Methods of Integration

1. Complexity of differentiation.

Compared to integration, differentiation is relatively easy. One reason is that there are simple rules for differentiation: rules for differentiating compositions of functions (the chain rule), products (the product rule), and quotients (the quotient rule).

The functions we deal with are usually built up out of simple functions (like polynomials, $\sin$, $\cos$, $\log$ or e^x) using multiplication or composition to get something more complicated.

It follows that if we know the differentiation rules, and know how to differentiate the simple functions, we can differentiate the complicated functions as well – it's a matter of following the rules and not getting lost.

As an analogy, differentiation is something like unfolding a tightly wrapped object. As long as we start from the outside, and follow some simple procedures, we can unwrap the whole thing without too much trouble.

2. Complexity of Integration.

In comparison, integration is the problem of starting with the unwrapped object, and trying to fold it back up again. It's much more difficult, and it's not as clear what to do, or what order to do it in.

One problem is that there are no simple rules like that for differentiation (and in fact there cannot be any such simple rules).

For instance, the product rule for differentiation tells us that if we have a product of two functions $f(x)g(x)$, then the derivative is $f'(x)g(x) + f(x)g'(x)$. Besides giving us an explicit formula for the derivative, the product rule also assures us that *if we know the derivatives of $f(x)$ and $g(x)$, we know the derivative of $f(x)g(x)$*, no further work is required.

The corresponding statement is simply not true for integration: If we know the integral of $f(x)$ and $g(x)$, we don't really know anything about the integral of $f(x)g(x)$, and it's not even clear that we can find the integral of $f(x)g(x)$.

3. Methods of Integration.

In order to try and integrate functions, there are two main approaches

Develop methods to try and simplify the problem, one step at a time.

This is comparable to taking our unwrapped object and trying out a single fold, to see if it helps us any.

Methods in this category include substitution and integration by parts. We hope that after doing this we've made the problem simpler.

Of course, it may happen that after doing one of these steps the problem still complicated, and we may have to try one or more of the steps again, repeating until we get down to a problem that we can solve.

Or, it may happen that after several such simplifications, we realize that we've gone down the wrong path, and that we have to back up and try again.

There is nothing that can be done about this, it's just the nature of the problem.

The second approach is

Forget about trying to find an antiderivative, and use numerical methods to get a number.

This is only useful when we are actually looking for a number as an answer, for instance when we are trying to find the area under a graph. It doesn't help if we are trying to find the antiderivative of a function.

There are many, many numerical methods for trying to compute a definite integral. The simplest are the trapezoid rule and Simpson's rule.

This approach may seem a bit defeatist, but it is actually sometimes necessary. It was discovered in the 19th century (by Liouville) that functions like e^{x^2} and $\sin(x)/x$ have *no* antiderivative that can be written in terms of the simple functions we use. That is, no matter how clever you are, and no matter how complicated an expression you write down involving $\sin$, $\cos$, e^x, $\log$, and things like that, you will never be able to write down the antiderivative of those two functions – their antiderivatives simply can't be written that way.

For these kinds of functions the only hope for an answer is to use some kind of numerical method.

4. Description of integration by parts as a "guess and fix" method.

Integration by parts is a simplification technique we can apply when we are trying to integrate the product of two functions. As always with the simplification methods we only use it if it actually does help to make the problem easier.

There are various ways to explain where the integration by parts formula comes from. One way is to use the fact that integration is the reverse process to differentiation, and ask what the product rule for differentiation means for integrals. This approach is described in the class notes.

Another method can be explained along the following lines: guess at a possibility for the antiderivative, see that it is not quite right, and then add in a correction term to fix it.

For instance, suppose that we're trying to find the integral of a product

$$\int f(x)g(x)\ dx.$$

Suppose that we know an antiderivative for $f(x)$; that is, we've found some function $F(x)$ with $F'(x) = f(x)$.

Apparently out of the blue, let's consider what the derivative of $F(x)g(x)$ is. By the product rule:

$$\begin{aligned} \frac{d}{dx}F(x)g(x) &= F'(x)g(x) + F(x)g'(x) \\ &= f(x)g(x) + F(x)g'(x). \end{aligned}$$

(the last step comes from $F'(x) = f(x)$.)

What's the use of that? Well, we're looking for something which has as derivative $f(x)g(x)$. What we've found is something which has as derivative $f(x)g(x)$ plus $F(x)g'(x)$. If only we could get rid of the second term, we'd be in business.

So, what we do is look for something whose derivative is $F(x)g'(x)$. Hopefully this is easy to find, or at least easier than finding an antiderivative for $f(x)g(x)$.

Suppose we succeed in finding a function, say called $H(x)$, whose derivative is $F(x)g'(x)$ (i.e., $H'(x) = F(x)g'(x)$). Then if we compute the derivative of $F(x)g(x) - H(x)$, we see that we get:

$$\begin{aligned} \frac{d}{dx}\left(F(x)g(x) - H(x)\right) &= F'(x)g(x) + F(x)g'(x) - H'(x) \\ &= f(x)g(x) + F(x)g'(x) - F(x)g'(x) \\ &= f(x)g(x), \end{aligned}$$

exactly what we were looking for.

One way to express this little trick is to say that

$$\int f(x)g(x)\ dx = F(x)g(x) - H(x).$$

Where $F(x)$ is an antiderivative for $f(x)$, and $H(x)$ is an antiderivative for $F(x)g'(x)$.

Another way to say that "$H(x)$ is the antiderivative of $F(x)g'(x)$" is by the symbols

$$H(x) = \int F(x)g'(x)\ dx.$$

Substituting this into the above formula, we can write our trick as

$$\int f(x)g(x)\ dx = F(x)g(x) - \int F(x)g'(x)\ dx.$$

which is exactly integration by parts.

5. How to use integration by parts.

Whenever we're faced with the problem of integrating a product of functions, one thing to think about is whether integration by parts will help you simplify the problem.

To use integration by parts, we need to give the pieces of the product names. One of the pieces we call $f(x)$. That's the piece we'll try and integrate on its own, to find an antiderivative $F(x)$.

The other piece we'll call $g(x)$. That piece we'll differentiate, and then try and find an antiderivative for $F(x)g'(x)$.

There is no hard and fast rule to decide which part of the product gets to be $f(x)$ and which gets to be $g(x)$, but, keeping in mind that our goal is to either solve or simplify the integral, we usually try to choose $f(x)$ and $g(x)$ so that

- We can find the antiderivative $F(x)$ of $f(x)$.
- We can find the antiderivative of $F(x)g'(x)$. Or, failing that, that the problem of finding the antiderivative of $F(x)g'(x)$ seems easier than the original problem of finding the antiderivative of $f(x)g(x)$.

Choices of $f(x)$ and $g(x)$ which satisfy the two rules above are ones that are going to make our problem simpler, and so are probably good choices.

A little practice using integration by parts will make is seem much clearer. The knack of choosing $f(x)$ and $g(x)$ is also something which becomes easier with some experience in using integration by parts.

This handout can (soon) be found at

http://www.mast.queensu.ca/∼mikeroth/calculus/calculus.html

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