Riemann Sums and Finding Integrals.

1. The Riemann Sum as an approximation to an integral.

When we're trying to find the area under a graph, we know that the exact answer is given by an integral. In fact, by definition the symbol $\int_a^b f(x)\ dx$ means "that number which is the exact area under the graph of $y = f(x)$ from $x = a$ to $x = b$".

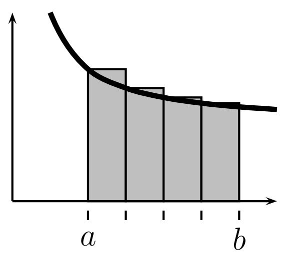
Sometimes, instead of finding the exact area, we approximate it by using a Riemann sum. (Up until now we've only done this in situations where we couldn't compute the area exactly. Today there will be a different reason to do this.)

Given a and b, a function $f(x)$, and a positive integer n, the symbol

$$\sum_{i=1}^{n} f(x_i)\ dx$$

is code for "divide the interval from $x = a$ to $x = b$ up into n pieces, for each piece approximate the area with a rectangle, then add up the areas of the rectangles". In pictures,

$$\sum_{i=1}^{4} f(x_i)\ dx \qquad \text{means}$$



If we let n get larger and larger, the number in the approximation gets closer and closer to the actual area under the graph, that is, to the actual integral. In symbols, we'd write this as

$$\lim_{n\to\infty} \left(\sum_{i=1}^{n} f(x_i)\ dx \right) \quad = \quad \int_a^b f(x)\ dx.$$

We're going to use the above fact that the limit of Riemann sums is an integral in a new way. Rather than use it as an approximation to the area, we're going to use it to help us figure out which integral we should be doing.

2. The Riemann Sum as an intermediate step.

The starting point is this: we'll be given a physical problem, whose answer is given by solving an integral. The problem is that we won't know *which* integral it is. (What function do we integrate?, and between which limits?).

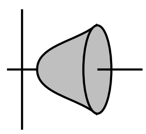
In order to figure this out, we'll write down an approximation to the answer for the physical problem. We'll do this by slicing the problem into small pieces, and approximating the answer on each slice. We then add up the estimate on the slices to get our approximation to the problem.

Our approximation will be given in the form of a sum. The approximation isn't the exact answer (of course), but by increasing the number of slices we can make the approximation better and better, so that what we really want to do is take the limit of the sums.

Staring at the sum, we'll recognize it as the Riemann sum associated to an integral. Since we want to take the limit of the sums, this is the same as computing the integral. Once we realize what integral it is, we'll simply compute that and solve our problem.

Schematically, our thought process will follow the following sequence:

Physical Problem $\longrightarrow$ Riemann Sum Approximation $\longrightarrow$ Integral



$$\longrightarrow \quad \sum_{i=1}^{n} \text{(Something)} \ \Delta x \quad \longrightarrow \quad \int_a^b \text{(Something)} \ dx$$

Our goal today is to learn how to follow this process: to go from the problem to the sum to the integral which gives us the answer. After we find the integral, we can forget about the Riemann sum. Its only purpose is to serve as a bridge to lead us from the statement of the problem to the correct integral solving it.

This handout can (soon) be found at

http://www.mast.queensu.ca/∼mikeroth/calculus/calculus.html

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