INVARIANTS OF THE DIAGONAL C_p-ACTION ON V_3 — ADDITIONAL DETAILS

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Abstract. In this document we provide extra details of the proofs and computations given in [0] where a finite SAGBI basis for $\mathbf{F}[V_3 \oplus V_3]^{C_p}$ is given.

The purpose of this document is to provide additional details to the interested reader of [0]. These details are omitted from [0] for reasons of clarity and brevity. This document is not intended to stand on its own and we assume the reader has already read [0]. All the notation and definitions used here are given in [0].

1. Lead Monomials of Transfers

Here we prove Lemma 5.1 which is needed to identify the lead monomials of some of the transfers lying in the set B. This lemma is a simple extension of the results found in [14, § 3].

It is easily seen that for $j = 1, 2$ and $0 \le c \le p-1$

$$\begin{aligned} \sigma^c(z_j) &= z_j + cy_j + \binom{c}{2} x_j \\ \sigma^c(y_j) &= y_j + cx_j \ . \end{aligned}$$

Furthermore it is well known (for a proof see for example [5, 9.4]) that

$$\sum_{c=0}^{p-1} c^i = \begin{cases} 0 \text{ in } \mathbf{F} & \text{if } i < p-1 \\ -1 \text{ in } \mathbf{F} & \text{if } i = p-1 \end{cases}$$

In particular, if $f(c)$ is any polynomial of degree less than $p-1$ then $\sum_{c=0}^{p-1} f(c) = 0$ in $\mathbf{F}$.

Lemma 5.1. *Suppose* $0 \le t, s \le p-1$ *and* $2s + t \ge p-1$. *Then*

$$\mathrm{LM}(\mathrm{Tr}(z_1^t z_2^s)) = \begin{cases} z_1^t x_2^{p-1-s} y_2^{2s-(p-1)} & \text{if } s \ge (p-1)/2 \\ y_1^{p-1-2s} z_1^{t+2s-(p-1)} x_2^s & \text{if } s \le (p-1)/2 \end{cases}$$

Proof. If $s \ge (p-1)/2$ then this result is a minor modification of [14, Theorem 3.2]. Therefore we suppose that $s \le (p-1)/2$. We have $\sigma^c(z_1^t z_2^s) = (z_1 + cy_1 + \binom{c}{2} x_1)^t (z_2 + cy_2 + \binom{c}{2} x_2)^s$. Expanding this we

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obtain an expression $\mathrm{Tr}(z_1^t z_2^s) = \sum_{c=0}^{p-1} \sum_w f_w(c)\mathbf{x}^w$ where $f_w \in \mathbf{F}[c]$ for each w. The largest monomial w here (with respect to the monomial ordering) for which the polynomial f_w has degree at least $p-1$ is is the monomial $\mathbf{x}^w = y_1^{p-1-2s} z_1^{t+2s-(p-1)} x_2^s$. By the above lemma, all larger monomials have coefficient 0 in $\mathrm{Tr}(z_1^t z_2^s)$. For $w = y_1^{p-1-2s} z_1^{t+2s-(p-1)} x_2^s$ we get $f_w(c) = \binom{t}{p-1-2s} c^{p-1-2s} \binom{c}{2}^{2s} = 2^{-2s} c^{p-1} \left(\binom{t}{p-1-2s} - \binom{t}{p-1-2s} c^{-s} \right)$ and thus $\sum_{c=0}^{p-1} f_w(c) = -\binom{t}{p-1-2s} 2^{-2s} \neq 0$ since $0 \leq p-1-2s \leq t \leq p-1$. Therefore $\mathrm{LT}(\mathrm{Tr}(z_1^t z_2^s)) = -2^{-2s} \binom{t}{p-1-2s} y_1^{p-1-2s} z_1^{t+2s-(p-1)} x_2^s$. □

Applying Lemma 5.1 together with simple generalizations of [14, Theorem 3.3] and [14, Theorem 3.6] we find that M is generated over A by the following monomials.

(0) 1
(1) $z_1^s x_2^s$ for $1 \leq s \leq (p-3)/2$
(2) $y_1 z_1^s x_2^{s+1}$ for $0 \leq s \leq (p-1)/2$
(3a) $y_1^{p-2s} z_1^{2s} x_2^s$ for $0 \leq s \leq (p-1)/2$
(3b) $y_1 z_1^{p-1} x_2^{p-1-s} y_2^{2s-(p-1)}$ for $(p+1)/2 \leq s \leq p-1$
(4) $z_1^s y_2^p$ for $0 \leq s \leq p-1$
(5a) $y_1^{p-1-2s} z_1^{t+2s-(p-1)} x_2^s$ for $1 \leq s \leq (p-1)/2, (p+1)/2 \leq t \leq p-1$ and $p \leq t+s$
(5b) $z_1^t x_2^{p-1-s} y_2^{2s-(p-1)}$ for $(p+1)/2 \leq s \leq p-1, 1 \leq t \leq p-1$ and $p \leq t+s$
(6a) $y_1^{p-2s} z_1^{t+2s-(p-1)} x_2^s$ for $1 \leq s \leq (p-1)/2, (p-1)/2 \leq t \leq p-2$ and $p-1 \leq t+s$
(6b) $y_1 z_1^t x_2^{p-1-s} y_2^{2s-(p-1)}$ for $(p+1)/2 \leq s \leq p-1, 0 \leq t \leq p-2$ and $p-1 \leq t+s$
(7) $y_1 z_1^s y_2^p$ for $0 \leq s \leq p-1$

The numbering of these families of lead monomials corresponds to the numbering of the families of invariants given in Theorem 4.1.

2. Computation of $\mathcal{H}(M, \lambda)$

In this section we give addition details of the computation of the Hilbert series of M.

Decompose M by multi-degree (with respect to the H-grading) as follows:

$$M = \bigoplus_{\omega \in C_2 \times C_2 \times C_p} M_\omega = \bigoplus_{i=0}^{1} \bigoplus_{j=0}^{1} \bigoplus_{k=0}^{p-1} M_{(i,j,k)} .$$

By this direct sum decomposition,

$$\mathcal{H}(M, \lambda) = \sum_{i=0}^{1} \sum_{j=0}^{1} \sum_{k=0}^{p-1} \mathcal{H}(M_{(i,j,k)}, \lambda)$$

and we compute $\mathcal{H}(M,\lambda)$ by computing each of the individual $\mathcal{H}(M_{(i,j,k)},\lambda)$.

To do this we begin by sorting the monomials in C according to their H-degree.

For $i=0$ and all $0 \le j \le 1$ and $0 \le k \le p-1$, if $k+j \le (p-1)/2$ we have the following generators of $M_{(0,j,k)}$.

(1) $y_1^{p-1+j-2t} z_1^k x_2^t$ for $\lceil k/2 \rceil \le t \le k+j-1$.
(2) $y_1^j z_1^k x_2^{k+j}$
(3) $y_1^j z_1^k x_2^{p-1-t} y_2^{2t-(p-1)}$ for $p-j-k \le t \le p-1$

For $i=0$ and all $0 \le j \le 1$ and $0 \le k \le p-1$ if $k+j > (p-1)/2$ we have the following generators of $M_{(0,j,k)}$.

(1) $y_1^{p-1+j-2t} z_1^k x_2^t$ for $\lceil k/2 \rceil \le t \le (p-1)/2$.
(2) $y_1^j z_1^k x_2^{p-1-t} y_2^{2t-(p-1)}$ for $(p+1)/2 \le t \le p-1$

For $i=1$, and all $0 \le j \le 1$ and $0 \le k \le p-1$ we have the following generator of $M_{(1,j,k)}$.

(1) $y_1^j z_1^k y_2^p$

Since each $M_{(1,j,k)}$ is generated over A by a single element, it is a free rank one A-module. Therefore

$$\mathcal{H}(M_{(1,j,k)},\lambda) = \mathcal{H}(A,\lambda) \cdot \lambda^{p+k+j} \tag{2.0.1}$$

for all $0 \le j \le 1$ and $0 \le k \le p-1$.

In Equations (5.1.3) and (5.1.4) we give expressions for $\mathcal{H}(M_{(0,j,k)},\lambda)$. Here is an expanded description of the derivation of these expressions.

Examining the points in $V_{(0,j,k)}$ we find the following values for the $\mathrm{LCM}(\Delta_1)$ and the $\mathrm{LCM}(\Delta_2)$.

For $i=0$ and $0 \le j \le 1$ and $0 \le k \le p-1$, if $j+k \le (p-1)/2$ then we have the following points in $\{\mathrm{LCM}(\Delta_1) \mid \Delta_1 \in E_{(0,j,k)}\}$.

(1) $(0,j,k,p-t,2t-(p-1),0)$ for $p-j-k \le t \le p-1$
(2) $(0,p+j-1-2t,k,t+1,0,0)$ for $\lceil k/2 \rceil \le t \le k+j-1$
(3) $(0,p+j-1-2t,k,t,p-1-2t,0)$ for $\lceil k/2 \rceil \le t \le k+j-1$

For $i=0$ and $0 \le j \le 1$ and $0 \le k \le p-1$, if $j+k \ge (p+1)/2$ then we have the following points in $\{\mathrm{LCM}(\Delta_1) \mid \Delta_1 \in E_{(0,j,k)}\}$.

(1) $(0,j,k,p-t,2t-(p-1),0)$ for $(p+1)/2 \le t \le p-1$
(2) $(0,p+j-1-2t,k,t+1,0,0)$ for $\lceil k/2 \rceil \le t \le (p-3)/2$
(3) $(0,p+j-1-2t,k,t,p-1-2t,0)$ for $\lceil k/2 \rceil \le t \le (p-3)/2$

For $i=0$ and $0 \le j \le 1$ and $0 \le k \le p-1$, if $j+k \le (p-1)/2$ then we have the following points in $\{\mathrm{LCM}(\Delta_2) \mid \Delta_2 \in F_{(0,j,k)}\}$.

(1) $(0,p+j-1-2t,k,t+1,p-1-2t,0)$ for $\lceil k/2 \rceil \le t \le k+j-1$

For $i=0$ and $0 \le j \le 1$ and $0 \le k \le p-1$, if $j+k \ge (p+1)/2$ then we have the following points in $\{\mathrm{LCM}(\Delta_2) \mid \Delta_2 \in F_{(0,j,k)}\}$.

(1) $(0,p+j-1-2t,k,t+1,p-1-2t,0)$ for $\lceil k/2 \rceil \le t \le (p-3)/2\}$

Therefore if $j+k \leq (p-1)/2$ then

$$\begin{aligned}
\mathcal{H}(K_0,\lambda) &= \big(\lambda^{2k+2j} + \sum_{t=\lceil k/2\rceil}^{k+j-1} \lambda^{p-1+j+k-t} + \sum_{t=p-j-k}^{p-1} \lambda^{j+k+t}\big)\mathcal{H}(A,\lambda)\\
\mathcal{H}(K_1,\lambda) &= \big(\sum_{t=p-j-k}^{p-1} \lambda^{j+k+t+1} + \sum_{t=\lceil k/2\rceil}^{k+j-1} (\lambda^{p+j+k-t} + \lambda^{2p+j+k-3t-2})\big)\mathcal{H}(A,\lambda)\\
\mathcal{H}(K_2,\lambda) &= \big(\sum_{t=\lceil k/2\rceil}^{k+j-1} \lambda^{2p+j+k-3t-1}\big)\mathcal{H}(A,\lambda)
\end{aligned}$$

Conversely if $k+j \geq (p+1)/2$ then

$$\begin{aligned}
\mathcal{H}(K_0,\lambda) &= \big(\sum_{t=\lceil k/2\rceil}^{(p-1)/2} \lambda^{p-1+j+k-t} + \sum_{t=(p+1)/2}^{p-1} \lambda^{j+k+t}\big)\mathcal{H}(A,\lambda)\\
\mathcal{H}(K_1,\lambda) &= \big(\sum_{t=(p+1)/2}^{p-1} \lambda^{j+k+t+1} + \sum_{t=\lceil k/2\rceil}^{(p-3)/2} (\lambda^{p+j+k-t} + \lambda^{2p+j+k-3t-2})\big)\mathcal{H}(A,\lambda)\\
\mathcal{H}(K_2,\lambda) &= \big(\sum_{t=\lceil k/2\rceil}^{(p-3)/2} \lambda^{2p+j+k-3t-1}\big)\mathcal{H}(A,\lambda)
\end{aligned}$$

From the above, using (5.1.2) and observing that the resulting sums telescope, we obtain the expressions given in (5.1.3) and (5.1.4).

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