Abelian Subvarieties and the Shimura Construction

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Abstract. In 1971 Shimura showed that each weight 2 Hecke eigenfunction f gives rise to both an abelian subvariety and an abelian quotient of the Jacobian variety of the modular curve $X_1(N)/\mathbb{Q}$. The purpose of this paper is to show that both these constructions follow from a general "dictionary" that translates statements about subvarieties and quotients of an abelian variety into statements about ideals of the associated endomorphism algebra. This dictionary is, in fact, a special case of more general dictionary which applies to subobjects and quotients in a general semi-simple abelian category.

Keywords: Abelian variety, abelian category, endomorphism algebra, modular forms, subobject.

1 Introduction

In his fundamental book, Shimura [Sh1] showed that each (Hecke) $\mathbb{T}$-eigenfunction $f \in S_2(N)$ on $\Gamma_1(N)$ gives rise to an abelian subvariety $A'_f \subset J_1(N)$ of the Jacobian variety of the modular curve $X_1(N)/\mathbb{Q}$, and in a subsequent paper [Sh2] he explained that such Hecke eigenfunctions give more naturally rise to *quotient varieties* A_f of $J_1(N)$. The purpose of this paper is to show that both these constructions (and more) follow from a general "dictionary" that translates statements about subvarieties of abelian varieties into statements about ideals of the associated endomorphism algebras.

To explain this more precisely, let A be an abelian variety over an arbitrary field K, and let $\mathbf{Sub}(A/K) = \{B \leq A\}$ denote the set of abelian subvarieties B of A (which are defined over K). Then the aforementioned dictionary translates this set into the set $\mathbf{Id}_{\mathbb{E}}$ of right ideals of the endomorphism algebra $\mathbb{E} = \mathrm{End}_K(A) \otimes \mathbb{Q}$ of A/K as follows:

Theorem 1.1 *The map $B \mapsto I(B) := \{f \in \mathbb{E} : \mathrm{Im} f \subset B\}$ defines an inclusion-preserving bijection*

$$I_{A/K} : \mathbf{Sub}(A/K) \xrightarrow{\sim} \mathbf{Id}_{\mathbb{E}}$$

between the set of abelian subvarieties of A/K and the set of right ideals of $\mathbb{E} = \mathrm{End}^0_K(A)$. Furthermore, if $B_1, B_2 \in \mathbf{Sub}(A/K)$ are any two abelian subvarieties, then there is a canonical (functorial) isomorphism

$$\mathrm{Hom}^0(B_1, B_2) := \mathrm{Hom}_K(B_1, B_2) \otimes \mathbb{Q} \xrightarrow{\sim} \mathrm{Hom}_{\mathbb{E}}(I(B_1), I(B_2)).$$

This theorem is deduced in §3 from a general fact (Theorem 2.7) about semi-simple abelian categories which is presented in §2. It is interesting to note that both Theorem A of [KR] (cf. Remark 3.2(a)) and a result of Lange[Lan] (Corollary 3.5) are easily deduced from this theorem.

Another consequence of Theorem 1.1 is the following. If V is any faithful (left) $\mathbb{E}$-module, then there is a natural bijection between $\mathbf{Sub}(A/K)$ and certain *algebraic subspaces* of V; cf. Theorem 4.1. In the case that V is finitely generated, this yields (via the *Morita theorems*) the following result.

Corollary 1.2 *Let V be a faithful, finitely generated left $\mathbb{E}$-module. Then the map $B \mapsto I(B)V$ defines an inclusion-preserving bijection*

$$S_{A/K,V} : \mathbf{Sub}(A/K) \xrightarrow{\sim} \mathbf{Sub}_{\tilde{\mathbb{E}}}(V)$$

between the set of abelian subvarieties of A/K and the set of left $\tilde{\mathbb{E}}$-submodules of V, where $\tilde{\mathbb{E}} = \mathrm{End}_{\mathbb{E}}(V)$. Thus, for each left $\tilde{\mathbb{E}}$-submodule $W \subset V$ there is a unique abelian subvariety $B_W \subset A$ such that $S_{A/K,V}(B_W) = W$, and we have a ring isomorphism $\theta_W : \mathrm{End}_{\tilde{\mathbb{E}}}(W) \xrightarrow{\sim} \mathrm{End}^0(B_W)$.

Two natural examples for which this result can be applied are $V = H_1(A^{an}, \mathbb{Q})$, the first *homology group* of the underlying analytic space A^{an} (if $K = \mathbb{C}$) and $V = T_0(A)$, the *tangent space* of A at 0 (if $K = \mathbb{Q}$). Now it is the second example which gives rise to Shimura's (first) construction because the latter may be deduced from the following general statement.

Corollary 1.3 *Let $A/\mathbb{Q}$ be an abelian variety, and suppose that there exists a commutative subring $\mathbb{T} \subset \mathbb{E} = \mathrm{End}^0(A)$ such that both $T_0(A)$ and $T_0(A)^*$ are free $\mathbb{T}$-modules of rank 1. Fix a $\mathbb{T}$-module isomorphism $\varphi : T_0(A)^* \xrightarrow{\sim} T_0(A)$, and let $W \subset T_0(A)^*$ be a $\mathbb{T}$-submodule. Then $\varphi(W) \subset T_0(A)$ does not depend on the choice of φ and there is a unique abelian subvariety $B_W \in \mathbf{Sub}(A/\mathbb{Q})$ such that $T_0(B_W) = \varphi(W)$. Moreover, $\dim B_W = \dim_{\mathbb{Q}} W$.*

Indeed, as will be explained in section 5, this can be applied directly to the case $A = J_1(N)$, the Jacobian of the modular curve $X_1(N)$; cf. Theorem 5.1.

As Shimura explained in [Sh2], it is, however, more natural to work with *abelian quotients* of A in place of abelian subvarieties because then we can work with the dual module $\Omega(A) = H^0(A, \Omega^1_{A/K}) \simeq T_0(A)^*$ directly. More precisely, we have the following result (which is a special case of Corollary 4.4):

Theorem 1.4 *If A is an abelian variety over a number field K and $p : A \to C$ is a quotient of A, then the assignment $(C, p) \mapsto p^*\Omega(C) \subset \Omega(A) := H^0(A, \Omega^1_{A/K})$ induces a bijection*

$$Q_{A/K,\Omega} : \mathbf{Quot}(A/K) \xrightarrow{\sim} \mathbf{Sub}_{\tilde{\mathbb{E}}}(\Omega(A))$$

between the set of abelian quotients of A/K *and the set of left* $\tilde{\mathbb{E}}$*-submodules of* $\Omega(A)$*, where* $\tilde{\mathbb{E}} = \mathrm{End}_{\mathbb{E}}(\Omega(A))$*. In particular, if there exists a commutative subring* $\mathbb{T} \subset \mathbb{E}$ *such that* $\Omega(A)$ *is a free* $\mathbb{T}$*-module of rank* 1*, then for every* $\mathbb{T}$*-submodule* $W \subset \Omega(A)$ *there exists a unique abelian quotient* $p_W : A \to C_W$ *such that* $p_W^*\Omega(C_W) = W$*. In addition,* $\dim C_W = \dim_K W$*.*

As will be explained in section 5, the above theorem applies in particular to the case that $A = J_1(N)$ is the Jacobian of the modular curve $X_1(N)/\mathbb{Q}$ and $\mathbb{T} = \mathbb{T}_{\mathbb{Q}}$ is the Hecke algebra of $J_1(N)$. We thus obtain a different characterization (and proof) of the (second) Shimura construction [Sh2].

Moreover, we can use the above dictionary to construct other interesting subvarieties and/or abelian quotients of $J_1(N)$. These will be used to show how the Atkin-Lehner decomposition of $S_2(\Gamma, \mathbb{Q})$ gives rise to an isogeny decomposition of $J_1(N)$ of the form

$$J_1(N) \sim \prod_{f \in \mathcal{N}(\Gamma_1(N))/G_{\mathbb{Q}}} A_f^{n_f}. \tag{1}$$

Here $\mathcal{N}(\Gamma_1(N))$ denotes the set of all weight 2 newforms (of all levels) on $\Gamma_1(N)$, A_f is the abelian variety attached to (the Galois orbit of) $f \in \mathcal{N}(\Gamma_1(N))$ by the Shimura construction, and $n_f = \sigma_0(N/N_f)$ denotes the number of divisors of N/N_f, where $N_f|N$ denotes the level of the newform f.

This paper is organized as follows. In section 1 we shall explain the general categorical setting for the above results and prove them in that generality. Thus, all the above results also hold for *motives*; cf. [Ja]. In sections 3 and 4 we shall specialize them to the case of abelian varieties, and in section 5 we shall apply them to generalize the results of Shimura [Sh1], [Sh2] for Jacobians of modular curves.

2 The categorical background

Let $X \in \mathrm{ob}(\mathcal{C})$ be an object of a category $\mathcal{C}$, and let $\mathbf{Sub}_{\mathcal{C}}(X) = \mathrm{Sub}_{\mathcal{C}}(X)/\!\sim$ denote the classes of *subobjects of* X. Here $\mathrm{Sub}_{\mathcal{C}}(X)$ denotes the class of pairs (Y, f) where $Y \in \mathrm{ob}(\mathcal{C})$ and $f : Y \to X$ is a monic (=monomorphism) in $\mathcal{C}$, and $\sim$ is the equivalence relation induced by the pre-order on $\mathrm{Sub}_{\mathcal{C}}(X)$. By definition, the latter is given by $(Y_1, f_1) \leq (Y_2, f_2) \Leftrightarrow f_1 = f_2 g$, for some $g \in \mathrm{Hom}(Y_1, Y_2)$. Thus, $(Y_1, f_1) \sim (Y_2, f_2) \Leftrightarrow (Y_1, f_1) \leq (Y_2, f_2) \leq (Y_1, f_1) \Leftrightarrow f_1 = f_2 g$, for some isomorphism $g : Y_1 \xrightarrow{\sim} Y_2$ (cf. [Mac], p. 122), and the pre-order on $\mathrm{Sub}_{\mathcal{C}}(X)$ induces a partial order on $\mathbf{Sub}_{\mathcal{C}}(X)$.

Similarly, let $\mathbf{Quot}_{\mathcal{C}}(X) = \mathrm{Quot}_{\mathcal{C}}(X)/\!\sim$ denote the class of *quotients of* X, which is defined by duality: $\mathrm{Quot}_{\mathcal{C}}(X) = \mathrm{Sub}_{\mathcal{C}^{op}}(X)$.

In the sequel we shall be particularly interested in the subclass $\mathbf{Sub}'_{\mathcal{C}}(X) = \mathrm{Sub}'_{\mathcal{C}}(X)/\!\sim$ of $\mathbf{Sub}_{\mathcal{C}}(X)$ consisting of the *split subobjects of* X, i.e. $\mathrm{Sub}'_{\mathcal{C}}(X)$ consists of pairs (Y, f) where $f : Y \to X$ is a *split monic* in the sense of [Mac], p. 19: there exists $g : X \to Y$ such that $gf = 1_Y$. Analogously, the class $\mathbf{Quot}'_{\mathcal{C}}(X) = \mathrm{Quot}'_{\mathcal{C}}(X)/\!\sim$ of *split quotients* (or *retracts*) of X is defined. We observe:

Remark 2.1 If $F : \mathcal{C}_1 \to \mathcal{C}_2$ is a functor, then F maps split monics and split epis in $\mathcal{C}_1$ to split monics and split epis in $\mathcal{C}_2$, respectively. Thus, for each $X \in \mathrm{ob}(\mathcal{C}_1)$, we have induced maps

$$F_X : \mathrm{Sub}'_{\mathcal{C}_1}(X) \to \mathrm{Sub}'_{\mathcal{C}_2}(F(X)), \quad {}_XF : \mathrm{Quot}'_{\mathcal{C}_1}(X) \to \mathrm{Quot}'_{\mathcal{C}_2}(F(X)),$$

given by the rules $F_X((Y, f)) = (F(Y), F(f))$ and ${}_XF((Y, f)) = (F(Y), F(f))$. It is clear that F_X is pre-order preserving in the sense that

$$(Y_1, f_1) \leq (Y_2, f_2) \;\Rightarrow\; F_X((Y_1, f_1)) \leq F_X((Y_2, f_2)), \tag{2}$$

for all $(Y_i, f_i) \in \mathrm{Sub}'_{\mathcal{C}_1}(X))$, $i = 1, 2$, because $f_1 = f_2 g \Rightarrow F(f_1) = F(f_2)F(g)$. Similarly, ${}_XF$ is pre-order preserving, and hence F_X and ${}_XF$ induces order-preserving maps

$$\overline{F}_X : \mathbf{Sub}'_{\mathcal{C}_1}(X) \to \mathbf{Sub}'_{\mathcal{C}_2}(F(X)) \quad \text{and} \quad {}_X\overline{F} : \mathbf{Quot}'_{\mathcal{C}_1}(X) \to \mathbf{Quot}'_{\mathcal{C}_2}(F(X))$$

on the equivalence classes. Note that if F is fully faithful, then the converse of (2) holds and hence $\overline{F}_X$ and ${}_X\overline{F}$ are injective in this case. Moreover, if F is an equivalence of categories, then it is easy to see that $\overline{F}_X$ and ${}_X\overline{F}$ are bijections.

We now suppose that $\mathcal{C}$ is a *preadditive* category, so $E_X = \mathrm{End}_{\mathcal{C}}(X)$ is naturally a ring; cf. [Mac], p. 28. In that case we can identify $\mathbf{Sub}'_{\mathcal{C}}(X)$ with a certain subset of the set $\mathbf{Id}_{E_X}$ of right ideals of E_X, and similarly $\mathbf{Quot}'_{\mathcal{C}}(X)$ can be identified with subset of the set ${}_{E_X}\mathbf{Id}$ of left ideals of E_X, as we shall now see.

Proposition 2.2 *Let $X \in \mathrm{ob}(\mathcal{C})$ be an object of a preadditive category $\mathcal{C}$ and put $E_X = \mathrm{End}_{\mathcal{C}}(X)$. Then the rules $(Y, f) \mapsto I_X(Y, f) := f\mathrm{Hom}_{\mathcal{C}}(X, Y)$ and $(Y, f) \mapsto {}_XI(Y, f) := \mathrm{Hom}_{\mathcal{C}}(Y, X)f$ induce order-preserving bijections*

$$I_X : \mathbf{Sub}'_{\mathcal{C}}(X) \xrightarrow{\sim} \mathbf{SpId}^{\mathcal{C}}_{E_X} \quad \textit{and} \quad {}_XI : \mathbf{Quot}'_{\mathcal{C}}(X) \xrightarrow{\sim} {}_{E_X}\mathbf{SpId}^{\mathcal{C}},$$

where $\mathbf{SpId}^{\mathcal{C}}_{E_X}$ (respectively, ${}_{E_X}\mathbf{SpId}^{\mathcal{C}}$) denotes the set of right (respectively, left) ideals of E_X which are generated by split idempotents.

Proof. It is enough to verify these assertions for I_X because those for ${}_XI$ follow from this by duality. Now if $(Y, f) \in \mathrm{Sub}'_{\mathcal{C}}(X)$ is a split subobject with splitting g, i.e. if $gf = 1_Y$, then $\varepsilon_{f,g} := fg$ is a split idempotent of E_X ([Mac], p. 20) and we have

$$I_X(Y, f) \,=\, \varepsilon_{f,g}E_X. \tag{3}$$

Indeed, $I_X(Y,f) \subset \varepsilon_{f,g}E_X$ because $h \in I_X(Y,f) \Rightarrow h = fg'$, $g' \in \mathrm{Hom}(Y,X) \Rightarrow gh = gfg' = g' \Rightarrow \varepsilon_{f,g}h = fg' = h$, so $h \in \varepsilon_{f,g}E_X$. On the other hand, since $gE_X \subset \mathrm{Hom}(X,Y)$, we have $\varepsilon_{f,g}E_X \subset f\mathrm{Hom}(X,Y) = I_X(Y,f)$, and so (3) follows. From this we deduce further that

$$I_X(\mathrm{Sub}'_{\mathcal{C}}(X)) = \mathbf{SpId}^{\mathcal{C}}_{E_X}. \tag{4}$$

Indeed, one inclusion follows from (3). Conversely, if $\varepsilon E_X \in \mathbf{SpId}^{\mathcal{C}}_{E_X}$, then $\varepsilon = fg$ with $gf = 1_Y$. Thus $(Y,f) \in \mathbf{Sub}'_{\mathcal{C}}(X)$ and $I_X(Y,f) = \varepsilon E_X$ by (3), and so (4) follows.

We next observe that

$$(Y_1,f_1) \le (Y_2,f_2) \Leftrightarrow I_X(Y_1,f_1) \subset I_X(Y_2,f_2), \quad \forall (Y_i,f_i) \in \mathrm{Sub}'_{\mathcal{C}}(X). \tag{5}$$

Indeed, if $f_1 = f_2 g$, then $I_X(Y_1,f_1) = f_2 g\mathrm{Hom}(X,Y_1) \subset f_2\mathrm{Hom}(X,Y_2) = I_X(Y_2,f_2)$. Conversely, if $I_X(Y_1,f_1) = \varepsilon_1 E_X \subset \varepsilon_2 E_X = I'_X(Y_2,f_2)$, where $\varepsilon_i = \varepsilon_{f_i,g_i}$, then $\varepsilon_1 = \varepsilon_2 h$ for some $h \in E_X$ and then $f_1 = \varepsilon_1 g_1 = \varepsilon_2 h g_1 = f_2(g_2 h g_1)$, so $(Y_1,f_1) \le (Y_2,f_2)$.

From (5) it follows immediately that I_X induces an order-preserving injection $I_X : \mathbf{Sub}'_{\mathcal{C}}(X) \to \mathbf{SpId}^{\mathcal{C}}_{E_X}$ which is bijection because of (4).

Remark 2.3 (a) Since the above proof did not make use of the hypothesis that $\mathcal{C}$ is a preadditive category, we see that Proposition 2.2 is valid for any category $\mathcal{C}$ if we define $\mathbf{SpId}^{\mathcal{C}}_{E_X} := \{\varepsilon E_X : \varepsilon \in E \text{ is a split idempotent}\}$. Thus, $\mathbf{Sub}'_{\mathcal{C}}(X)$ and $\mathbf{Quot}'_{\mathcal{C}}(X)$ are always sets.

(b) Note that the above proof gives the following description of the inverse maps: $I_X^{-1}(\varepsilon E_X) \sim (Y,f)$ and ${}_XI^{-1}(E_X\varepsilon) \sim (Y,g)$ if $\varepsilon = fg$ with $gf = 1_Y$. In particular, the equivalence classes of (Y,f) and (Y,g) do not depend on the choice of ε nor on the choice of f,g such that $\varepsilon = fg$.

The above maps I_X and ${}_XI$ are closely related to the basic representation functors $h^X : \mathcal{C} \to \underline{\mathrm{Sets}}$ and $h_X : \mathcal{C}^{op} \to \underline{\mathrm{Sets}}$ defined by X. To explain this in more detail, let us first introduce the following notation.

Notation. Let $\mathcal{A}$ be an additive category and $X \in \mathrm{ob}(\mathcal{A})$. For $n \ge 1$, let $X^n = X \oplus \ldots \oplus X$ denote the n-fold direct sum with injections $e_i = e_i^n : X \to X^n$ and projections $p_i = p_i^n : X^n \to X$, $1 \le i \le n$. Furthermore, let $\underline{\mathrm{Sub}}_X = \underline{\mathrm{Sub}}_{X,\mathcal{A}}$ denote the full subcategory of $\mathcal{A}$ whose objects are subobjects of X^n for some $n \ge 1$, i.e. $\mathrm{ob}(\underline{\mathrm{Sub}}_X)$ consists of all $Y \in \mathrm{ob}(\mathcal{A})$ which admit a monic $f : Y \to X^n$, for sone n. Similarly, let $\underline{\mathrm{Quot}}_X$ and $\underline{\mathrm{Ret}}_X$ denote the full subcategories of $\mathcal{A}$ whose objects are quotients and retractions of X^n for some $n \ge 1$, respectively. Thus, $\underline{\mathrm{Ret}}_X$ is a subcategory of both $\underline{\mathrm{Sub}}_X$ and of $\underline{\mathrm{Quot}}_X$.

Proposition 2.4 *(a) X is a generator of $\underline{\mathrm{Quot}}_X$ and a cogenerator of $\underline{\mathrm{Sub}}_X$.*

(b) The restriction of the functors h_X and h^X to $\underline{\mathrm{Ret}}_X$ define fully faithful, additive functors

$$h^X : \underline{\mathrm{Ret}}_X \to \underline{\mathrm{Mod}}_{E_X} \quad \textit{and} \quad h_X : \underline{\mathrm{Ret}}_X^{op} \to {}_{E_X}\underline{\mathrm{Mod}}$$

where $\underline{\mathrm{Mod}}_{E_X}$ and ${}_{E_X}\underline{\mathrm{Mod}}$ denote the categories of right and left E_X-modules, respectively.

Proof. (a) Let $f_1, f_2 : Y_1 \to Y_2$, where $f_1 \neq f_2$ and $Y_1, Y_2 \in \mathrm{ob}(\underline{\mathrm{Sub}}_X)$. By definition, there is a monic $g : Y_2 \to X^n$ for some $n \geq 1$. Then $gf_1 \neq gf_2$, and hence $p_i^n gf_1 \neq p_i^n gf_2$ for some i, $1 \leq i \leq n$. Since $p_i^n g \in \mathrm{Hom}(Y_2, X)$, this proves that X is a cogenerator of $\underline{\mathrm{Sub}}_X$ in the sense of [Mac], p. 123. The proof that X is a generator of $\underline{\mathrm{Quot}}_X$ is analogous.

(b) Since $\mathcal{A}$ is additive, it is immediate that $h^X : \mathcal{A} \to \underline{\mathrm{Sets}}$ factors over the category $\underline{\mathrm{Mod}}_{E_X}$ of right E_X-modules to yield an additive functor $h^X : \mathcal{A} \to \underline{\mathrm{Mod}}_{E_X}$; cf. [Sch], p. 143. (Note that $h^X(Y) = \mathrm{Hom}_{\mathcal{A}}(X, Y)$ is naturally a right E_X-module, and for any $f \in \mathrm{Hom}_{\mathcal{A}}(Y_1, Y_2)$, the map $h^X(f) : h^X(Y_1) \to h^X(Y_2)$ defined by $h^X(f)(g) = fg$ is clearly E_X-linear.) Similarly, the functor $h_X : \mathcal{A}^{op} \to \underline{\mathrm{Sets}}$ factors additively over ${}_{E_X}\underline{\mathrm{Mod}}$.

By (a) we know that X is both a generator and a cogenerator of $\underline{\mathrm{Ret}}_X$, and hence the restriction of h_X and h^X to $\underline{\mathrm{Ret}}_X$ is faithful. To prove that h^X is full, let $\varphi \in \mathrm{Hom}_{E_X}(h^X(Y_1), h^X(Y_2))$ be E_X-linear, where $Y_1, Y_2 \in \mathrm{ob}(\underline{\mathrm{Ret}}_X)$. We claim that $\exists f \in \mathrm{Hom}(Y_1, Y_2)$ such that $h^X(f) = \varphi$.

For this, we first consider the case that $Y_1 = X^m$ and $Y_2 = X^n$. Since $e_i^m \in h^X(Y_1) = \mathrm{Hom}(X, X^m)$, $\varphi(e_i^m) \in h^X(X^n)$, and so $\varphi(e_i^m) = \sum_j e_j^n f_{ij}$, for some $f_{ij} \in E_X$. Then $\exists! f \in \mathrm{Hom}(Y_1, Y_2)$ such that $p_j^n f e_i^m = f_{ij}$, $\forall i, j$. Since φ is E_X-linear, we have $\forall (x_1, \ldots, x_m) \in E_X^m$ that $\varphi\left(\sum e_i^m x_i\right) = \sum\sum e_j^n f_{ij} x_i = f\left(\sum e_i^m x_i\right) = h^X(f)\left(\sum e_i^m x_i\right)$. Thus $h^X(f) = \varphi$, because every $g \in h^X(Y_1)$ has the form $g = \sum e_i^m x_i$.

Now assume that $Y_1, Y_2 \in \mathrm{ob}(\underline{\mathrm{Ret}}_X)$ are arbitrary, and let $f_i \in \mathrm{Hom}(Y_i, X^{n_i})$ be such that $g_i f_i = 1_{Y_i}$ for some g_i. For φ as above, define $\tilde{\varphi} : h^X(X^{n_1}) \to h^X(X^{n_2})$ by $\tilde{\varphi}(g) = f_2 \varphi(g_1 g)$, where $g \in h^X(X^{n_1})$. It is immediate that $\tilde{\varphi}$ is E_X-linear, and so by the above $\tilde{\varphi} = h^X(\tilde{f})$, for some $\tilde{f} \in \mathrm{Hom}(X^{n_1}, X^{n_2})$. Put $f = g_2 \tilde{f} f_1 \in \mathrm{Hom}(Y_1, Y_2)$. Then $h^X(f) = \varphi$ because if $g \in h^X(Y_1)$, then $f_1 g \in h^X(X^{n_1})$ and hence $h^X(f)(g) = g_2 \tilde{h} f_1 g = g_2 \tilde{\varphi}(f_1 g) = g_2(f_2 \varphi(g_1 f_1 g)) = \varphi(g)$. This proves that h^X is fully faithful. The proof for h_X is similar.

Corollary 2.5 *If $(Y, f) \in \mathbf{Sub}'_{\mathcal{A}}(X)$ and $(Z, g) \in \mathbf{Quot}'_{\mathcal{A}}(X)$, then*

$$I_X(Y, f) \simeq h^X(Y) \quad \textit{and} \quad {}_XI(Z, g) \simeq h_X(Z), \tag{6}$$

as right and left E_X-modules, respectively. Moreover, for any $(Y_1, f_1), (Y_2, f_2) \in \mathbf{Sub}_{\mathcal{A}}(X)$ we have a (functorial) isomorphism

$$\tilde{h}^X_{Y_1, Y_2} : \mathrm{Hom}_{\mathcal{A}}(Y_1, Y_2) \xrightarrow{\sim} \mathrm{Hom}_{E_X}(I_X(Y_1, f_1), I_X(Y_2, f_2))$$

such that $\tilde{h}^X_{Y_1,Y_2}(f)(f_1 g) = f_2 f g$, *for all* $f \in \mathrm{Hom}(Y_1, Y_2), g \in \mathrm{Hom}(X, Y_1)$.

Proof. Since f is monic, the map $g \mapsto fg$ induces a bijection $\varphi_f : h^X(Y) \xrightarrow{\sim} fh^X(Y) = I_X(Y, f)$. Since φ_f is E_X-linear, this is the desired isomorphism of right E_X-modules. Similarly, the map $f \mapsto fg$ defines an isomorphism $h_X(Z) \xrightarrow{\sim} {}_XI(Z, g)$ of left E_X-modules.

By Proposition 2.4(b), the map $h^X_{Y_1,Y_1} : \mathrm{Hom}_{\mathcal{A}}(Y_1, Y_2) \to \mathrm{Hom}_{E_X}(h^X(Y_1), h^X(Y_2))$ induced by h^X is an isomorphism (of abelian groups), and hence the same is true for $\tilde{h}^X_{Y_1,Y_2}$ which is defined by the rule $\tilde{h}^X_{Y_1,Y_2}(f) = \varphi_{f_2} h^X_{Y_1,Y_2}(f)\varphi^{-1}_{f_1}$. Thus by definition we have $\tilde{h}^X_{Y_1,Y_2}(f)(f_1 g) = \varphi_{f_2} h^X_{Y_1,Y_2}(f)(g) = \varphi_{f_2}(fg) = f_2 f g$.

Remark 2.6 If $(Y_1, f_1), \ldots, (Y_1, f_r) \in \mathbf{Sub}'_{\mathcal{A}}(X)$, then for any $s < r$ we have

$$\begin{aligned} Y_1 \oplus \ldots \oplus Y_s &\simeq Y_{s+1} \oplus \ldots \oplus Y_r \Leftrightarrow \\ I(Y_1) \oplus \ldots \oplus I(Y_s) &\simeq I(Y_{s+1}) \oplus \ldots \oplus I(Y_r), \end{aligned} \tag{7}$$

where $I(Y_i) = I_X(Y, f_i)$. (Similarly, if $(Y, f_i) \in \mathbf{Quot}'_{\mathcal{A}}(X)$, then (7) holds with $I(Y_i) = {}_XI(Y, f_i)$.) Indeed, put $Y := Y_1 \oplus \ldots \oplus Y_s$ and $Y' = Y_{s+1} \oplus \ldots \oplus Y_r \in \mathrm{ob}(\underline{\mathrm{Ret}}_X)$. Since h^X is fully faithful, we have $Y \simeq Y' \Leftrightarrow h^X(Y) \simeq h^X(Y')$. But since h^X is additive, we have $h^X(Y) = h^X(Y_1) \oplus \ldots \oplus h^X(Y_s) \simeq I(Y_1) \oplus \ldots \oplus I(Y_s)$, and similarly $h^X(Y') \simeq I(Y_{s+1}) \oplus \ldots \oplus I(Y_r)$. Thus (7) follows.

In the sequel we shall be interested the following special case of the above results.

Theorem 2.7 *Let* $\mathcal{A}$ *be an additive category. If* $X \in \mathrm{ob}(\mathcal{A})$ *satisfies the conditions that*

(i) every idempotent $\varepsilon \in E_X = \mathrm{End}_{\mathcal{A}}(X)$ *splits,*

(ii) E_X *is a semi-simple ring,*

then we have order-preserving bijections

$$I_X : \mathbf{Sub}'_{\mathcal{A}}(X) \xrightarrow{\sim} \mathbf{Id}_{E_X} \quad \text{and} \quad {}_XI : \mathbf{Quot}'_{\mathcal{A}}(X) \xrightarrow{\sim} {}_{E_X}\mathbf{Id},$$

and the functor h^X *induces an equivalence of categories* $h^X : \underline{\mathrm{Ret}}_X \to \underline{\mathrm{Mod}}^f_{E_X}$ *between the category* $\underline{\mathrm{Ret}}_X$ *and the category* $\underline{\mathrm{Mod}}^f_{E_X}$ *of finitely generated right* E_X*-modules. In particular,* $\underline{\mathrm{Ret}}_X$ *is an abelian category.*

Proof. Let $\mathfrak{a} \in \mathbf{Id}_{E_X}$. Since E_X is semi-simple, $\mathfrak{a} = \varepsilon E_X$ is generated by an idempotent $\varepsilon \in E_X$ ([BA], p. 47). By (i), ε is split in $\mathcal{A}$, and so $\mathfrak{a} \in \mathbf{SpId}^{\mathcal{A}}_{E_X}$. Thus $\mathbf{Id}_{E_X} = \mathbf{SpId}^{\mathcal{A}}_{E_X}$, and similarly ${}_{E_X}\mathbf{Id} = {}_{E_X}\mathbf{SpId}^{\mathcal{A}}$, so the first assertion follows from Proposition 2.2.

To prove that h^X is an equivalence, we first note that if $Y \in \mathrm{ob}(\underline{\mathrm{Ret}}_X)$, then $h^X(Y)$ is a right E_X-submodule of E^n_X for some $n \geq 1$, and hence is finitely generated. Thus,

h^X maps $\underline{\mathrm{Ret}}_X$ into the full subcategory $\underline{\mathrm{Mod}}^f_{E_X}$ of $\underline{\mathrm{Mod}}_{E_X}$. Since h^X is fully faithful by Proposition 2.4(b), it is enough to show that if $M \in \mathrm{ob}(\underline{\mathrm{Mod}}^f_{E_X})$ is a finitely generated E_X-module, then $M \simeq h^X(Y)$, for some $Y \in \mathrm{ob}(\underline{\mathrm{Ret}}_X)$. Since E_X is semi-simple, $M \simeq \mathfrak{a}_1 \oplus \ldots \oplus \mathfrak{a}_r$, for some ideals $\mathfrak{a}_1, \ldots, \mathfrak{a}_r \in \mathbf{Id}_{E_X}$ (cf. [BA], p. 47). By the above (and Corollary 2.5) we know that there exist $Y_1, \ldots, Y_r \in \mathbf{Sub}'_{\mathcal{A}}(X)$ such that $h^X(Y_i) \simeq \mathfrak{a}_i$, for $1 \le i \le r$, and so $h^X(Y_1 \oplus \ldots \oplus Y_r) \simeq M$. Thus, h^X is an equivalence and hence $\underline{\mathrm{Ret}}_X$ is an abelian category because $\underline{\mathrm{Mod}}^f_{E_X}$ is abelian.

Remark 2.8 (a) We thus have (in the situation of Theorem 2.7) that for every right E_X-ideal $\mathfrak{a} \in \mathbf{Id}_{E_X}$ there is a unique subobject $(Y_\mathfrak{a}, f_\mathfrak{a}) \in \mathbf{Sub}_{\mathcal{A}}(X)$ such that $I_X(Y_\mathfrak{a}, f_\mathfrak{a}) = \mathfrak{a}$. Moreover, by Corollary 2.5 we have a ring isomorphism

$$\theta_\mathfrak{a} = (\tilde{h}^X_{Y_\mathfrak{a}, Y_\mathfrak{a}})^{-1} : \mathrm{End}_{E_X}(\mathfrak{a}) \xrightarrow{\sim} \mathrm{End}_{\mathcal{A}}(Y_\mathfrak{a})$$

which can be characterized by the formula $f_\mathfrak{a}\theta_\mathfrak{a}((\lambda_g)_{|\mathfrak{a}}) = gf_\mathfrak{a}, \forall g \in E_X$ with $g\mathfrak{a} \subset \mathfrak{a}$, where $\lambda_g(f) = gf$, $\forall f \in E_X$. [To see this, note first that for any $\varphi \in \mathrm{End}_{E_X}(\mathfrak{a})$ we have $\tilde{h}(\theta_\mathfrak{a}(\varphi)) = \varphi$, where $\tilde{h} = \tilde{h}^X_{Y_\mathfrak{a}, Y_\mathfrak{a}}$, and so $\varphi(f_\mathfrak{a} g') = \tilde{h}(\theta_\mathfrak{a}(\varphi))(f_\mathfrak{a} g') = f_\mathfrak{a}\theta_\mathfrak{a}(\varphi)g'$, $\forall g' \in \mathrm{Hom}(X, Y_\mathfrak{a})$. Thus, if $g' = g_\mathfrak{a}$ satisfies $g_\mathfrak{a} f_\mathfrak{a} = 1_{Y_\mathfrak{a}}$, then $\varphi(\varepsilon) = f_\mathfrak{a}\theta_\mathfrak{a}(\varphi)g_\mathfrak{a}$ where $\varepsilon = f_\mathfrak{a} g_\mathfrak{a}$, and hence $\varphi(\varepsilon)f_\mathfrak{a} = f_\mathfrak{a}\theta_\mathfrak{a}(\varphi)$. Thus, taking $\varphi = (\lambda_g)_{|\mathfrak{a}}$, we get $f_\mathfrak{a}\theta_\mathfrak{a}(\varphi) = \lambda_g(\varepsilon)f_\mathfrak{a} = g\varepsilon f_\mathfrak{a} = gf_\mathfrak{a}$, as claimed.]

(b) The condition (i) of Theorem 2.7 is equivalent to the condition that every idempotent $\varepsilon \in E_X$ has a factorization $\varepsilon = fg$ into a monic f and epi g, for the condition $\varepsilon^2 = \varepsilon$ implies that $fgfg = fg$ and so $gf = 1$ as f is monic and g is epi. Thus, (i) holds in any abelian category; cf. [Mac], p. 195. It also holds in any *pseudo-abelian category* (as defined in [Man]), for if $\varepsilon \in E_X$ is an idempotent, then $X = \mathrm{Ker}(\varepsilon) \oplus \mathrm{Ker}(1 - \varepsilon)$ and we have $\varepsilon = i_2 p_2$, where $i_2 : \mathrm{Ker}(1 - \varepsilon) \to X$ and $p_2 : X \to \mathrm{Ker}(1 - \varepsilon)$ denote the canonical injection and projection, respectively. Thus, the following corollary may be viewed as a sharpening of Lemma 2 of [Ja].

Corollary 2.9 *If $\mathcal{A}$ is an additive category, then the above conditions (i) and (ii) hold for all $X \in \mathrm{ob}(\mathcal{A})$ if and only if $\mathcal{A}$ is a semi-simple abelian category. In particular, every monic and epi splits in such a category.*

Proof. If $\mathcal{A}$ is abelian and semi-simple, then condition (i) holds by Remark 2.8(b) and (ii) holds by definition.

To prove the converse, we first observe that if $X, Y \in \mathrm{ob}(\mathcal{A})$, then $X = X_1 \oplus X_2$, where $X_1 \in \mathrm{ob}(\underline{\mathrm{Ret}}_Y)$ and $\mathrm{Hom}_{\mathcal{A}}(X_2, Y) = 0, \mathrm{Hom}_{\mathcal{A}}(Y, X_2) = 0$. Indeed, since $X, Y \in \mathrm{ob}(\underline{\mathrm{Ret}}_Z)$ with $Z = X \oplus Y$ and since $\underline{\mathrm{Ret}}_Z$ is equivalent to $\underline{\mathrm{Mod}}^f_{E_Z}$ by Theorem 2.7, this observation follows from the corresponding one in $\underline{\mathrm{Mod}}^f_{E_Z}$.

Now suppose $f \in \mathrm{Hom}_{\mathcal{A}}(X, Y)$, where $X, Y \in \mathrm{ob}(\mathcal{A})$. Them f has a kernel $\mathrm{Ker}(f) \in \mathrm{ob}(\underline{\mathrm{Ret}}_Z)$, where $Z = X \oplus Y$, because $\underline{\mathrm{Ret}}_Z$ is abelian. By the above observation, $\mathrm{Ker}(f)$ is also kernel of f in $\mathcal{A}$. Similarly, f has a cokernel. In same way it follows that every monic (resp. epi) $f : X \to Y$ is a kernel (resp. cokernel) because this is true in $\underline{\mathrm{Ret}}_Z$. Thus, $\mathcal{A}$ is an abelian category. Moreover, since E_Z is semi-simple, every monic and epi in $\underline{\mathrm{Mod}}^f_{E_Z}$ splits (cf. [BA], p. 46 and 32), and so the same is true for any monic and epi in $\underline{\mathrm{Ret}}_Z$ and hence also in $\mathcal{A}$.

Remark 2.10 (a) Since every map of an abelian category $\mathcal{A}$ has a canonical decomposition $f = \mathrm{im}(f)\mathrm{coim}(f)$ (cf. [Mac], p. 195), we have the following formula for the inverse of I_X (and that of ${}_XI$):

$$(8) \qquad I_X^{-1}(fE_X) = (\mathrm{Im}(f), \mathrm{im}(f)), \quad {}_XI^{-1}(E_Xf) = (\mathrm{Coim}(f), \mathrm{coim}(f)),$$

for all $f \in E_X$. To see this, recall that $m := \mathrm{im}(f) : Y := \mathrm{Im}(f) \to X$ is monic and $p := \mathrm{coim}(f) : X \to \mathrm{Coim}(f) = Y$ is epi, and so there exist splittings g, h such that $gm = 1_Y = ph$. Then $f = m(gm)p = (mg)mp = \varepsilon f$, where $\varepsilon = mg$, and $\varepsilon = m(ph)g = f(hg)$. Thus, $fE_X = \varepsilon E_X$ and so $I_X^{-1}(fE_X) = I_X^{-1}(\varepsilon E_X) = (\mathrm{Im}(f), \mathrm{im}(f))$, the latter by Remark 2.3(b). The proof for the second formula is analogous.

For later usage, let us also observe that since $\ker(\mathrm{coim}(f)) = \ker(f)$, we have

$$(9) \qquad \mathrm{Coim}(f) = X/\mathrm{Ker}(f) \quad \text{with} \quad \mathrm{Ker}(f) = I_X^{-1}(r_{E_X}(E_Xf)),$$

where $r_{E_X}(\mathfrak{a}) = \{g \in E_X : \mathfrak{a}g = 0\} \in \mathbf{Id}_{E_X}$ denotes the *right annihilator* of a left E_X-ideal $\mathfrak{a} \in {}_{E_X}\mathbf{Id}$. Here, the second formula holds because by the universal property of kernels ([Mac], p. 188) we have $I_X(\mathrm{Ker}(f)) = \ker(f)\mathrm{Hom}(X, \mathrm{Ker}(f)) = \{g \in E_X : fg = 0\} = r_{E_X}(E_Xf)$.

(b) Note that (8) implies that

$$(10) \qquad hI_X(Y, f) = I_X(h(Y), \mathrm{im}(hf)), \quad \forall h \in E_X,$$

where $h(Y) = \mathrm{Im}(hf)$ denotes the image of (Y, f) under h (cf. [Sch], p. 134), because $hI_X(Y, f) = h\varepsilon_{f,g}E_X$ and $\mathrm{im}(h\varepsilon_{f,g}) = \mathrm{im}(hf)$. More generally, we see that each $h \in \mathrm{Hom}_{\mathcal{A}}(X, X')$ induces maps

$$h_* : \mathbf{Sub}_{\mathcal{A}}(X) \to \mathbf{Sub}_{\mathcal{A}}(X') \quad \text{and} \quad h^* : \mathbf{Quot}_{\mathcal{A}}(X') \to \mathbf{Quot}_{\mathcal{A}}(X)$$

by the rules $h_*(Y, f) = (h(Y), \mathrm{Im}(hf))$ and $h^*(Y', f') = (\mathrm{Coim}(f'h), \mathrm{coim}(f'h))$. We then have $I_{X'}(h_*(Y, f)) = hI_X(Y, f)\mathrm{Hom}(X', X)$ and ${}_XI(h^*(Y', f')) = \mathrm{Hom}(X', X)\cdot {}_{X'}I(Y', f')h$, as is easy to verify.

(c) From (10) it follows that $I_X(Y, f)$ is a two-sided E_X-ideal if and only if (Y, f) is E_X-*stable* in the sense that $\mathrm{im}(hf)$ factors over f, for all $h \in E_X$. Indeed, $hI_X(Y, f) \overset{(10)}{=} I_X(\mathrm{Im}(hf), \mathrm{im}(hf)) \subset I_X(Y, f) \Leftrightarrow (\mathrm{Im}(hf), \mathrm{im}(hf)) \le (Y, f) \Leftrightarrow \mathrm{im}(hf) = fg$, for some $g : \mathrm{Im}(hf) \to Y$.

(d) Note that $\mathbf{Id}_{E_X}$ and ${}_{E_X}\mathbf{Id}$ are *lattices*, i.e. that each collection $\mathfrak{a}_1, \ldots, \mathfrak{a}_r$ of ideals has a least upper bound $\sum \mathfrak{a}_i$ and a greatest lower bound $\cap \mathfrak{a}_i$. Thus, by

Theorem 2.7 and Corollary 2.9, the same is true for $\mathbf{Sub}_{\mathcal{A}}(X) = \mathbf{Sub}'_{\mathcal{A}}(X)$ and $\mathbf{Quot}_{\mathcal{A}}(X) = \mathbf{Quot}'_{\mathcal{A}}(X)$. We now describe these upper and lower bounds explicitly.

As is explained in [Mac], p. 122, the greatest lower bound of $(Y_1, f_1), \ldots, (Y_r, f_r) \in \mathbf{Sub}_{\mathcal{A}}(X)$ is $(\cap Y_i, \cap f_i)$, where $\cap Y_i = Y_1 \times_X \ldots \times_X Y_r$ (which exists in $\mathcal{A}$ by [Sch], p. 110) with canonical monic $\cap f_i = (f_1, \ldots, f_r) : \cap Y_i \to X$. Moreover, the least upper bound is obtained by factoring the canonical map $f : Y := Y_1 \oplus \ldots \oplus Y_r \to X$ (defined by $fe_i = f_i$) into $f = \mathrm{im}(f)\mathrm{coim}(f)$; the image $(\mathrm{Im}(f), \mathrm{im}(f)) \in \mathbf{Sub}_{\mathcal{A}}(X)$ is denoted by $\sum(Y_i, f_i) = (\sum Y_i, \sum f_i)$. In particular, we have unique maps $g_i : Y_i \to \sum Y_i$ and $h_i : \cap Y_i \to Y_i$ such that $f_i = (\sum f_j)g_i$ and $f_i h_i = \cap f_i$, for $1 \le i \le r$. Thus:

$$(11) \qquad I_X\left(\sum(Y_i, f_i)\right) = \sum I_X(Y_i, f_i), \quad I_X\left(\bigcap(Y_i, f_i)\right) = \bigcap I_X(Y_i, f_i).$$

From this we can deduce easily that

$$(12) \qquad \begin{aligned} & I_X(Y_0, f_0) = I_X(Y_1, f_1) \oplus \ldots \oplus I_X(Y_r, f_r) \\ \Leftrightarrow \quad & \exists g : Y_1 \oplus \ldots \oplus Y_r \xrightarrow{\sim} Y_0 \text{ such that } f_0 g e_i = f_i, 1 \le i \le r, \end{aligned}$$

where $e_i : Y_i \to Y_1 \oplus \ldots \oplus Y_r$ denotes the ith inclusion map.

By duality, similar statements hold for $\mathbf{Quot}_{\mathcal{A}}(X) = \mathbf{Sub}_{\mathcal{A}^{op}}(X)$ in place of $\mathbf{Sub}_{\mathcal{A}}(X)$. For example, if $(Y_i, g_i) \in \mathbf{Quot}_{\mathcal{A}}(X)$, $1 \le i \le r$, then their least upper bound is $\sum(Y_i, g_i) = (\mathrm{Coim}(g), \mathrm{coim}(g))$, where $g = (g_1, \ldots, g_r) : X \to Y_1 \oplus \ldots \oplus Y_r$ is the unique map such that $p_i g = g_i$.

(e) By using the notation in (d), we can give an alternate definition for the inverse maps $S_X := I_X^{-1}$ and $Q_X := {}_X I^{-1}$: if $\mathfrak{a}$ is a right (respectively, left) E_X-ideal, then

$$(13) \qquad I_X^{-1}(\mathfrak{a}) = \sum_{f \in \mathfrak{a}} (\mathrm{Im}(f), \mathrm{im}(f)) \quad \text{and} \quad {}_X I^{-1}(\mathfrak{a}) = \sum_{f \in \mathfrak{a}} (\mathrm{Coim}(f), \mathrm{coim}(f)).$$

Indeed, since each $\mathfrak{a} \in \mathbf{Id}_{E_X}$ has the form $\mathfrak{a} = \varepsilon E_X$ (cf. proof of Theorem 2.7), we have by (8) that $I_X^{-1}(\mathfrak{a}) = (\mathrm{Im}(\varepsilon), \mathrm{im}(\varepsilon)) = \sum_{f \in \mathfrak{a}} (\mathrm{Im}(f), \mathrm{im}(f))$, and the second formula is proved similarly.

Example 2.11 Let k be a semi-simple ring, and let $\mathcal{A} = {}_k\underline{\mathrm{Mod}}^f$ be the (abelian) category of finitely generated left k-modules. Then Corollary 2.9 shows that $\mathcal{A}$ is a semi-simple abelian category because for every $M \in \mathrm{ob}(\mathcal{A})$ the ring $E_M = \mathrm{End}_k(M)$ is semi-simple by [BA], p. 47.

In addition, from Theorem 2.7 and Remark 2.10(e), (a) we see that the maps $S_M : \mathfrak{a} \mapsto \mathfrak{a}M = \sum_{f \in \mathfrak{a}} \mathrm{Im}(f)$ and $Q_M : \mathfrak{a} \mapsto M/r_{E_M}(\mathfrak{a})M$ induce order-preserving bijections

$$S_M : \mathbf{Id}_{E_M} \xrightarrow{\sim} \mathbf{Sub}_k(M) = \mathbf{Sub}'_k(M) \quad \text{and} \quad Q_M : {}_{E_M}\mathbf{Id} \xrightarrow{\sim} \mathbf{Quot}_k(M) = \mathbf{Quot}'_k(M),$$

where $\mathbf{Sub}_k(M) = \mathbf{Sub}_{\mathcal{A}}(M)$ and $\mathbf{Sub}_k(M) = \mathbf{Quot}_{\mathcal{A}}(M)$ denote the sets of k-submodules and k-quotients of M, respectively.

Furthermore, if M is a *faithful* k-module, then $k \subset M^n$ for some $n > 0$ (by [BA], p. 26) and so $\underline{\mathrm{Ret}}_M = {}_k\underline{\mathrm{Mod}}^f$ (because every $M' \in {}_k\underline{\mathrm{Mod}}^f$ is isomorphic to a (finite) direct sum of ideals of k). Thus, by Theorem 2.7 we see that the functor h^M induces an equivalence of categories $h^M : {}_k\underline{\mathrm{Mod}}^f \to \underline{\mathrm{Mod}}^f_{E_M} = {}_{E_M^{op}}\underline{\mathrm{Mod}}^f$. Note that since each $M' \in \mathrm{ob}(\mathcal{A})$ is projective, we have $h^M \simeq M^* \otimes_k *$, i.e. h^M is equivalent to the Morita functor of [CR], p. 60.

In order to be able to apply the above results to the Shimura construction, it is useful to extend Theorem 2.7 as follows.

Theorem 2.12 *In the situation of Theorem 2.7, suppose that M is a faithful left E_X-module, and put $\tilde{E}_X = \mathrm{End}_{E_X}(M)$. Then the rule $(Y, f) \mapsto I_X(Y, f)M$ defines an order-preserving bijection*

$$S_{X,M} : \mathbf{Sub}'_{\mathcal{A}}(X) \xrightarrow{\sim} \mathbf{Alg}_{E_X}(M)$$

where $\mathbf{Alg}_{E_X}(M) = \{aM : a \in E_X\} \subset \mathbf{Sub}_{\tilde{E}_X}(M)$. Moreover, if M is a finitely generated E_X-module, then $\mathbf{Alg}_{E_X}(M) = \mathbf{Sub}_{\tilde{E}_X}(M)$, and we have an equivalence of categories

$$h^{X,M} := h^X \otimes_{E_X} M : \underline{\mathrm{Ret}}_X \to {}_{\tilde{E}_X}\underline{\mathrm{Mod}}^f$$

with the property that $h^{X,M}(Y) \simeq S_{X,M}(Y, f)$, for all $(Y, f) \in \mathbf{Sub}_{\mathcal{A}}(X)$. In particular, for every $N \in \mathbf{Sub}_{\tilde{E}_X}(M)$ there is a unique $(Y_N, f_N) \in \mathbf{Sub}_{\mathcal{A}}(X)$ such that $S_{X,M}(Y_N, f_N) = N$ and we have a ring isomorphism

$$\theta_N : \mathrm{End}_{\tilde{E}_X}(N) \xrightarrow{\sim} \mathrm{End}_{\mathcal{A}}(Y_N)$$

such that $f_N\theta_N((\lambda_g)_{|N}) = gf_N, \forall g \in E_X$ with $gN \subset N$, where $\lambda_g(x) = gx$, $\forall x \in M$.

Proof. Since E_X is semi-simple and M is faithful, it follows that M is a faithfully flat E_X-module. (Use Exercise 1 of [BCA], p. 49). Thus, the map $\mathfrak{a} \mapsto \mathfrak{a}M$ defines an (order-preserving) injection $\mu_M : \mathbf{Id}_{E_X} \hookrightarrow \mathbf{Sub}_{\tilde{E}_X}(M)$. (Here we use the obvious fact that each $\mathfrak{a}M$ is a left $\tilde{E}_X$-submodule of M.) It thus follows from Theorem 2.7 that the map $S_{X,M} = \mu_M \circ I_X$ is an order-preserving bijection onto its image $\mathbf{Alg}(M) := \mathrm{Im}(S_{X,M}) \subset \mathbf{Sub}_{\tilde{E}_X}(M)$. Note that $\mathbf{Alg}(M) = \mathbf{Alg}_{E_X}(M)$ because $I_X(Y, f) = \varepsilon E_X$ for some (idempotent) $\varepsilon \in E_X$ and then $S_{X,M}(Y, f) = \varepsilon M$.

Now suppose that M is finitely generated. Then $\tilde{E}_X$ is again semi-simple ([BA], p. 47) and $E_{\tilde{E}_X} \simeq E_X$ ([BA], p. 50). Thus, by Example 2.11 it follows that the map $S_M = \mu_M : \mathbf{Id}_{E_X} \to \mathbf{Sub}_{\tilde{E}_X}(M)$ is a bijection, and so $\mathbf{Alg}_{E_X}(M) = \mathbf{Sub}_{\tilde{E}_X}(M)$. Furthermore, since M is faithful and projective, it follows from Morita's Theorem ([CR], p. 60) that the functor $* \otimes_{E_X} M : \underline{\mathrm{Mod}}^f_{E_X} \to \underline{\mathrm{Mod}}^f_{\tilde{E}_X^{op}} = {}_{\tilde{E}_X}\underline{\mathrm{Mod}}^f$ is an equivalence of categories, and so it follows from Theorem 2.7 that $h^{X,M} := (* \otimes_{E_X} M) \circ h^X$ is also an equivalence. Moreover, since M is E_X-flat, we have by Corollary 2.5 that $S_{X,M}(Y,f) = I_X(Y,f)M \simeq I_X(Y,f) \otimes_{E_X} M \simeq h^X(Y) \otimes_{E_X} M = h^{X,M}(Y)$.

Finally, if $N \in \mathbf{Sub}_{\tilde{E}_X}(M)$, then the existence and uniqueness of (Y_N, f_N) is clear since $S_{X,M}$ is a bijection. Moreover, since $N = \mathfrak{a}M \simeq \mathfrak{a}_{E_X} M$, for some (unique) $\mathfrak{a} \in \mathbf{Id}_{E_X}$, we see from Remark 2.8(a) that $\theta_N = \theta_{\mathfrak{a}} \otimes_{E_X} M$ is the desired isomorphism.

As we shall see in section 5, the following corollary can be viewed as an abstract version of the *Shimura construction*.

Corollary 2.13 *In the situation of Theorem 2.12, suppose that M is finitely generated and that there is a commutative subring $T \subset E_X$ such that M is a free T-module of rank 1. Then for every T-submodule $N \subset M$ there is a unique subobject $(Y_N, f_N) \in \mathbf{Sub}_{\mathcal{A}}(X)$ such that $S_{X,M}(Y_N, f_N) = N$ and we have a natural ring embedding*

$$\theta'_N : \mathrm{End}_T(N) \hookrightarrow \mathrm{End}_{\mathcal{A}}(Y_N)$$

such that $f_N \theta_N((\lambda_t)_{|N}) = t f_N$, for all $t \in T$.

Proof. Since M is a free T-module of rank 1, we have $\mathrm{End}_T(M) = T_M$, where T_M denotes the image of T in $\mathrm{End}(M)$. Thus $\tilde{E}_X = \mathrm{End}_{E_X}(M) \subset \mathrm{End}_T(M) = T_M$, and so every T-submodule N of M is *a fortiori* an $\tilde{E}_X$-submodule. Thus, the assertions follow directly from Theorem 2.12, if we let θ'_N be the restriction of θ_N to $\mathrm{End}_T(N) \subset \mathrm{End}_{\tilde{E}_X}(N)$.

In practice, most examples of E_X-modules arise via additive functors in the following way.

Corollary 2.14 *Let $\mathcal{A}$ be a semi-simple abelian category and let k be any ring. If $F : \mathcal{A} \to {}_k\underline{\mathrm{Mod}}$ is a faithful additive functor, then for every $X \in \mathrm{ob}(\mathcal{A})$ the map F_X induces an order-preserving bijection*

$$\overline{F}_X = S_{X,F(X)} : \mathbf{Sub}_{\mathcal{A}}(X) \ \tilde{\to}\ \mathbf{Alg}_F(F(X)),$$

where $\mathbf{Alg}_F(F(X)) = \{\mathrm{Im}(F(a)) : a \in E_X\} \subset \mathbf{Sub}_{\tilde{E}_X}(F(X))$. Moreover, the inverse of $\overline{F}_X$ is given by $\overline{F}_X^{-1}(\mathrm{Im}(F(a))) = (\mathrm{Im}(a), \mathrm{im}(a))$.

Proof. By functoriality, the rule $a \mapsto F(a)$ defines a ring homomorphism $\rho_{X,F} : E_X \to \mathrm{End}_k(F(X))$ which is injective since F is faithful. Thus, $M = F(X)$ is a faithful left E_X-module via $\rho_{X,F}$, and so $S_{X,F(X)} : \mathbf{Sub}_{\mathcal{A}}(X) \tilde{\to} \mathbf{Alg}_{E_X}(F(X))$ is a bijection by Theorem 2.12. Since $I_X(Y,f) = fgE_X$, where $g \in \mathrm{Hom}(X,Y)$ satisfies $gf = 1_Y$, we see that

(14) $$S_{X,F(X)}(Y,f) = F(fg)F(X) = \mathrm{Im}(F(f)F(g)) = \mathrm{Im}(F(f)).$$

Here the last equality holds because $F(g)$ is a retract (and hence is surjective). We thus see that $\overline{F}_X(Y,f) := (F(Y), F(f)) \sim \mathrm{Im}(F(f)) = S_{X,F(X)}(Y,f)$ and that $\mathbf{Alg}_F(F(X)) = \mathbf{Alg}_{E_X}(F(X)) \subset \mathbf{Sub}_{\tilde{E}_X}(F(X))$. Finally, it is clear from (14) that $S^{-1}_{X,F(X)}(\mathrm{Im}(F(a))) = (\mathrm{Im}(a), \mathrm{im}(a))$.

On the other hand, contravariant functors can also be used in the following mannner.

Corollary 2.15 *If $\mathcal{A}$ is a semi-simple abelian category and $G : \mathcal{A}^{op} \to {}_k\underline{\mathrm{Mod}}$ is a faithful (contravariant) additive functor, then for every $X \in \mathrm{ob}(\mathcal{A})$ the map G_X defines an order-preserving bijection*

$$Q_{X,G} := \overline{G}_X : \mathbf{Quot}_{\mathcal{A}}(X) = \mathbf{Sub}_{\mathcal{A}^{op}}(X) \stackrel{\sim}{\to} \mathbf{Alg}_G(G(X)) = \{\mathrm{Im}(G(a)) : a \in \mathrm{End}(A)\},$$

and $Q_{X,G}(Y,p) = G(X)_X I(Y,p) = \mathrm{Im}(G(p))$, for any $(Y,p) \in \mathbf{Quot}_{\mathcal{A}}(X)$. Moreover, if $G(X)$ is a finitely generated E_X-module, then $\mathbf{Alg}_G(G(X)) = \mathrm{End}_{\tilde{E}_X}(G(X))$, where $\tilde{E}_X = \mathrm{End}_{E_X}(G(X)) = \{f \in \mathrm{End}(G(X)) : fG(a) = G(a)f, \forall a \in E_X\}$. Thus, for each (left) $\tilde{E}_X$-module $W \subset G(A)$, there is a unique $(Z_W, p_W) \in \mathbf{Quot}_{\mathcal{A}}(X)$ such that $\mathrm{Im}(G(p_W)) = W$, and we have $\mathrm{Ker}(p_W) = I_X^{-1}(r_{E_X}(W))$.

Proof. Since $\mathcal{A}^{op}$ is again a semi-simple abelian category, the first assertion is clear from Corollary 2.14, and the second follows from equation (14) together with the fact that $Q_{X,G}(Y,p) = S_{X,G(X)}(Y, p^{op})$. Moreover, since $G(X)$ is naturally a left E_X^{op}-module or, equivalently, a right E_X-module, the third assertion follows from Theorem 2.12. To prove the last assertion, put $\varepsilon = fp_W$, where $p_W f = 1_{Z_W}$. Then $W = G(X)\varepsilon$ and $r_{E_X}(W) = r_{E_X}(E_X\varepsilon) = (1-\varepsilon)E_X$, and hence $(Z_W, p_W) = (\mathrm{Coim}(\varepsilon), \mathrm{coim}(\varepsilon))$. Thus, by (9) we have $I_X^{-1}(r_{E_X}(W)) = \mathrm{Ker}(\varepsilon) = \mathrm{Ker}(p_W)$, as claimed.

3 Subvarieties and Quotients of an Abelian Variety

As in the introduction, fix an arbitrary ground field K and let A/K be an abelian variety defined over K. Throughout, we shall freely use the basic facts about abelian varieties as presented in Milne[Mi] and Mumford[Mu].

Notation. Let $\mathbf{Sub}(A/K)$ denote the set of *abelian subvarieties* B of A/K in the usual sense, i.e. B is an abelian variety over K together with a closed immersion $j_B : B \hookrightarrow A$ which is a K-homomorphism of abelian varieties. Since closed immersions are monics in the category $\underline{\mathrm{Var}}_K$ of K-varieties, they are also monics in the subcategory $\underline{\mathrm{Ab}}_K$ of abelian varieties over K with K-homomorphisms, and so we have an inclusion $\mathbf{Sub}(A/K) \subset \mathbf{Sub}_{\underline{\mathrm{Ab}}_K}(A)$. However, these sets are rarely equal, as we shall see; cf. Remark 3.2(a).

Similarly, let $\mathbf{Quot}(A/K) \subset \mathbf{Quot}_{\underline{\mathrm{Ab}}_K}(A)$ denote the set of equivalence classes of *abelian quotients* (C, p) of A; the latter means that $p : A \to C$ is a surjective K-homomorphism such that its (schematic) kernel $\mathrm{Ker}(p)$ is an abelian variety.

Furthermore, let $\underline{\mathrm{Ab}}_K^0$ denote the category of abelian K-varieties up to isogeny (cf. [Mu], p. 172): we have $\mathrm{ob}(\underline{\mathrm{Ab}}_K^0) = \mathrm{ob}(\underline{\mathrm{Ab}}_K)$ and $\mathrm{Hom}_{\underline{\mathrm{Ab}}_K^0}(A, B) = \mathrm{Hom}^0(A, B) := \mathrm{Hom}(A, B) \otimes \mathbb{Q}$. Thus, since $\mathrm{Hom}(A, B)$ is torsionfree (cf. [Mi], p. 122), $\underline{\mathrm{Ab}}_K$ is naturally a subcategory of $\underline{\mathrm{Ab}}_K^0$. We observe:

Proposition 3.1 *$\underline{\mathrm{Ab}}_K^0$ is a semi-simple abelian category, and the embedding $i_K : \underline{\mathrm{Ab}}_K \to \underline{\mathrm{Ab}}_K^0$ induces bijections*

$$\begin{aligned} i_{K,A} : \mathbf{Sub}(A/K) &\xrightarrow{\sim} \mathbf{Sub}_{\underline{\mathrm{Ab}}_K^0}(A) = \mathbf{Sub}'_{\underline{\mathrm{Ab}}_K^0}(A), \\ {}_A i_K : \mathbf{Quot}(A/K) &\xrightarrow{\sim} \mathbf{Quot}_{\underline{\mathrm{Ab}}_K^0}(A) = \mathbf{Quot}'_{\underline{\mathrm{Ab}}_K^0}(A). \end{aligned}$$

Proof. The first assertion is well-known, but we shall give a quick proof below using Corollary 2.9. For this, we first observe that

$$f \in \mathrm{Hom}(A, B) \text{ is a monic } \Leftrightarrow \ i_K(f) \text{ is a monic in } \underline{\mathrm{Ab}}_K^0 \tag{15}$$

(and similarly for "monic" replaced by "epi"). Indeed, since i_K is an embedding (hence faithful), $i_K(f)$ monic (epi) $\Rightarrow f$ monic (epi). Conversely, if f is monic and $i_K(f)g = i_K(f)h$ with $h, g \in \mathrm{Hom}^0(C, A)$, then $\exists n > 0$ such that $g[n]_C = i_K(g')$ and $h[n]_C = i_K(h')$ with $g', h' \in \mathrm{Hom}(C, A)$, where $[n]_C$ denotes the mulipication by n map on C. Then $fg' = fh'$, so $g' = h'$ and hence $g[n]_C = h[n]_C$. Thus $g = h$ because $[n]_C$ is an isomorphism in $\underline{\mathrm{Ab}}_K^0$, and so f is monic. The proof for epis is similar.

Now since every map in $\underline{\mathrm{Ab}}_K$ factors as $f = gh$ where h is a surjection (hence epi) and g is a closed immersion (hence monic), it follows from (15) that every map in $\underline{\mathrm{Ab}}_K^0$ factors as $f = gh$ with g monic, h epi. Thus condition (i) of Theorem 2.7 holds; cf. Remark 2.8(b). Moreover, $\underline{\mathrm{Ab}}_K^0$ is clearly an additive category, and condition (ii) holds for every $A \in \mathrm{ob}(\underline{\mathrm{Ab}}_K^0)$ by [Mi], p. 122. Thus, by Corollary 2.9 we see that $\underline{\mathrm{Ab}}_K^0$ is a semi-simple abelian category and that $\mathbf{Sub}_{\underline{\mathrm{Ab}}_K^0}(A) = \mathbf{Sub}'_{\underline{\mathrm{Ab}}_K^0}(A)$ and $\mathbf{Quot}_{\underline{\mathrm{Ab}}_K^0}(A) = \mathbf{Quot}'_{\underline{\mathrm{Ab}}_K^0}(A)$, for every A. By (15) we see that the rule $(B, j_B) \mapsto (i_K(B), i_K(j_B))$ defines a map $i_{K,A} : \mathbf{Sub}(A/K) \to \mathbf{Sub}_{\underline{\mathrm{Ab}}_K^0}(A)$ and similarly we have a map ${}_A i_K : \mathbf{Quot}(A/K) \to \mathbf{Quot}_{\underline{\mathrm{Ab}}_K^0}(A)$.

To see that $i_{K,A}$ is injective, suppose that $i_{K,A}(B, j_B) = i_{K,A}(B', j_{B'})$, i.e. that $i_K(j_B) = i_K(j_{B'})h$, for some isomorphism $h : i_K(B) \xrightarrow{\sim} i_K(B')$. Then $\exists n > 0$ such that $h[n]_B = i_K(h')$, for some isogeny $h' : B \to B'$, and so $j_B[n]_B = j_{B'}h'[n]_B$, or $j_B = j_{B'}h'$ since $[n]_B$ is epi. But since $\mathrm{Ker}(j_B) = 0$, it follows that $\mathrm{Ker}(h') = 0$, and so $h' : B \xrightarrow{\sim} B'$ is an isomorphism. Thus $(B, j_B) \sim (B', j'_B)$, i.e. $i_{K,A}$ is injective.

It remains to show that $i_{K,A}$ is surjective. Let $(B, f) \in \mathbf{Sub}'_{\underline{\mathrm{Ab}}_K^0}(A)$, and let g be such that $gf = 1_B$. Then $\exists n > 0$ such that $fn = i_K(f')$, $gn = i_K(g')$ with

$f' \in \mathrm{Hom}(B,A), g' \in \mathrm{Hom}(A,B)$, and so $g'f' = [n^2]_B$. Thus, f' has finite kernel and and so there exists an isogeny $\pi : B \to B' = B/\mathrm{Ker}(f)$ such that $f' = f''\pi$, where $f'' : B' \to A$ is a closed immersion (use [Mu], p. 118). Thus $(B', f'') \in \mathbf{Sub}(A/K)$ and $i_{K,A}(B', f'') \sim (B, f)$ because $i_K(f'')i_K(\pi) = f[n]_B$ (and $i_K(\pi)$ and $[n]_B$ are isomorphisms in $\underline{\mathrm{Ab}}^0_K$. Thus $i_{K,A}$ is surjective.

The proof for ${}_Ai_K$ is entirely analogous. Alternately, we can deduce it directly from what was proved above by observing that the duality functor $D_K : \underline{\mathrm{Ab}}^{op}_K \to \underline{\mathrm{Ab}}_K$ defined by $D(A) = \hat{A}$, $D(f) = \hat{f}$ induces for every A/K an order-reversing bijection

$$D_{K,A} : \mathbf{Quot}_{\underline{\mathrm{Ab}}_K}(A) = \mathbf{Sub}_{\underline{\mathrm{Ab}}^{op}_K}(A) \xrightarrow{\sim} \mathbf{Sub}_{\underline{\mathrm{Ab}}_K}(\hat{A}). \tag{16}$$

Indeed, the latter assertion is equivalent to the well-known fact (cf. [La], p. 216) that if $p : A \to C$ is any homomorphism of abelian varieties, then p is an abelian quotient $\Leftrightarrow \hat{p} : \hat{C} \to \hat{A}$ is a closed immersion.

Remark 3.2 (a) From the above proof we see that $f : A \to B$ is a monic in $\underline{\mathrm{Ab}}_K$ if and only if $f = f_0 g$, where $(A', f_0) \in \mathbf{Sub}(B/K)$ and $g : A \to A'$ is an isogeny. Thus f is monic if and only if $\mathrm{Ker}(f)$ is finite. Similarly, f is epi in $\underline{\mathrm{Ab}}_K$ if and only if $f = gf_0$, where $(C, f_0) \in \mathbf{Quot}(A/K)$ and $g : C \to B$ is an isogeny; for later reference let us write $(B, f)^0 := (C, f_0)$. Thus, f is epi if and only if f is surjective.

(b) Since $\underline{\mathrm{Ab}}^0_K$ is an abelian category, every $f \in \mathrm{Hom}^0(A,B)$ has a canonical factorization $f = jhp$ where $j : \mathrm{Im}(f) \to B$ is a monic, $p : A \to \mathrm{Coim}(f)$ is epi and $h : \mathrm{Coim}(f) \xrightarrow{\sim} \mathrm{Im}(f)$ is an isomorphism. Thus, by (a), every $f \in \mathrm{Hom}(A,B)$ has a canonical factorization $f = jhp$ where j is a closed immersion (and $\mathrm{Im}(f)$ is an abelian subvariety of B), $p : A \to \mathrm{Coim}(f) = A/\mathrm{Ker}(p)$ is an abelian quotient and $h : \mathrm{Coim}(f) \to \mathrm{Im}(f)$ is an isogeny.

We are now ready to prove Theorem 1.1 of the introduction, as well as a dual version. More precisely, we shall prove the following result.

Theorem 3.3 *Let A/K be an abelian variety and put $\mathbb{E} = \mathrm{End}^0(A)$. Then the maps $B \mapsto I(B) := j_B\mathrm{Hom}^0(A,B) = \{f \in \mathbb{E} : \mathrm{Im} f \subset B\}$ and $(C,p) \mapsto I'(C) = \mathrm{Hom}^0(C,A)p$ define lattice-preserving bijections*

$$I_{A/K} : \mathbf{Sub}(A/K) \xrightarrow{\sim} \mathbf{Id}_{\mathbb{E}} \quad \textit{and} \quad I'_{A/K} : \mathbf{Quot}(A/K) \xrightarrow{\sim} {}_{\mathbb{E}}\mathbf{Id}$$

whose inverses are given by $I^{-1}_{A/K}(\mathfrak{a}) = \mathfrak{a}A := \sum_{f\in\mathfrak{a}} \mathrm{Im}(f)$ and by $I'^{-1}_{A/K}(\mathfrak{a}) = (C_{\mathfrak{a}}, p_{\mathfrak{a}})$, respectively. Here $C_{\mathfrak{a}} = A/r_{\mathbb{E}}(\mathfrak{a})A$, where $r_{\mathbb{E}}(\mathfrak{a}) = \{f \in \mathbb{E} : \mathfrak{a}f = 0\}$ is the right annihilator of $\mathfrak{a}$, and $p_{\mathfrak{a}} : A \to C_{\mathfrak{a}}$ is the quotient map. Furthermore, the functor h^A induces an equivalence of categories

$$h^A : \underline{\mathrm{Sub}}^0_{A/K} = \underline{\mathrm{Quot}}^0_{A/K} \xrightarrow{\sim} \underline{\mathrm{Mod}}^f_{\mathbb{E}}$$

where $\underline{\mathrm{Sub}}^0_{A/K}$ *(respectively,* $\underline{\mathrm{Quot}}^0_{A/K}$*) is the full subcategory of* $\underline{\mathrm{Ab}}^0_K$ *consisting of those abelian varieties which are isogenous to a subvariety (respectively, to a quotient) of* A^n*, for some* $n \geq 1$*. In particular, for any* $B_1, B_2 \in \mathbf{Sub}(A/K)$ *and* $C_1, C_2 \in \mathbf{Quot}(A/K)$ *we have functorial isomorphisms*

$$\mathrm{Hom}^0(B_1, B_2) \xrightarrow{\sim} \mathrm{Hom}_{\mathbb{E}}(I(B_1), I(B_2)),\ \mathrm{Hom}^0(C_1, C_2) \xrightarrow{\sim} \mathrm{Hom}_{\mathbb{E}}(I'(C_2), I'(C_2)).$$

Proof. Put $I_{A/K} := I_A \circ i_{A,K}$ and $I'_{A/K} = {}_AI \circ {}_Ai_K$, so by definition $I_{A/K}(B) = j_B\mathrm{Hom}^0(A, B)$ and $I'_{A/K}(C, p) = \mathrm{Hom}^0(A, C)p$, for $B \in \mathbf{Sub}(A/K)$ and $(C, p) \in \mathbf{Quot}(A/K)$. Note that $I_{A/K}(B) = \{f \in \mathbb{E} : \mathrm{Im} f \subset B = \mathrm{Im}(j_B)\}$ by the universal property of an image (viewed as a kernel). From Proposition 3.1 and Theorem 2.7 (with $\mathcal{A} = \underline{\mathrm{Ab}}^0_K$) we see that $I_{A/K}$ and $I'_{A/K}$ are order-preserving bijections and that h^A is an equivalence of categories. (Note that since $\underline{\mathrm{Ab}}^0_K$ is semi-simple, we have $\underline{\mathrm{Sub}}^0_{A/K} = \underline{\mathrm{Quot}}^0_{A/K} = \underline{\mathrm{Ret}}^0_{A/K}$ by Corollary 2.9.) Moreover, by Remark 2.10(e) together with formula (9) we see that the inverses are given by the indicated formulae. The last assertion is an immediate consequence of Corollary 2.5.

Remark 3.4 (a) The above results give a new proof of Theorem A of [KR], which may stated as follows: if $B_1, \ldots, B_r \in \mathbf{Sub}(A/K)$ and $s < r$, then

(17) $B_1 \times \ldots \times B_s \sim B_{s+1} \times \ldots \times B_r \Leftrightarrow I(B_1) \oplus \ldots I(B_s) \simeq I(B_{s+1}) \oplus \ldots \oplus I(B_r).$

Indeed, via Proposition 3.1, this is just a restatement of (7). [To see that (17) is equivalent to Theorem A, note that $B_i = \mathrm{Im}(\varepsilon_i) = \varepsilon_i(A)$, for some idempotents $\varepsilon_i \in \mathbb{E}$ (cf. (3)), and then $I(B_i) = \varepsilon_j\mathbb{E}$. Since $\varepsilon_1\mathbb{E} \oplus \ldots \oplus \varepsilon_s\mathbb{E} \simeq \varepsilon_{s+1}\mathbb{E} \oplus \ldots \oplus \varepsilon_r\mathbb{E} \Leftrightarrow \varepsilon_1 + \ldots + \varepsilon_s \sim \varepsilon_{s+1} + \ldots + \varepsilon_r$, the equivalence of (17) and Theorem A is clear.]

(b) Recall from Remark 3.2(b) that each $f \in \mathrm{End}(A)$ has a factorization $f = jhp$, where $h : C \to B$ is an isogeny, $(B, j) \in \mathbf{Sub}(A/K)$, and $(C, p) \in \mathbf{Quot}(A/K)$. It then follows from (8) that $I_{A/K}(B) = f\mathbb{E}$ and $I'_{A/K}(C, p) = \mathbb{E}f$.

(c) Note that it follows from Proposition 3.1 that each closed immersion $j_B : B \to A$ is split in $\underline{\mathrm{Ab}}^0_K$, i.e. that $\exists h : A \to B$ such that hj_B is an isogeny. Such a map can be chosen *canonically* if we fix a polarization $\lambda = \lambda_{\mathcal{L}} : A \to \hat{A}$. Indeed, let $\lambda_B := \hat{j}_B\lambda h = \lambda_{j_B^*\mathcal{L}} : B \to \hat{B}$ denote the induced polarization on B. Then there exists a unique λ'_B such that $\lambda'_B\lambda_B = [n]_B$, for some (minimal) $n = n_B > 0$, and $N_{B,\lambda} := \lambda'_B(j_B)\hat{}\lambda : A \to B$ satisfies $N_{B,\lambda}j_B = [n_B]$. Thus $\varepsilon_{B,\lambda} := \frac{1}{n_B}N_{B,\lambda}j_B$ is a canonical generator of $I(B) = \varepsilon_{B,\lambda}\mathbb{E}$.

Using Remark 3.4(c), we see that the above theorem implies the following result of Lange[Lan] (cf. [LB], p. 126, for the case $K = \mathbb{C}$).

Corollary 3.5 (Lange) *Fix a polarization $\lambda : A \to \hat{A}$. Then the map $B \mapsto \varepsilon_{B,\lambda}$ defines a bijection*

$$\mathbf{Sub}(A/K) \quad \xrightarrow{\sim} \quad \mathbf{SymId}(\mathbb{E}) := \{\textit{symmetric idempotents of } \mathbb{E}\}$$

between the set of abelian subvarieties of A/K and the set $\mathbf{SymId}(\mathbb{E})$ of idempotents $\varepsilon \in \mathbb{E} = \mathrm{End}^0(A)$ which are symmetric with respect to the Rosati involution $a \mapsto a^* = \lambda^{-1} \circ \hat{a} \circ \lambda$ on $\mathbb{E}$.

Proof. We first show that $\varepsilon_{B,\lambda}$ is symmetric. Indeed, since λ and $\lambda_B = \hat{j}_B \lambda h$ are polarizations, we have that $\hat{\lambda}\kappa_A = \lambda$ and $\hat{\lambda}_B \kappa_B = \lambda_B$, where $\kappa_A : A \xrightarrow{\sim} \hat{\hat{A}}$ and $\kappa_B : B \xrightarrow{\sim} \hat{\hat{B}}$ are the canonical identifications. Thus $(\lambda'_B)\hat{} = \kappa_B \lambda'_B$ and $\hat{p}_B \kappa_B = \hat{\lambda}\hat{\hat{j}}_B \kappa_B = \hat{\lambda}\kappa_A j_B = \lambda j_B$. From this we obtain that $e_B \hat{\varepsilon}_B \lambda = \hat{N}_B \hat{j}_B \lambda = \hat{N}_B p_B = \hat{p}_B (\lambda'_B)\hat{}\, p_B = \hat{p}_B \kappa_B \lambda'_B p_B = \lambda j_B N_B = e_B \lambda \varepsilon_B$, i.e. that $\varepsilon_B^* = \varepsilon_B$, as claimed.

In view of Theorem 1.1 (or Theorem 3.3) it is therefore enough to show:

Each right $\mathbb{E}$-ideal $\mathfrak{a}$ is generated by a unique symmetric idempotent.

To prove this, note first that it is enough to verify the uniqueness assertion because by Theorem 1.1 and Remark 3.4(c) we know that $\mathfrak{a} = \varepsilon_{B,\lambda}\mathbb{E}$ for some $B \in \mathbf{Sub}(A/K)$. To prove uniqueness, let $\varepsilon_1, \varepsilon_2 \in \mathbf{SymId}(\mathbb{E})$ be such that $\varepsilon_1\mathbb{E} = \varepsilon_2\mathbb{E}$. Then $(1 - \varepsilon_2)\varepsilon_1\mathbb{E} = (1 - \varepsilon_2)\varepsilon_2\mathbb{E} = 0$, so $\varepsilon_2\varepsilon_1 = \varepsilon_1$, and similarly $\varepsilon_1\varepsilon_2 = \varepsilon_2$. Since ε_1 and ε_2 are symmetric, we have from the second equation that $\varepsilon_2\varepsilon_1 = \varepsilon_2^*\varepsilon_1^* = (\varepsilon_1\varepsilon_2)^* = \varepsilon_2^* = \varepsilon_2$, and hence $\varepsilon_1 = \varepsilon_2\varepsilon_1 = \varepsilon_2$.

Corollary 3.6 *Let (C, p) be an abelian quotient of A and let $\mathfrak{a} = I'_{A/K}(C, p)$ be its associated left $\mathbb{E}$-ideal. Let $B \subset A$ be the subvariety corresponding to the right $\mathbb{E}$-ideal $\mathfrak{a}^*$ which is the image of $\mathfrak{a}$ with respect to the Rosati involution * of A (associated to a polarization $\lambda : A \to \hat{A}$). Then $p_{|B} : B \to C$ is an isogeny.*

Proof. By Corollary 3.5 we have $\mathfrak{a}^* = \varepsilon_{B,\lambda}\mathbb{E}$, so $\mathfrak{a} = \mathfrak{a}^{**} = \mathbb{E}\varepsilon_{B,\lambda}^* = \mathbb{E}\varepsilon_{B,\lambda}$ since $\varepsilon_{B,\lambda}$ is symmetric. With the notation of Remark 3.4(c), write $N_{B,\lambda} = hp'$ where $(C', p') \in \mathbf{Quot}(A/K)$ and $h : C' \to B$ is an isogeny. By Remark 3.4(b) we have $I'(C', p') = \mathbb{E}\varepsilon_{B,\lambda} = I'(C, p)$, and hence by Theorem 3.3 there exists an isomorphism $f : C \xrightarrow{\sim} C'$ such that $p' = fp$. Thus $hfpj_B = hp'j_B = N_{B,\lambda}j_B = [n_B]$ is an isogeny, and hence so is $p_{|B} = pj_B$.

4 Algebraic Subspaces of $\mathbb{E}$-Modules

As before, let A/K be an abelian variety and $\mathbb{E} = \mathrm{End}^0(A)$. If V is any *faithful* left $\mathbb{E}$-module, then the results of the previous sections show that we have a canonical bijection between the set $\mathbf{Sub}(A)$ of abelian subvarieties of A/K and the set $\mathbf{Alg}_{\mathbb{E}}(V) = \{aV : a \in \mathbb{E}\}$ of *algebraic subspaces* of V. More precisely:

Theorem 4.1 *If V is a* faithful *left $\mathbb{E}$-module, then the map $B \mapsto I_{A/K}(B)V$ induces an order-preserving bijection*

$$S_{A/K,V} : \mathbf{Sub}(A/K) \overset{\sim}{\to} \mathbf{Alg}_{\mathbb{E}}(V).$$

Moreover, if V is also finitely generated, then $\mathbf{Alg}_{\mathbb{E}}(V) = \mathrm{End}_{\tilde{\mathbb{E}}}(V)$, where $\tilde{\mathbb{E}} = \mathrm{End}_{\mathbb{E}}(V)$, and hence for each left $\tilde{\mathbb{E}}$-submodule $W \subset V$ there is a unique abelian subvariety $B_W \subset A$ such that $S_{A/K,V}(B_W) = W$, and we have a ring isomorphism

$$\theta_W : \mathrm{End}_{\tilde{\mathbb{E}}}(W) \overset{\sim}{\to} \mathrm{End}^0(B_W)$$

such that $j_B\theta_W((\lambda_f)_{|W}) = f_{|B_W}$, for all $f \in \mathbb{E}$ with $f(W) \subset W$.

Proof. This follows immediately from Theorem 2.12 and Proposition 3.1 by putting $S_{A/K,V} = S_{A,V} \circ i_{A,K}$.

Most of the interesting examples of faithful $\mathbb{E}$-modules arise from faithful (covariant) functors in the following way.

Corollary 4.2 *If $F : \underline{\mathrm{Ab}}_K \to \underline{\mathrm{Vec}}_k$ is a faithful functor, where $\underline{\mathrm{Vec}}_k$ is the category of (finite dimensional) vector spaces over some field k, then for every abelian variety A/K the map $B \mapsto F(B)$ induces an order preserving bijection*

$$S_{A/K,F} : \mathbf{Sub}(A/K) \overset{\sim}{\to} \mathbf{Alg}(F(A)) = \{\mathrm{Im}(F(a)) : a \in \mathrm{End}(A)\}.$$

Moreover, if k is a finite extension of $\mathbb{Q}$, then $\mathbf{Alg}(F(A)) = \mathrm{End}_{\tilde{\mathbb{E}}}(F(A))$, where $\tilde{\mathbb{E}} = \mathrm{End}_{\mathbb{E}}(F(A))$.

Proof. Since F is faithful, it follows that $\mathrm{char}(k) = 0$ and so F extends uniquely to a (faithful) functor $F : \underline{\mathrm{Ab}}^0_K \to \underline{\mathrm{Vec}}_k$. Thus, the first assertion follows immediately from Corollary 2.14. (Note that if $a \in \mathbb{E}$, then $na \in \mathrm{End}(A)$ for some $n > 0$ and then $\mathrm{Im}(F(a)) = \mathrm{Im}(F(na))$ and hence $\mathbf{Alg}(F(A)) = \mathbf{Alg}_F(F(A))$.) Moreover, if k is a finite extension of $\mathbb{Q}$, then $F(A)$ is a finitely generated $\mathbb{E}$-module, and hence $\mathbf{Alg}(F(A)) = \mathrm{End}_{\tilde{\mathbb{E}}}(F(A))$ by Theorem 2.12.

Some examples of functors satisfying the above hypotheses are the following.

Example 4.3 (a) (Homology functor) Suppose that $K \subset \mathbb{C}$. Then we can view $A_{\mathbb{C}} := A \otimes_K \mathbb{C}$ as a complex analytic space, and so homology theory yields a faithful functor $H_1 : \underline{\mathrm{Ab}}_K \to \underline{\mathrm{Vec}}_{\mathbb{Q}}$ which is defined by $H_1(A) = H_1(A_{\mathbb{C}}^{an}, \mathbb{Q})$; cf. [Mu], p. 176. Note that $\dim_{\mathbb{Q}} H_1(A) = 2\dim(A)$.

(b) (Tangent space functor) Suppose that $\mathrm{char}(K) = 0$. Then the tangent space $T_0(A)$ of A at the origin is a K-vector space of dimension $d = \dim(A)$, and we obtain a faithful functor $T_0 : \underline{\mathrm{Ab}}_K \to \underline{\mathrm{Vec}}_K$. (To see that T_0 is faithful, reduce to the case $K = \mathbb{C}$ and use [Mu], p. 176(top).)

(c) (Tate module functor) Let K be any field and fix a prime $\ell \neq \mathrm{char}(K)$. For any abelian variety A/K, its Tate module $T_\ell(A) := T_\ell(A \otimes \overline{K})$ a free $\mathbb{Z}_\ell$-module of rank $2d$ and so $T_\ell^0(A) = T_\ell \otimes \mathbb{Q}_\ell$ is a $\mathbb{Q}_\ell$-vector space of dimension $2d$. Moreover, the induced functor $T_\ell^0 : \underline{\mathrm{Ab}}_K \to \underline{\mathrm{Vec}}_{\mathbb{Q}_\ell}$ is faithful by [Mu], p. 176ff.

Proof of Corollary 1.3. By hypothesis, $\mathrm{Hom}_{\mathbb{T}}(T_0^*(A), T_0(A)) = \mathbb{T}\varphi$ because $T_0(A)^* \simeq T_0(A) \simeq \mathbb{T}$ as $\mathbb{T}$-modules. Thus, if φ' is another isomorphism, then $\varphi' = t\varphi$ with $t \in \mathbb{T}^\times$ and so $\varphi'(W) = \varphi(tW) = \varphi(W)$. This proves the first assertion. Moreover, by Example 4.3(b), the tangent space functor T_0 satisfies the hypotheses of Corollaries 4.2 and Corollary 2.13, and so the second assertion follows. Finally, we note that $\dim_{\mathbb{Q}} W = \dim_{\mathbb{Q}} T_0(B_W) = \dim B_W$, as asserted.

For contravariant functors, we have the following dual version of Corollary 4.2.

Corollary 4.4 *If $G : \underline{\mathrm{Ab}}_K^{op} \to \underline{\mathrm{Vec}}_k$ is a faithful, contravariant functor, then for each abelian variety A/K the map $(C,p) \mapsto G(A)I'(C,p) = \mathrm{Im}(G(p)) \subset G(A)$ defines an order preserving bijection*

$$Q_G = Q_{A/K,G} : \mathbf{Quot}(A/K) \xrightarrow{\sim} \mathbf{Alg}(G(A)) = \{\mathrm{Im}(G(a)) : a \in \mathrm{End}(A)\}.$$

Moreover, if k is a finite extension of $\mathbb{Q}$, then $\mathbf{Alg}(F(A)) = \mathrm{End}_{\tilde{\mathbb{E}}}(F(A))$, where $\tilde{\mathbb{E}} = \mathrm{End}_{\mathbb{E}}(G(A)) = \{f \in \mathrm{End}(G(A)) : fG(a) = G(a)f, \forall a \in \mathbb{E}\}$. Thus, for each (left) $\tilde{\mathbb{E}}$-module $W \subset G(A)$, there is a unique $(C_W, p_W) \in \mathbf{Quot}(A/K)$ such that $\mathrm{Im}(G(p_W)) = W$, and we have $\mathrm{Ker}(p_W) = r_{\mathbb{E}}(W)A$.

Proof. Using the same reasoning as in the proof of Corollary 4.2, we see that these assertions follow immediately from Corollary 2.15.

The following are examples of such functors.

Example 4.5 (a) (Duals of covariant functors) Let $\mathcal{D}_k : \underline{\mathrm{Vec}}_k \xrightarrow{\sim} \underline{\mathrm{Vec}}_k^{op}$ denote the (contravariant) duality functor defined by $\mathcal{D}_k(V) = V^* = \mathrm{Hom}(V,k)$. Clearly, if $F : \underline{\mathrm{Ab}}_K \to \underline{\mathrm{Vec}}_k$ is any faithful *covariant* functor, then its "dual" $F^* = \mathcal{D}_k \circ F : \underline{\mathrm{Ab}}_K \to \underline{\mathrm{Vec}}_k^{op}$ is a faithful *contravariant* functor. In particular, the duals of the functors H_1, T_0 and T_ℓ^0 considered in Example 4.3 are faithful and are called the *cohomology functor* $H^1 = (H_1)^*$, the *cotangent functor* $T_0^* = (T_0)^*$ and the *étale cohomology functor* $H^1_{et}(\cdot, \mathbb{Q}_\ell) = (T_\ell^0(\cdot))^*$, respectively.

(b) (The functor of holomorphic differentials) Let $\mathrm{char}(K) = 0$ and let $\Omega : \underline{\mathrm{Ab}}_K^{op} \to \underline{\mathrm{Vec}}_K$ denote the *functor of holomorphic differentials* defined by $\Omega(A) = H^0(A, \Omega^1_{A/K})$. Since the (restriction) map $\omega \mapsto \omega_0 \in T_0^*(A)$ defines an isomorphism of functors $\Omega \simeq T_0^*$, this functor is again faithful.

Proof of Theorem 1.4. The first assertion follows from Corollary 4.4 because the functor Ω satisfies the required hypotheses by Example 4.5(b). The second assertion follows from Corollary 2.14. Finally, since $p_W^* : \Omega(C_W) \to \Omega(A)$ is injective (cf. Remark 4.6), we have $\dim C_W = \dim_K \Omega(C_W) = \dim_K p_W^*\Omega(C_W) = \dim W$, as claimed.

Remark 4.6 Let $G : \underline{\mathrm{Ab}}_K^{op} \to \underline{\mathrm{Vec}}_k$ be as in Corollary 4.4, and let $f : A \to A'$ be a homomorphism of abelian varieties. If f is surjective, then f is a split epi in $\underline{\mathrm{Ab}}_K^0$ (cf. Proposition 3.1) and hence $G(f) : G(A') \to G(A)$ is injective because G maps split epis in $\underline{\mathrm{Ab}}_K^0$ to monics in $\underline{\mathrm{Vec}}_k$. (Similarly, if f has finite kernel, then $G(f)$ is surjective, and if f is an isogeny, then $G(f)$ is bijective.) Thus, $G(f)$ induces an injection $G(f) : \mathbf{Sub}_k(G(A')) \hookrightarrow \mathbf{Sub}_k(G(A))$ which maps algebraic subspaces to algebraic subspaces because we have

$$(18) \qquad G(f)(Q_G(C',p')) = Q_G((C',p'f)^0), \quad \text{for all } (C',p') \in \mathbf{Quot}(A'/K).$$

Indeed, if we write $(C,p) := (C',p'f)^0 \in \mathbf{Quot}(A/K)$ (cf. Remark 3.2(a)), then there is an isogeny $g : C \to C'$ such that $gp = p'f$, and we have $G(f)(Q_G(C',p')) = G(f)(\mathrm{Im}(G(p'))) = \mathrm{Im}(G(p'f)) = \mathrm{Im}(G(p)G(g)) = \mathrm{Im}(G(p)) = Q_G((C',p'f)^0)$ because $G(g)$ is bijective.

The following corollary is frequently useful in applications.

Corollary 4.7 *If $(C,p),(C_1,p_1),\ldots,(C_n,p_n) \in \mathbf{Quot}(A/K)$ are abelian quotients such that $Q_G(C,p) = \sum Q_G(C_i,p_i)$, where G is as in Corollary 4.4, then there is a unique homomorphism $\nu : C \to C_1\times\ldots\times C_n$ with finite kernel such that $(p_1,\ldots,p_n) = \nu \circ p$. Furthermore, ν is an isogeny if and only if the sum of the $Q_G(C_i,p_i)$'s is a direct sum.*

Proof. Since Q_G is a lattice isomorphism by Corollary 4.4, it follows from the hypothesis that (C,p) is the maximum of the (C_i,p_i)'s, and so the first assertion follows from Remark 2.10(d). Moreover, since $Q_G(C_i,p_i) = Q_G(A)I'(C_i,p_i)$ and since $G(A)$ is a faithfully flat $\mathbb{E}$-module (cf. proof of Theorem 2.12), we see that the sum of the $Q_G(C_i,p_i)$'s is a direct sum if and only if the $I'(C_i,p_i)$'s are a direct sum, and so the assertion follows immediately from (the dual version of) (12).

5 Applications to Modular Curves

Let $X_{\Gamma,\mathbb{C}} = \Gamma\backslash\mathfrak{H}^*$ be the complex modular curve attached to a subgroup $\Gamma \leq \mathrm{SL}_2(\mathbb{Z})$ *of level* $N \geq 1$, i.e. $\Gamma_0(N) \leq \Gamma \leq \Gamma_1(N)$. (We could also take $\Gamma = \Gamma(N)$.) By Shimura[Sh1], $X_{\Gamma,\mathbb{C}}$ has a "canonical" model $X = X_\Gamma/\mathbb{Q}$ over $\mathbb{Q}$ such that the map $f \mapsto f dz$ induces a natural identification $S_2(\Gamma,\mathbb{Q}) \xrightarrow{\sim} H^0(X,\Omega^1_{X/\mathbb{Q}})$ where $S_2(\Gamma,\mathbb{Q}) \subset$

$S_2(\Gamma)$ denotes the subspace of all cusp forms of weight 2 on Γ whose Fourier expansion (at the cusp ∞) have rational coefficients; cf. [Sh1], p. 156 and p. 140 or [DDT], p. 35. Thus, if $J = J_\Gamma/\mathbb{Q}$ denotes the Jacobian variety of X_Γ, then we have a canonical identification

$$\Omega(J) := H^0(J, \Omega^1_{J/\mathbb{Q}}) = H^0(X, \Omega^1_{X/\mathbb{Q}}) = S_2(\Gamma, \mathbb{Q}).$$

By Hecke's theory, there is a commutative subring (called the *Hecke algebra*) $\mathbb{T} = \mathbb{Q}[\{T_n\}_{n\geq 1}] \subset \mathbb{E} = \mathrm{End}^0_{\mathbb{Q}}(J)$ such that $S_2(\Gamma, \mathbb{Q})^* \simeq T_0(J)$ is a free $\mathbb{T}$-module of rank 1; cf. [Sh1], Theorem 3.51 or [DDT], Lemma 1.34. Furthermore, $\Omega(J) = S_2(\Gamma, \mathbb{Q})$ is also a free $\mathbb{T}$-module of rank 1, as can be seen either from [Sh1], Theorem 3.51 or by Atkin-Lehner Theory (cf. [DDT], Lemma 1.35), and so there is a $\mathbb{T}$-module isomorphism $\varphi : \Omega(J) \xrightarrow{\sim} T_0(J)$. We then have the following generalization of the Shimura construction:

Theorem 5.1 *Let $W \subset S_2(\Gamma, \mathbb{Q})$ be any $\mathbb{T}$-submodule. Then:*

(a) There exists a unique abelian quotient $(A_W, p_W) \in \mathbf{Quot}(J_\Gamma)$ such that $p_W^*\Omega(A_W) = W$. Furthermore, $\dim A_W = \dim_{\mathbb{Q}} W$ and we have an injective ring homomorphism $\theta_W : \mathrm{End}_{\mathbb{T}}(W) \hookrightarrow \mathrm{End}^0_{\mathbb{Q}}(A_W)$ such that $\theta_W(t_{|W}) \circ p_W = p_W \circ t$, for all $t \in \mathbb{T}$.

(b) Fix a $\mathbb{T}$-module isomorphism $\varphi : S_2(\Gamma, \mathbb{Q}) \xrightarrow{\sim} T_0(J)$. Then $\varphi(W)$ does not depend on the choice of φ, and there exists a unique abelian subvariety $A'_W \in \mathbf{Sub}(J_\Gamma)$ such that $\varphi_\Gamma(W) = T_0(A'_W) \subset T_0(J)$. Furthermore, $\dim A'_W = \dim_{\mathbb{Q}} W$ and there exists a ring injection $\theta'_W : \mathrm{End}_{\mathbb{T}}(W) \hookrightarrow \mathrm{End}^0(A'_W)$ such that $\theta'_W(t_{|W}) = t_{|A'_W}$, for all $t \in \mathbb{T}$.

(c) The restriction $p'_W = (p_W)_{|A'_W} : A'_W \to A_W$ of p_W to A'_W is an isogeny provided that $(W \otimes \mathbb{C})|w_n = W \otimes \mathbb{C}$, where $w_N = \left(\begin{smallmatrix} 0 & -1 \\ N & 0 \end{smallmatrix}\right)$.

Proof. (a) By the above discussion, this follows immediately from Theorem 1.4.

(b) Via the identification $S_2(\Gamma, \mathbb{Q}) = T_0(J)^*$, all assertions except the last follow from Corollary 1.3. The last assertion follows from Corollary 2.14.

(c) Since W is an $\tilde{\mathbb{E}}$-module, we have by Corollary 4.4 that $W = \Omega(J)\mathfrak{a}$ for some left ideal $\mathfrak{a}$ of $\mathbb{E}$. Since w_N induces an automorphism of $\Omega(J \otimes \mathbb{C})$, the hypothesis implies that $\Omega(J \otimes \mathbb{C})w_N^{-1}\mathfrak{a}w_N = \Omega(J \otimes \mathbb{C})\mathfrak{a}$. Thus, since $w_N^{-1}\mathfrak{a}w_N$ is a left ideal of $\mathbb{E}$ (cf. Lemma 5.2 below), it follows that $w_N^{-1}\mathfrak{a}w_N = \mathfrak{a}$. Thus, using the map φ of Lemma 5.2, we have by (19) that $\varphi(W) = \varphi(\Omega(J)w_N\mathfrak{a}w_N^{-1}) = \mathfrak{a}^*T_0(J)$ (because $w_N^* = w_N^{-1}$). This shows that $\varphi(W) = \mathfrak{a}^*T_0(J)$, and so the assertion follows from Corollary 3.6.

Lemma 5.2 *The map $\alpha \mapsto w_N^{-1}\alpha w_N$ defines an automorphism of $\mathbb{E} = \mathrm{End}^0_{\mathbb{Q}}(J)$. Thus, if $\alpha \mapsto \alpha^*$ denotes the Rosati involution on $\mathbb{E}$ induced by the canonical polarization of J, then there exists a $\mathbb{T}$-module isomorphism $\varphi : S_2(\Gamma, \mathbb{Q}) \xrightarrow{\sim} T_0(J)$ such that*

(19) $$\varphi(f\alpha) = (w_N^{-1}\alpha^* w_n)\varphi(f), \quad \textit{for all } f \in S_2(\Gamma, \mathbb{Q}),\ \alpha \in \mathbb{E}.$$

Proof. We first verify that $w_N^{-1}\alpha w_n \in \mathbb{E}$, if $\alpha \in \mathbb{E}$. For this we observe that the automorphism w_N is defined over $K := \mathbb{Q}(\zeta_N)$ and satisfies $w_N^{\tau_a} = w_N\sigma_a^{-1}$, where $\tau_a \in \mathrm{Gal}(K/\mathbb{Q})$ is given by $\zeta_N^{\tau_a} = \zeta_N^a$ and $\sigma_a \equiv \left(\begin{smallmatrix} a^{-1} & 0 \\ 0 & a \end{smallmatrix}\right) \pmod N$. Since $\sigma_a \in \mathbb{T}'$ is in the centre of $\mathbb{E}$ (cf. [Ka], Corollary 2), it follows that $w_N^{-1}\alpha w_n \in \mathrm{End}_K^0(J)$ is τ_a-invariant and hence lies in $\mathbb{E}$.

It thus follows that $T_0(J)$ becomes a *right* $\mathbb{E}$-module via the rule $v\alpha = (w_N^{-1}\alpha^* w_N)v$ (for $v \in T_0(J)$). Moreover, $T_0(J)$ is a faithful right $\mathbb{E}$-module under this twisted action because $T_0(J)$ is a faithful left $\mathbb{E}$-module under the usual action. Thus, since $\mathbb{E}$ is semisimple, it follows that every irreducible $\mathbb{E}$-module apears at least once in the decomposition of $T_0(J)$ into irreducible submodules. Now since $\Omega(J) = S_2(\Gamma, \mathbb{Q})$ is also faithful right $\mathbb{E}$-module, the same is true for $\Omega(J)$. Moreover, since $\mathrm{End}_{\mathbb{E}}(\Omega(J)) \subset \mathrm{End}_{\mathbb{T}}(\Omega(J)) = \mathbb{T}$ is commutative, it follows that every irreducible $\mathbb{E}$-module appears only once in the $\mathbb{E}$-module decomposition of $\Omega(J)$, and so we see that we have an $\mathbb{E}$-module embedding $\varphi : \Omega(J) \to T_0(J)$, which is an isomorphism because $\dim \Omega(J) = \dim T_0(J)$. Moreover, since $w_N^{-1} t w_N = t^*$, for all $t \in \mathbb{T}$, it follows that twisted action of $\mathbb{T}$ on $T_0(J)$ is the same as the usual action of $\mathbb{T}$ on $T_0(J)$. In particular, φ is a $\mathbb{T}$-module isomorphism satisfying (19).

The classical Shimura construction is the following special case of the above theorem.

Example 5.3 (Shimura) Let $f \in S_2(\Gamma)$ be a (normalized) $\mathbb{T}$-eigenfunction, and put $W_{f,\mathbb{C}} = \sum \mathbb{C} f^\sigma$, where the sum is over all $\mathrm{Aut}(\mathbb{C})$-conjugates of f. Clearly, $W_{f,\mathbb{C}}$ is $\mathrm{Aut}(\mathbb{C})$-invariant and hence is of the form $W_{f,\mathbb{C}} = W_f \otimes \mathbb{C}$ for a unique subspace $W_f \subset S_2(\Gamma, \mathbb{Q})$. Moreover, $\dim_{\mathbb{Q}} W_f = \dim_{\mathbb{C}} W_{f,\mathbb{C}} = [K_f : \mathbb{Q}]$, where K_f is the field generated by the Fourier coefficients $a_n(f)$ of f. Let $\lambda_f : \mathbb{T} \to K_f$ denote the canonical surjective homomorphism defined by $f|t = \lambda_f(t)f$, for $t \in \mathbb{T}$. (In particular, $\lambda_f(T_n) = a_n(f)$, where T_n is the n-th Hecke operator.) It is then immediate that $\mathrm{Ann}_{\mathbb{T}}(W_f) = \mathrm{Ker}(\lambda_f)$, and thus we have a natural injection $K_f = \mathbb{T}/\mathrm{Ann}_{\mathbb{T}}(W_f) \hookrightarrow \mathrm{End}_{\mathbb{T}}(W)$. Thus, by the above theorem there exists an abelian subvariety $A'_f = A'_{W_f} \leq J = J_\Gamma$ and an abelian quotient $p = p_f : J \to A_f := A_{W_f}$ together with maps $\theta'_f : K_f \hookrightarrow \mathrm{End}^0_{\mathbb{Q}}(A'_f)$ and $\theta_f : K_f \hookrightarrow \mathrm{End}^0_{\mathbb{Q}}(A_f)$ such that (A'_f, θ'_f) and (A_f, p_f, θ_f) satisfy the following conditions (which are in fact identical to those of Theorems 1 and 2 of [Sh2]):

(i) $A'_f \in \mathbf{Sub}(J/\mathbb{Q})$ and $(A_f, p_f) \in \mathbf{Quot}(J/\mathbb{Q})$.

(ii) $\theta'_f : K_f \hookrightarrow \mathrm{End}^0_{\mathbb{Q}}(A'_f)$ and $\theta_f : K_f \hookrightarrow \mathrm{End}^0_{\mathbb{Q}}(A'_f)$ are injective ring homomorphisms such that $\theta'_f(a_n(f)) = (T_n)_{|A'_f}$ and $\theta_f(a_n(f)) \circ p_f = p_f \circ T_n$, for all $n \geq 1$.

(iii) $\dim A'_f = \dim A_f = [K_f : \mathbb{Q}]$.

(iv) $T_0(A'_f) = \varphi_\Gamma(W_f)$ and $p_f^* \Omega(A_f) = W_f$.

There are other natural $\mathbb{T}$-submodules to which the (generalized) Shimura construction can be applied, as the following example shows.

Example 5.4 Let $f \in S_2(\Gamma)$ be a newform of some level $N_f|N$ and let $S_f = \sum_{d|(N/N_f)} \mathbb{C}f|\beta_d$ denote the Atkin-Lehner eigenspace of f, where $\beta_d = \left(\begin{smallmatrix} d & 0 \\ 0 & 1 \end{smallmatrix}\right)$. Moreover, let $S_{[f]} = (\sum_\sigma S_{f^\sigma}) \cap S_2(\Gamma, \mathbb{Q})$, where $\sigma \in G_\mathbb{Q}$ runs over all elements of the absolute Galois group $G_\mathbb{Q} = \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$. Then by Atkin-Lehner theory (cf. [DDT], p. 36) each $S_{[f]}$ is a $\mathbb{T}$-module of dimension $\sigma_0(N/N_f)[K_f : \mathbb{Q}]$ and we have the following $\mathbb{T}$-module decomposition in which the sum extends over the set $\mathcal{N}(\Gamma)$ of all newforms of all levels modulo the action of $G_\mathbb{Q}$:

$$S_2(\Gamma, \mathbb{Q}) = \bigoplus_{f \in \mathcal{N}(\Gamma)/G_\mathbb{Q}} S_{[f]}$$

Thus, by Theorem 5.1 there exists a unique abelian quotient $(\tilde{A}_{[f]}, p_{[f]})$ (respectively, abelian subvariety $\tilde{A}'_{[f]}$) of J_Γ such that $p^*_{[f]}\Omega(\tilde{A}_{[f]}) = S_{[f]}$ (respectively, such that $\varphi_\Gamma(S_{[f]}) = T_0(\tilde{A}'_{[f]}) \subset T_0(J)$), and $\dim \tilde{A}_{[f]} = \tilde{A}'_{[f]} = \dim_\mathbb{Q} S_{[f]}$. Moreover, by Corollary 4.7 it follows from the above decomposition that we have isogenies

$$J_\Gamma \sim \prod \tilde{A}_{[f]} \sim \prod \tilde{A}'_{[f]}. \tag{20}$$

Since in general $\mathbb{T} \neq \tilde{\mathbb{E}}$ (because $\tilde{\mathbb{E}}$ is semi-simple whereas in general $\mathbb{T}$ is not), we cannot expect that every abelian subvariety and/or quotient can be obtained by the Shimura construction (Theorem 5.1). For example, the following varieties $A_{[f],d}$ cannot be obtained from the Shimura construction.

Example 5.5 As before, let $f \in \mathcal{N}(\Gamma)$ be a newform of level $N_f|N$ and assume for simplicity that $\Gamma = \Gamma_1(N)$. For a divisor $d|\frac{N}{N_f}$ put $S_{f,d} = \mathbb{C}f|\beta_d$ and $S_{[f],d} = \sum \sigma(S_{f^\sigma,d}) \cap S_2(\Gamma, \mathbb{Q})$; thus $S_f = \oplus_d S_{f,d}$ and $S_{[f]} = \oplus_d S_{[f],d}$. In general, $S_{[f],d}$ is not a $\mathbb{T}$-module because $S_{[f],d}|T_p \subset S_{[f],d/p}$, if $p|d$. However, $S_{[f],d}$ is an algebraic subspace of $S_2(\Gamma, \mathbb{Q}) = \Omega(J_\Gamma)$ (as we shall see below) and hence corresponds to an abelian quotient $p_{[f],d} : J_\Gamma \to A_{[f],d}$.

To see that $S_{[f],d}$ is algebraic, we shall use the "degeneracy map" $\pi_{N_f,d} : X_1(N) \to X_1(N_f)$ which is induced by the map $g \mapsto g|\beta_d$ on the function fields. Thus, if $\tilde{\pi}_{N_f,d} = (\pi_{N_f,d})_* : J_1(N) \to J_1(N_f)$ denotes the Albanese map induced by autoduality of the Jacobian, then we have $\tilde{\pi}^*_{N_f,d} W_f = S_{[f],d}$. Since W_f is algebraic by Example 5.3 (because f is a $\mathbb{T}$-eigenform in $S_2(\Gamma_1(N_f))$) with corresponding quotient $(A_f, p_f) \in$ **Quot**$(J_1(N_f))$, we see from Remark 4.6 that $S_{[f],d}$ is also algebraic with corresponding abelian quotient $(A_{[f],d}, p_{[f],d}) := (A_f, p_f\tilde{\pi}_{N_f,d})^0$. In particular, there is an isogeny $\nu_{[f],d} : A_{[f],d} \to A_f$ such that $p_f \circ \tilde{\pi}_{N_f,d} = \nu_{[f],d} \circ p_{[f],d}$. Thus, since $S_{[f]} = \oplus_d S_{[f],d}$, it follows from Corollary 4.7 that

$$\tilde{A}_{[f]} \sim A_f^{n_f}, \quad \text{where } n_f = \sigma_0(N/N_f). \tag{21}$$

Combining this with the relation (20), we thus obtain the isogeny relation

$$J_1(N) \sim \prod_{f \in \mathcal{N}(\Gamma_1(N))/G_{\mathbb{Q}}} A_f^{n_f}. \tag{22}$$

Remark 5.6 (a) It follows from the work of Ribet that the above isogeny relation (22) is the isogeny decomposition of $J_1(N)$, for by Ribet we have that each A_f is a $\mathbb{Q}$-simple abelian variety and that $A_f \sim A'_f \Leftrightarrow [f] = [f']$.

(b) In [Ka] it is shown that in fact $\tilde{\mathbb{E}} := \mathrm{End}_{\mathbb{E}}(\Omega(J_\Gamma)) = \mathbb{T}'$, where $\mathbb{T}' = \mathbb{Q}[\{T_n : (n, N) = 1\}] \subset \mathbb{T}$. Thus, by the dictionary of Corollary 4.4 we have that the map $(C, p) \mapsto p^*\Omega(C) \subset \Omega(J_\Gamma) = S_2(\Gamma, \mathbb{Q})$ induces a bijection

$$\mathbf{Quot}(J_\Gamma) \overset{\sim}{\to} \mathbf{Sub}_{\mathbb{T}'}(S_2(\Gamma, \mathbb{Q}))$$

between the set of abelian quotients of $J_\Gamma/\mathbb{Q}$ and the set of $\mathbb{T}'$-submodules of $S_2(\Gamma, \mathbb{Q})$. Furthermore, if (C_i, p_i) $i = 1, 2$ are two such quotients, then we have a canonical isomorphism

$$\mathrm{Hom}^0_{\mathbb{Q}}(C_1, C_2) \simeq \mathrm{Hom}_{\mathbb{T}'}(p^*\Omega(C_2), p^*\Omega(C_1)).$$

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