pairs (E, E') we mean that the associated j-invariants of E and E' depend rationally on a parameter $t \in \mathbb{Q}$.

9. Néron-Severi Groups

Recall: The geometric Conjectures 1 and 2 ask whether all curves on Z_N of a certain type are modular. Somewhat related to this is the following question.

Question. How large is the subgroup generated by the modular curves in the Néron-Severi group of $Z_{N,\varepsilon}$?

Recall: The Néron-Severi Group of a surface Z is

$$\mathrm{NS}(Z) = \mathrm{Div}(Z)/\mathrm{Div}^0(Z),$$

where $\mathrm{Div}^0(Z)$ denotes the subgroup of divisors $D \equiv 0$ which are algebraically equivalent to 0. Recall that $NS(Z)$ is finitely generated (Theorem of the Base) and that hence $\mathrm{NS}^0(Z) := \mathrm{NS}(Z) \otimes \mathbb{Q}$ is a finite-dimensional $\mathbb{Q}$-vector space.

Remark: A similar question was asked by Hirzebruch/Zag for the HMS's. Harder/Langlands/Rapoport gave a partial answer to this in 1986, which was then completed in 1987 by K.Murty/D.Ramanakrishnan and by C. Klingenberg (independently).

Notation: Let $\mathrm{NS}^0(Z_{N,\varepsilon}, \mathbb{T}) \subset \mathrm{NS}^0(Z_{N,\varepsilon})$ denote the subspace generated by all modular curves $T_{n,k}$ (together with the "basic curves"). Furthermore, let

$$\mathrm{NS}^0(Z_{N,\varepsilon}, \mathbb{M}) \subset \mathrm{NS}^0(Z_{N,\varepsilon} \otimes \overline{\mathbb{Q}})$$

denote the subspace generated by all the twisted modular curves $\varphi((id \times g)(T_n))$ (together with the "basic curves").

Remark: Let $\mathrm{NS}^0(Z_{N,\varepsilon}, \mathbb{B}) \simeq \mathbb{Q}^2$ denote the subgroup generated by the "basic curves". Then there is a natural embedding

$$\gamma : \mathrm{NS}^0(Z_{N,\varepsilon} \otimes \overline{\mathbb{Q}})/\mathrm{NS}^0(Z_{N,\varepsilon}, \mathbb{B}) \hookrightarrow \mathbb{E},$$

where $\mathbb{E} := \mathrm{End}^0(J(N))$ denotes the endomorphism algebra of the Jacobian $J(N) = J_{X(N)\otimes\overline{\mathbb{Q}}}$ of the curve $X(N) \otimes \overline{\mathbb{Q}}$.

Theorem 9: If $\varepsilon = 1$ and $N \geq 3$ then

$$\begin{aligned}
\gamma(\mathrm{NS}^0(Z_{N,\varepsilon} \otimes \overline{\mathbb{Q}})) &= C_{\mathbb{E}}(G) \\
\gamma(\mathrm{NS}^0(Z_{N,\varepsilon})) &= C_{\mathbb{E}}(G)^{G_{\mathbb{Q}}} \\
\gamma(\mathrm{NS}^0(Z_{N,\varepsilon}, \mathbb{M})) &= C_{\mathbb{M}}(G) \\
\gamma(\mathrm{NS}^0(Z_{N,\varepsilon}, \mathbb{T})) &= Z(\mathbb{M})^{sym},
\end{aligned}$$

where $\mathbb{M} = \langle G, \mathbb{T} \rangle \subset \mathbb{E}$ denotes the algebra of all modular correspondences.

Remark: The first three assertions of Theorem 9 are relatively straightforward, but the last requires a deeper understanding of the algebra $\mathbb{M}$.

Theorem 10: If $\varepsilon = 1$ and $N = p$ is prime, then

$$\begin{aligned}
\dim Z(\mathbb{M})^{sym} &= \tfrac{1}{24}(p-1)(p-5) + \tfrac{1}{2}h(p), \\
\dim C_{\mathbb{M}}(G) &= \dim Z(\mathbb{M})^{sym} + \tfrac{1}{2}(y + h(p)), \\
\dim C_{\mathbb{E}}(G) &= \dim C_{\mathbb{M}}(G) + 2h(p)(h(p)-1), \\
\dim C_{\mathbb{E}}(G)^{G_{\mathbb{Q}}} &= \dim Z(\mathbb{M})^{sym} \; (\text{if } p \equiv 1(4)),
\end{aligned}$$

where

$$h(p) = \begin{cases} h(\mathbb{Q}(\sqrt{-p})) & \text{if } p \equiv 3 \pmod 4 \\ 0 & \text{if } p \equiv 1 \pmod 4, \end{cases}$$

$$y = \tfrac{1}{2}\left(g(X_0(p)) - \tfrac{1}{2}\left(\tfrac{-1}{p}\right)\left(1 + \left(\tfrac{2}{p}\right)\right)\right).$$

Thus: If $N = p$ (and $p \equiv 1(4)$), then

$$\mathrm{NS}^0(Z_{p,1}) = \mathrm{NS}^0(Z_{p,1}, \mathbb{T}),$$

i.e. the modular curves generate the Neron-Severi group of $Z_{p,1}/\mathbb{Q}$. Furthermore:

$$\begin{array}{cl}
\mathrm{NS}^0(Z_{p,1} \otimes \overline{\mathbb{Q}}) & \\
| & \Big\} \; 2h(p)(h(p)-1) \\
\mathrm{NS}^0(Z_{p,1}, \mathbb{M}) & \\
| & \Big\} \; \tfrac{1}{2}(y + h(p)) \\
\mathrm{NS}^0(Z_{p,1}) &
\end{array}$$

Remark: The case $\varepsilon \neq 1$ "can be reduced to" the case $\varepsilon = 1$.

10. Zeta-functions

The methods of Theorem 1 generalize to prove:

Theorem 1′: The functors $\mathcal{Z}_N$ and $\mathcal{Z}_{N,\varepsilon}$ are coarsely representable by normal affine surfaces Z_N and $Z_{N,\varepsilon}$ over $\mathbb{Z}[\frac{1}{N}]$ which have natural compactifications $\bar{Z}_N$ and $\bar{Z}_{N,\varepsilon}$ (which are proper over $\mathbb{Z}[\frac{1}{N}]$). Moreover, the $\bar{Z}_{N,\varepsilon}$ are the connected components of $\bar{Z}_N$, and each $\bar{Z}_{N,\varepsilon}$ has geometrically irreducible fibres.

Theorem 11: (S. Mohit, 2001) If $N = p$ is a prime, then the zeta-function of the arithmetic threefold $\bar{Z}_N$ has the form:

$$\zeta(\bar{Z}_N, s) = F_N(s)\mathcal{L}(s)$$

where $F_N(s)$ is a product of Riemann zeta-functions and

$$\mathcal{L}(s) = \prod_{(f_1,f_2)} L_N(f_1 \times f_2, s)$$

is a product of Rankin convolutions associated to (certain) pairs (f_1, f_2) of $\mathbb{T}_N$-eigenfunctions $f_i \in S_2(X(N))$.

Consequences: 1) $\zeta(\bar{Z}_N, s)$ has a meromorphic analytic continuation to $\mathbb{C}$.

2) S. Mohit can compute the order of the zero/pole of $\zeta(\bar{Z}_N, s)$ at $s = 2$ and can thus compare it to the rank of the Néron-Severi group of $\bar{Z}_N$

$\Rightarrow$ Tate's Conjecture for $\bar{Z}_N$.