An example of integration by parts.

We want to find the antiderivative of

$$\int x^5 e^{x^2}\ dx.$$

There is more than one way to do this. The point of this handout is to try two different ways, and to practice combining substitution with integration by parts.

The first method is to use substitution to make the integral easier, and then use integration by parts.

The second is to use integration by parts directly. Here it might be a little harder to see how to choose the parts.

1. Substitution, then integration by parts.

Starting with $u = x^2$, we compute $du = 2x\,dx$. Solving for dx gives $dx = \dfrac{du}{2x}$. If we substitute the formula for dx into the original integral, we get

$$\int x^5 e^{x^2}\ \frac{du}{2x} \ = \ \int \frac{1}{2} x^4 e^{x^2}\ du.$$

We now want to write everything leftover in terms of u. Since $u = x^2$, we can write e^{x^2} as e^u, and x^4 as u^2. So, after substitution, the integral becomes

$$\int \frac{1}{2} u^2 e^u\ du.$$

This looks a lot like the integrals we've been considering, and seems a good candidate for integration by parts.

Depending on how you like to remember integration by parts, you might run into a little notational problem trying to solve the integral above. If you like the $u\,dv$ method, then it's going to be a bit awkward to apply – there's already a variable called "u" in the equation, so it's going to be confusing trying to keep track of which is the "old" u, and which is the "new" u.

A way out is to change the u to any other name (other than v, of course). For example, we could call it w, so that we're looking for the integral

$$\int \frac{1}{2} w^2 e^w\ dw;$$

after we solve the integral we'll go back and replace the w's by x^2, just as we would have done with the u's.

We've seen integrals like the one above before. We know that the way to apply integration by parts is to make sure that the w^2 gets differentiated, and the e^w integrated. The resulting integral will be simpler than the one we started with.

In the $u\,dv$ notation, that means that we set

$$\begin{array}{lcl} u = \frac{1}{2}w^2, & \overset{\text{diff}}{\rightsquigarrow} & du = w \\ dv = e^w, & \overset{\text{int}}{\rightsquigarrow} & v = e^w. \end{array}$$

and that gives us

$$\int \frac{1}{2}w^2 e^w \; dw = \frac{1}{2}w^2 e^w - \int w e^w \; dw. \tag{1}$$

To deal with the second integral, $\int we^w \; dw$, we use integration by parts again. We use the same kind of pattern – differentiate w to reduce the power of w in the integral.

Picking

$$\begin{array}{lcl} u = w, & \overset{\text{diff}}{\rightsquigarrow} & du = 1 \\ dv = e^w, & \overset{\text{int}}{\rightsquigarrow} & v = e^w. \end{array}$$

we get

$$\begin{aligned} \int we^w \; dw &= we^w - \int e^w \; dw \\ &= we^w - e^w. \end{aligned}$$

Now we can substitute the integral of we^w back into equation (1), to get

$$\int \frac{1}{2}w^2 e^w \; dw = \frac{1}{2}w^2 e^w - we^w + e^w.$$

Finally, now that we know how to solve the integral in w, we substitute back in $w = x^2$ to get the solution

$$\int x^5 e^{x^2} \; dx = \frac{1}{2}x^4 e^{x^2} - x^2 e^{x^2} + e^{x^2}.$$

2. Integration by parts directly.

The trouble here is to figure out how to pick the parts to start off.

It's tempting to try the same kind of trick – differentiate the x^5 part and integrate e^{x^2} to try and simplify the integral. One problem is that there is no way to integrate e^{x^2} by itself – it just can't be done (and it's a famous example of a function which can't).

There is something closely resembling it which we can integrate: xe^{x^2} does have an easy antiderivative, $\frac{1}{2}e^{x^2}$ (differentiate to see why).

That suggests that we split the product up as $x^5e^{x^2} = x^4 \cdot xe^{x^2}$ and differentiate the x^4 piece and then integrate the xe^{x^2} piece. So, let's set

$$\begin{array}{lcl} u = x^4, & \overset{\text{diff}}{\rightsquigarrow} & du = 4x^3 \\ dv = xe^{x^2}, & \overset{\text{int}}{\rightsquigarrow} & v = \frac{1}{2}e^{x^2}. \end{array}$$

so that we have

(2) $$\int x^5 e^{x^2}\ dx = \frac{1}{2}x^4e^{x^2} - \int 2x^3e^{x^2}\ dx.$$

To deal with $\int 2x^3e^{x^2}\ dx$, we use integration by parts again. Again we're faced with the problem of integrating e^{x^2}, and again we we'll have to include an extra factor of x with the e^{x^2} to make it work out.

$$\begin{array}{lcl} u = 2x^2, & \overset{\text{diff}}{\rightsquigarrow} & du = 4x \\ dv = xe^{x^2}, & \overset{\text{int}}{\rightsquigarrow} & v = \frac{1}{2}e^{x^2}. \end{array}$$

$$\begin{aligned} \int 2x^3e^{x^2}\ dx &= x^2e^{x^2} - \int 2xe^{x^2}\ dx \\ &= x^2e^{x^2} - e^{x^2}. \end{aligned}$$

Substituting this back into equation (2), we get

$$\int x^5e^{x^2}\ dx = \frac{1}{2}x^4e^{x^2} - x^2e^{x^2} + e^{x^2}$$

just like before.

The method of integration by parts directly may have seemed simpler, but it was perhaps less obvious how to choose the parts to make everything work out.

3. A common mistake.

The most common mistake in trying to solve this integral is becoming confused about the antiderivative of e^{x^2}. A common guess is:

$$\int e^{x^2}\ dx = \frac{e^{x^2}}{x^2+1},$$

which comes from misapplying the rule for integrating x^n – that rule only works when the exponent is a number, but doesn't work when the exponent is a function.

Another common guess is to note that $\frac{d}{dx}e^{x^2} = 2xe^{x^2}$ and try and "fix it up" by adding $2x$ to the denominator:

$$\int e^{x^2}\ dx = \frac{e^{x^2}}{2x}.$$

That isn't right either – when differentiating something like $\frac{e^{x^2}}{2x}$ the quotient rule has to be used, and its derivative is

$$\frac{d}{dx}\left(\frac{e^{x^2}}{2x}\right) = \frac{2xe^{x^2}\cdot 2x - e^{x^2}\cdot 2}{(2x)^2} = e^{x^2} - \frac{e^{x^2}}{2x^2},$$

which isn't e^{x^2}.

In fact, e^{x^2} *has* no antiderivative that we can write down using any functions we know. It's a fact that's worth remembering – if you see e^{x^2} in an integral, you know you're going to have to work around it somehow.

The only thing close to e^{x^2} which we can integrate is xe^{x^2}. It doesn't seem so different from e^{x^2}, but actually it is quite different, it has antiderivative we can find:

$$\int xe^{x^2}\ dx = \frac{1}{2}e^{x^2}.$$

That fact is why we had to pick the parts the way we did in section 2.

This handout can (soon) be found at

http://www.mast.queensu.ca/∼mikeroth/calculus/calculus.html

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