Errata to Complete Caps in Projective Space which are Disjoint from a Subspace of Codimension Two

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The published version of *Complete Caps in Projective Space which are Disjoint from a Subspace of Codimension Two* contains an error in the proof of Theorem 4.1. This error affects Proposition 7.1 and Proposition 8.1.

The correct statement and proof of Theorem 4.1 are as follows:

Theorem 4.1 *Suppose that A and B are such that Equation 1.3 is satisfied for all $i = 1, 2 \ldots, 2^{n-r-1}$. Suppose $0 < t + u < 2^{n-r-1}$, that $\boxed{t \neq u}$, and that either $\boxed{\overline{A_f} \text{ or } \overline{B_f} \text{ is non-periodic}}$. Further suppose that $\widehat{C} \setminus C \subseteq \cup_{i=1}^{2^{n-r-1}} (A(i) + B(i))$. Then S is a complete cap in Σ.*

Proof By construction, S is a cap in Σ. Also Equation 1.3 guarantees that every point of A' and every point of B' lies on at least one secant of S through a point of C. Thus we need to show that every point of H_∞ and every point of $H_C \setminus C$ lies on a secant of S.

Without loss of generality suppose that $\boxed{\overline{A_f} \text{ is non-periodic.}}$ Consider the cap $\overline{S_1}$ in $\overline{\Sigma}$ given by $\overline{S_1} = \overline{A_f} \sqcup \overline{B_{ne}} \sqcup \{\overline{C}\}$. Applying Theorem $\boxed{3.5}$ we see that $\boxed{\text{for every point}}$ $\overline{c_k}$ of $\overline{H_C}$ different from $\overline{C}$ there exists $i \leq t$ and $j \geq t+1$ such that $\overline{a_i} + \overline{b_j} = \overline{c_k}$. Since the coset $H_A(i)$ is entirely contained in S, and $H_B(j) \cap S \neq \emptyset$, this means that every point of the coset $H_C(k) = H_A(i) + H_B(j)$ lies on a secant to S.

By assumption, every point of $\widehat{C} \setminus C$ is contained in $A + B$.

Hence it only remains to prove that every point of H_∞ lies on at least one secant to S. Note that every point of F lies in $H_A(1) \oplus H_A(1)$ (or in $H_B(1) \oplus H_B(1)$ if $t = 0$). Thus we consider a point $z \in H_\infty \setminus F$ and write $\overline{z}$ for the point of $\overline{\Sigma}$ corresponding to the coset of F generated by z.

As above, if $\overline{z} \in \overline{A_f} + \overline{A_{ne}}$ or $\overline{z} \in \overline{B_f} + \overline{B_{ne}}$ then z lies on a secant line to S. Thus we assume, by way of contradiction, that $\overline{z} + \overline{A_f} \subset \overline{A_e}$ and $\overline{z} + \overline{B_f} \subset \overline{B_e}$. The first inclusion implies that $t \leq u$ while the second implies that $u \leq t$. Thus $t = u$ contradicting our hypothesis.

□

Here are the corrected versions of Remark 4.3 and Remark 4.4.

Remark 4.3 *Note that, if t or u is odd, then $\overline{A_f}$ or $\overline{B_f}$ respectively is non-periodic. Thus for a fixed value of $t+u$ with $1 \le t+u \le 2^{n-r-1}-1$ we may take $t=1$, for example, to arrange that $\overline{A_f}$ is non-periodic.*

Furthermore, if $t+u \neq 2$ we may simultaneously arrange that $t \neq u$.

Remark 4.4 *.4 There do exist complete caps for which $t+u=0$ (and thus $\overline{A_f}=\overline{B_f}=\emptyset$). See Example 5.4.*

The condition that $t \neq u$ is also sufficient but not necessary.

The corrected version of paragraph 4 of Section 5, as well as two new paragraphs which should follow paragraph 4 are give below:

Then $|S| = (2^r-1)+2s+2^r(2^{n-r-1}-s) = 2^{n-1}+2^r-1-(2^r-2)s$. By Remark 4.2, s cannot equal 0. Furthermore by Theorem 4.1 (and Remark 4.3) there do exist complete caps of this form for all $s = 1,2,\ldots,2^{n-r-1}-3$ and $s = 2^{n-r-1}-1$. Thus we find complete caps of this form of all cardinalities: $2^{n-r}+k(2^r-2)+1$ for $k = 2,3,\ldots,2^{n-r-1}-2$ and $k = 2^{n-r-1}$ where $2 \le r \le n-2$.

Next we consider the case $s = 2^{n-r-1}-2$. Then $u+v=2$ and either $u=v$ or else both $\overline{A_f}$ and $\overline{B_f}$ are periodic. Reordering the cosets we may suppose $|H_A(i)| = |H_B(i)| = 1$ for $i = 3,4,\ldots,2^{n-r-1}$. Write $\{\alpha_i\} = H_A(i)$ and $\{\beta_i\} = H_B(i)$ for $i = 3,4,\ldots,2^{n-r-1}$. We consider first the case that $t=u=1$. Examining the proof of Theorem 4.1 we see that every point of $\mathbb{PG}(n,2)$ lies on a secant to S except possibly the 2^r points in the coset $H_A(1)+H_A(2) = H_B(1)+H_B(2) = \{z_1,z_2,\ldots,z_{2^r}\} \subset H_\infty$. These points z_k can only lie on secants of the form $\alpha_i+\alpha_j = z_k$ or $\beta_i+\beta_j = z_k$. Since $\beta_i+\beta_j = \alpha_i+\alpha_j$ we may restrict our attention to the α_i. Thus we see that S is complete if and only if for every $k = 1,2,\ldots,2^r$ there exists $3 \le i(k) < j(k) \le 2^{n-r-1}$ such that $\alpha_{i(k)}+\alpha_{j(k)} = z_k$. Since the α_i all lie in different cosets, it is clear that $\alpha_{i(1)},\alpha_{j(1)},\ldots,\alpha_{i(2^r)},\alpha_{j(2^r)}$ must be distinct. Thus we must have $2(2^r) \le 2^{n-r-1}-2$ or equivalently $2r+3 \le n$.

The case $(t,u)=(2,0)$ (or $(0,2)$) is entirely similar to the above case. The only difference being that the 2^r points which may not lie on secants to S form a coset in H_C instead of in H_∞. In summary we see that there exists a complete cap S with $|\widehat{C}\setminus C| = 1$ and with $|S| = 2^{n-r}+2^{r+1}+2^r-5$ (and $s = 2^{n-r-1}-2$) if and only if $n \ge 2r+3$.

Here is the corrected version of Proposition 7.1:

Proposition 7.1 *Let S be a complete cap in $\mathbb{PG}(n,2)$ with $n \geq 4$. Further suppose that S meets a hyperplane K_C of $\mathbb{PG}(n,2)$ in exactly 3 points. Then $|S|$ is one of the numbers: $2^{n-2}+5, 2^{n-2}+9, 2^{n-2}+11, 2^{n-2}+13, \ldots, 2^{n-1}+1$ or, if $n \geq 7$, $|S|$ may also be $2^{n-2}+3$ $\boxed{\text{or } 2^{n-2}+7.}$ Moreover complete caps of all these sizes exist for all corresponding values of n.*

Here is the corrected version of Proposition 8.1

Proposition 8.1 *Let S be a complete cap in $\mathbb{PG}(n,2)$ with $n \geq 5$. Further suppose that S meets a hyperplane K_C of $\mathbb{PG}(n,2)$ in exactly 4 points. Then $2^{n-2}+8 \leq |S| \leq 2^{n-1}-2$ or else $|S| = 2^{n-1}+1$.*
Moreover $\boxed{\text{except for } n=6 \text{ for sizes 29 and 30,}}$ complete caps of all these sizes $\boxed{\text{and this structure}}$ exist for all $n \geq 5$.

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I also thank Alexander Davydov for suggesting I clarify the statement of Proposition 8.1 (by adding $\boxed{\text{and this structure}}$) and for pointing out an error on page 355 line -11 (=last sentence on page 7 of the corrected version). The valid values for k are $k = \boxed{2,4,5,\ldots,2^{n-r-1}}$ where $2 \leq r \leq n-2$.

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