Regular Lecture given by Peter Taylor and Nathalie Sinclair ICME 2000
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Reinventing the teacher

I'm Peter Taylor, I'm a theoretical biologist deeply involved in mathematics curriculum development at the high school level in Ontario, Canada. I have found the task of preparing this talk to be quite frustrating, and it has taken me on a long journey, little of which will be apparent here. Over all the work that we do here, it seems to me that there are some truths that shine brightly. The curriculum must have life, energy, wonder, depth. If you don't have that, nothing else that you do is worth much. We all know this; we have known it for decades. What more is there to say?

I'm Nathalie Sinclair I am a Phd student and, in contrast to Peter's many years of involvement and experience in mathematics education, I have just begun. Whereas Peter has concentrated on curriculum development and textbook writing, I have been mainly involved in technology development and implementation. We have found that it is when we sit down to design a real curriculum or write a real textbook, or stand in front of a real class, that our views and our ideals are most tried. Many of the concessions and compromises we make grow out of intrinsic difficulties of mathematics education but many also are caused by cultural and political pressures. And when we try to imagine a completely different set of goals and structure for mathematics education, we are struck by our own prejudices, habits and assumptions.

Peter: Our work has focused on the design or shape of a mathematics curriculum—what should it look like and how should it be structured. And to make a long story short, the view that we have come to is that it should be based in what we call method rather than content, that the content has to flow out of the method, flow naturally—as it will if you allow it to. By method here we mean ways of thought, general strategies, ways of cutting up the mathematical universe, and putting it back together. And the way that method emerges in the classroom is through significant mathematical investigations—investigations that lead to large places but at the same time are contained—grains of sand which can in time open up to mathematical universes. How are these investigations to be chosen, how is quality to be assessed? The standard here is mathematics itself, the methods are the methods that mathematicians use, the problems or investigations must be of interest to a mathematician. The challenge of the curriculum writer, who must be both a mathematician and a teacher, is to find these and orchestrate them.

Nathalie: Aristotle tells us that all of philosophy begins in wonder. Not to be confused with "I wonder whether he will call me tonight," Aristotle's wonder was one that indicated an awareness of the contingency of events, situations, relationships. Suppose that we built a curriculum on wonder, on what students were able to experience surprise, anticipation, attraction, and awe about. It may seem fanciful. But isn't the capacity to perceive relationships and dependencies at the heart of mathematics? Isn't mathematics hard enough that in order to play and work at it, you'd better have a strong intrinsic motivation– one that might come from a sense of wonder? I have often assumed that what makes me wonder in mathematics will make my students wonder. And although we both wonder at dragon curves and the four colour theorem, they are surprised by different things than I am.

They are not surprised that the three medians of a triangle meet. It's math class, of course it works out. But they are surprised when the orthocentre of the triangle all of a sudden pops out of the triangle. That cheeky orthocentre.

Peter: We now take time to investigate the following problem.

For which *N* is there a polygon in 3-space with *N* equal sides and all angles 90°?
[Math. Magazine. 55, p. 47, 1982]
[*We distribute a large supply of cut plastic straws and pipe cleaners for the* 90° *bends.]*

Nathalie: I had played with this example before, in the days that I was devouring all the problems in Peter's textbook. And it was interesting, but no more so than the other ones. But then I saw Peter conduct this problem with a group of 60 students in a mathematics and poetry class – the most diverse group of students you might ever find gathered in the same university classroom. Of the 10 or so problems that Peter did with them, this is the one that most fired them up, that almost effortlessly had them working fiendishly. Why? Well, perhaps three dimensions and straws are something that the students have a lot of experience with. Maybe they were keen about the possibility of something being impossible – this has certainly motivated children, mathematicians and amateurs alike. Maybe it was the way it unfolded as a story with an intrigue, and an Agatha Christie-like falling into place. Maybe Peter was able to conduct the problem so that the students were there but still felt carried towards more sophistication. Maybe the mathematical tools that we use to work through this problem suddenly seemed just right and wonderful rather than technical and dreary. Maybe this very fact just surprised them—that little straws could lead to such abstract mathematics. Maybe it's because, from time to time, we would tell them that this was a beautiful problem.

By the time we had done this problem, the students had had sufficient experience in tapping into their poetic tools of thinking with Peter's problems. They had stopped saying things like "This isn't 'mathematics' but here's what I think" or "Is it okay if I draw you a picture?" They had started trusting their feelings toward a problem, their spatial and visual metaphors. But it took time and trust to overcome the bias that their everyday styles of thinking weren't welcome in the mathematics classroom.

Peter: This problem is interesting in a number of ways, but most important for us of course, it is a problem that every mathematician will reach out to with pleasure and anticipation, not only to know the answer, but to see and feel the how and the why of the answer. It has elements of wonder, of mystery, tension and resolution…. an elusiveness which is quite tantalizing. It is also of interest to us because although it connects to many ideas and techniques that we might want our students to have, it does not really fit into any of the standard curriculum packages. At the moment this would be unlikely to appear in any North American high school classroom.

Let me summarize our recent experiences in Ontario. The government decided to rewrite the high school curriculum in all subjects. As part of this we moved from a 5-year to a 4-year high school program. There was a competition in each subject for the writing of the curriculum policy document, and the winning team in mathematics, which was a wonderful collection of experienced teachers from school, college and university, had what appeared to be an open mandate. It was a time of energy and idealism. The new millennium was going to get a grassroots curriculum informed by the big ideas of mathematics.

A top priority was to jettison "the list" that enormous bag of hierarchically ordered technical skills that almost inevitably drives the curriculum into an incessant turning of the crank.

Yes we were idealistic, and in the final analysis there were many constraints we had to pay attention to, but at the end of the day we had something pretty good, new, responsible, creative, and mathematical. But it didn't stay quite that way. Because the grades 11-12 curriculum are seen as preparing students to go on to a variety of other levels and specialties and jobs, these courses then went on to a process of validation, rethinking, tinkering, transformation, meddling, if you like, and it was another two years, just last June, before the final 11-12 document emerged. It did manage to retain some of the philosophy of the original writers but it was much altered. In particular, we got the list back. A different format from the old list, but again we got mathematics cut up into polynomials and trig functions and exponential functions and in each of these we got this property and this and this.

Why did this happen? It happened first of all because mathematics is considered important—too important to leave anything to chance or to the whims of the individual teacher, so everything must be detailed. [The irony there is that it's the importance of the subject which has killed it—don't miss that one. Music, when it is taught, which is rare, can be taught with complete freedom only because it is considered unimportant.] And secondly, mathematics is a linear subject and it is well known that you can't learn one thing until you've learned what comes before, so you can't afford to leave anything out or the students will be crippled down the line. Right? And thirdly, all these things that you have to know are not framed in terms of method because method is hard to get hold of, cannot be easily described, rather they are catalogued in terms of fragments of technique. And if you put all those three things together, you get what we in Ontario finally got.

Nathalie: Now the list isn't all bad. For one thing, it's familiar and easily recognizable by us all—students, teachers, and curriculum developers. When we set out last winter to design and teach a grade 12 course, the list turned out to be useful in a variety of ways. For example it gave us some familiar signposts along the way as we moved from one investigation to another. We needed to be able to point out to questioning colleagues that we weren't doing anything *too* crazy, and we were "covering" this and that. We needed to show it to the students so that they felt they were still in math class: working up and through a set of skills and techniques.

Peter: Nathalie is right. The list in itself isn't bad, though we once thought it was. It's certainly a very useful guide. What happens is that it gets abused. In fact there seems to be an inevitable tendency to do that.

The abuse we have in mind here is lack of restraint. It turns out that skill-based curricula tend to be full, allowing no time to push outside the technical arena they define. However rich investigations tend not to fit within narrow technical boundaries. There it is; if you put both of these together, you have a vexing configuration.

Nathalie: The irony is that there's really no need to cover all those skills. The kids don't learn them all anyway. But with the onset of "outcome based" learning, standard testing and assessment, which are becoming increasingly common in North America, it seems that only the independent schools are prepared to leave stuff out, are prepared not to grind every technical detail of the curriculum into the classroom hour. Certainly the publishers can't afford to leave anything out.

Peter: Ah, the publishers, the text-book publishers. That's a significant factor in the equation. In most classrooms the text book has a greater influence on what is taught and how it is taught than any other factor. But the above constraints make it virtually impossible for a responsible main-stream publisher to produce a lively sophisticated book—in spite of the fact that most publishers really do want to serve the cause of pedagogical reform and produce lively innovative books.

Nathalie The other day we received in the mail a catalogue of mathematics books. Here are excerpts from some of the reviews. Listen and speculate on what kind of books these might be.

“A spin through the human condition nobody else can provide”
“…another delightful collection…”
“…a jewel… the most interesting book that I have read this year”
“You will find here a collection of delightful stories…witty, amusing and instructive…”
“..together they lead us on an imaginative, often astonishing tour…”
“An awesome achievement. Accurate, clearly written, and charmingly illustrated…”
“This polished and readable account of some fascinating aspects…”
“…a rich and fascinating book. It has everything and everything that it has is delightful, curious, enlightening, engrossing, interesting, informative, funny, stirring, poignant…”
“This book reads like a mystery; it was difficult to put down.”

Peter: Yes these are math books, and the quotes are taken from an MAA flyer, but needless to say none of them belong to recent high school math textbooks. On the other hand, such accolades might well apply to books that are used in the classroom of a humanities course. Indeed, in such courses, teachers actively seek out book with these qualities. This is not the case in high school math. Why do the math teachers stand for it? The tragedy is that most of them believe that mathematics text books are what they have to be because of the very nature of the subject.

Nathalie: When we think about a math text book, we must be open to all kinds of possibilities and questions. The style, what does it look like, what does it “feel” like, what is the role of narrative, of history, of mystery? who is the audience?—the students or the teacher? What are the pictures and diagrams about? Do we really need pictures of happy fulfilled students hard at work? And most important of all—what should we leave out?

Peter: : Sometimes the best examples are works of art in which just enough lines are drawn and colours used to invite the viewer to enter the picture for herself and become the artist.

Nathalie: I was reading this story about Ramanujan. He had access to one textbook while he was going through his critical period of learning mathematics. It was one where the author, instead of providing careful, detailed proofs just announced a result as a little gem: it almost dared you to jump in and prove it for yourself. Ramanujan’s biographer, Robert Kanigel, writes that it has a sense of flow, you feel as though your are going somewhere. Now, our students aren't all Ramanujans of course but perhaps they suffer from 'over-explanation' in textbooks. Everything is so clear and so detailed and so perfectly coherent and organized. There is not ambiguity, no mystery, and no chance to see any of the messiness that might lie behind. Students come to think of mathematics as a something that always works out; it's no wonder they object to having to struggle with ‘ill-structured’ problems.

Peter: With all the vast potential of the web, I am still a strong believer in the book, and the shape and feel of the book plays a huge role in telling the student about the subject, what it is, how we regard it, what it means to us. I have suggested that the publishers would really like to know more about our feelings for the subject and out experience with it, and to respond to that in the books they produce for us. But we cripple them with a tight inflexible list. And as long as we do that, we limit their capacity to respond.

You might reply, well of course art thrives under constraint— witness the structure of the sonnet, the discipline of the canvas. That's true, but there must not be too much for the allotted space and time. An artist faced with interesting general objectives and just the right boundaries has a wonderful feeling of challenge, of creativity—let’s see what I can do with this. But if the constraints are too severe she becomes a prisoner. This is a simple enough principle but it is grievously violated in the North American system at all levels.

Nathalie: Hey!! What if every other grade, you got to just play with what you had "learned" the grade before. See it in new situations, combine it with different elements, look for new problems, make them up. Isn't this what mathematicians do? They often take what they know and recombine it to create something new, some new connection. This isn't only about creating something new, it's about acknowledging that learning is almost never complete. You haven't "done" equivalent fractions. You continue to think of them and use them in different ways.

Peter: That's something like what I wanted our grade 12 to be—a fun year of anything goes explorations in which we used our skills and ingenuity of the first 3 high school years to bring a bunch of ideas together—to truly move from skills to methods. But no—the "authorities" wanted calculus instead. The formulas of calculus. So the students will move right into the rules of calculus with hardly any exploratory experience. A sad situation.

Nathalie: The problem is that there is a deeply engrained high school academic culture in North America. We can make little changes, little tweaks. We can try to make things just a little more palatable. But it seems unlikely that will take us where we want to go. You cannot move in small steps from impressionism to cubism—that represents major shifts of perception.

Peter: Of course this type of major perceptual shift is much more feasible in university. For example I'm teaching linear algebra for the first time this coming fall. My plan is to take one or two of the big ideas of the subject, the big "methods," and to use a set of models or investigations to illustrate or construct these methods. And that's it. As concerns the basic skills, that list of technical fragments we know so well, I will pay close attention to any of those that arise. And ignore the rest. I'm pretty sure that at some point we will encounter the notion of linear independence, of the inverse of a matrix, of an eigenvector. Actually, the notion of eigenvector will be one of my big ideas. Indeed, often in mathematics we are looking for "solutions" of certain sets of equations. Or more generally for configurations that satisfy certain conditions. And a fundamental method is to relax your constraints and then find some simple solutions that aren't quite what you want, because you don't have all the constraints, and that's what eigenvectors really are. Then try to get the solution you want by using these special solutions as building blocks in some sense. For example, sometimes the conditions are simple enough that "linear combinations" of simple solutions will also be solutions. Indeed this will be where I will begin my course—where most of my colleagues will end theirs.

Nathalie: More about the shift in perception. In a methods-based curriculum we no longer assume that students need to know basic skills before being able to identify, appreciate and work with methods. In fact, methods encompass the very processes that make such skills relevant and learnable. They emphasize action over objects: looking for something that works given a set of constraints instead of fussing over what exactly an eigenvector is. They welcome analogical thinking: isn't finding a configuration that works like changing your coordinate system? They include motivation: where am I trying to go with this mess? In short, they have more to do with the actions, perceptions and feelings we learn with than the formal and rigorous logic so commonly portrayed as paradigmatic of mathematics. Most would agree that this is what mathematics is about, why do we hesitate at the suggestion that this is what school mathematics can be about?

Peter: The problem we are talking about here is serious. The only way out is to let go. It seems clear to us that if anything significant is to happen in the teaching of mathematics, there must be a real letting go at all levels. Teachers, students, administrators, politicians, mathematicians, text-book publishers, must all let go of an essential measure of control, must trust the system to work without trying to impose everything that just might be necessary. We must honour our teachers and our students by giving them material of the highest quality, and then trust them to make proper use of it.

Nathalie: What we haven't talked about is the whole professional atmosphere in which school teachers operate in North America. Peter refers to the considerable autonomy and freedom he has in his university life—opportunity for creative design and experiment, time to be a scholar, to come to a deep understanding of a few ideas. Of course the school system operates under a different set of constraints, but school teachers in North America are given almost no professional autonomy. How then are they to think of themselves?

Peter. Teachers tell us they find our investigations challenging—they might work well for us, but they are perhaps too hard for most teachers to "pull off." But here again a perceptual shift is needed. They have to imagine themselves as mathematicians in front of the class struggling with a problem. Ah, but that will require more than a perceptual shift will it not?—it will mean a cultural shift as well, for they will have to *reinvent themselves* as mathematicians, they will have to find the space and time and most importantly the community in which to actually *be* mathematicians. And then the fact that the problems might be hard need not present a barrier.

Nathalie. Actually, "hard" is not quite the right word, "sophisticated" might be better. The problems we have worked with, the problems that we believe should be in the school mathematics classroom, are more sophisticated than the problems that currently appear in the text-books. In this regard, mathematics exhibits a marked difference from other disciplines, especially the arts and humanities, in which the material put in front of the students is often at a high level of sophistication, high enough to challenge the teacher as well as the student. A poetry teacher would not bring a poem to class that she herself did not find challenging—why is mathematics so different in this regard?

Peter: Time for some conclusions. We are talking about changing the nature of high school mathematics. What are the ingredients?

Nathalie. We have made comparisons between mathematics and the humanities but do not forget that knowledge is one, and these divisions are man-made. Mathematics is a human endeavour and if the curriculum is to have meaning to our students, if it is to take part in their lived experience, we must be open to essentially human modes of investigation, for example, analogical and narrative ways of thought and argument, and human responses, for example to wonder, to seek wholeness.

Peter: And if we are to have any chance of moving in this direction, we have to be prepared to let go, not only at the school but also at the university level, because the universities unwittingly shape a huge component of the school curriculum. We've got to take hold of some better more demanding more sophisticated more lively more wonder-ful stuff and put it fearlessly in front of the students and have fun with it and take it very very seriously and stop for heaven's sakes worrying about whether they will be truly prepared for university or how well they do on international tests and all that stuff. Go in with high quality material and trust it to work its magic, and trust the students to respond to it, and trust the teachers to struggle to gain some mastery of it. Quality speaks with a compelling voice.