SOME PROBLEMS IN INVARIANT THEORY

DAVID L. WEHLAU

Abstract. We present summaries of a number of research problems in invariant theory. These problems were posed during the organized problem sessions held during the April 2002, Kingston Ontario, Invariant Theory Workshop and Conference.

0. Introduction

During the Invariant Theory Workshop held in Kingston, Ontario in April 2002 there were four hour long organized problem sessions. During these sessions a number of the workshop participants posed problems for future research. V. Popov and Z. Reichstein have written articles which appear elsewhere in this proceedings where they present detailed descriptions of the problems they posed during the workshop. Here we briefly summarize the other problems that were posed during the workshop. Two of these problems have been solved since they were posed. Notably, Question 1.1 was solved during the workshop. The other problem which has been settled since the workshop is Conjecture 12.2. At the time of this writing all the other problems remain open.

A few of the participants wrote the sections bearing their name which describe the problem(s) they posed. For the other sections, I wrote the description based upon my notes from the problem session. Many of these descriptions have been benefited from the comments of the original problem posers. I, however, bear the responsibility for any errors that may have have crept through. I thank everyone who participated in this project, and particularly acknowledge the anonymous referee for a great many useful comments and corrections.

Before proceeding to the summaries of the problems, we provide a few definitions and some notation which will prove useful in a number of the sections.

Let $\mathbb{F}$ be a field and consider a finitely generated Noetherian graded $\mathbb{F}$-algebra $A = \oplus_{i=0}^{\infty} A_i$. We denote by $\mathcal{H}(A,t)$ the Hilbert series of A, $\mathcal{H}(A,t) := \sum_{i=0}^{\infty}(\dim A_i)t^i$. We define the *Noether Number* of A, denoted $\beta(A)$, to be the maximum degree of an element of a homogeneous minimal generating set for A. It is not hard to show that $\beta(A)$ is independent of the choice of homogeneous minimal generating set. Indeed, if we consider the artinian algebra $B := A/(A_+)^2$ where $A_+ := \oplus_{i=1}^{\infty} A_i$ is the *augmentation ideal* of A, then $\beta(A)$ is the largest degree in which the graded algebra B is non-zero.

For an ideal I of A we let $\beta(I)$ denote the largest degree of an element in a homogeneous minimal set of ideal generators for I. Again considering I/I^2 we may show that $\beta(I)$ is well-defined.

We call a representation V of G a *permutation representation* if there exists a basis $B = \{v_1, \ldots, v_n\}$ of V which is stabilized by every element g of G. Thus V is a permutation

September 19, 2003.

representation if and only if G is (conjugate to) a subgroup of the symmetric group $S_n \subset$ GL(V). If V is a permutation representation then G also permutes $B^* = \{x_1, \ldots, x_n\}$, the basis of V^* which is dual to B. For $f \in \mathbb{F}[V]$ we define the orbit of f, $G \cdot f := \{g \cdot f \mid g \in G\}$. Further for $f \in \mathbb{F}[V]$ we define the *orbit sum* of f as $\sum_{h \in G \cdot f} h$. It is well known that for a permutation representation the set of orbit sums of the monomials (with respect to $\{x_1, x_2 \ldots, x_n\}$) in $\mathbb{F}[V]$ forms a vector space basis for $\mathbb{F}[V]^G$.

Let V be a G-module. The *Hilbert ideal*, J, is the ideal of $\mathbb{F}[V]$ generated by the augmentation ideal, $(\mathbb{F}[V]^G)_+$, of the algebra of G-invariants.

1. Karin Baur

Let $V = \mathbb{C}^{n+1}$ be the standard representation of $\mathrm{SL}_{n+1}(\mathbb{C})$. Denote by $\epsilon_1, \epsilon_2, \ldots, \epsilon_{n+1}$ the weights of $\mathrm{SL}_{n+1}(\mathbb{C})$ acting on V with corresponding weight vectors $e_1, e_2, \ldots, e_{n+1}$. Write $\omega_i := \epsilon_1 + \epsilon_2 + \ldots + \epsilon_i$ and let $v_i = v_{\omega_i} := e_1 \wedge e_2 \wedge \ldots \wedge e_i$ denote a highest weight vector in $\wedge^i V$. Let $i < j$ and consider $\wedge^i V \otimes \wedge^j V$. This decomposes as a direct sum of irreducible $\mathrm{SL}_{n+1}(\mathbb{C})$ representations:

$$\wedge^i V \otimes \wedge^j V = V_{\omega_i + \omega_j} \oplus V_\lambda \oplus \cdots \oplus V_\mu.$$

We are interested in knowing whether every decomposable tensor in the Cartan component, $V_{\omega_i+\omega_j}$, is contained in the orbit closure $\overline{\mathrm{SL}_{n+1}(\mathbb{C}) \cdot (v_i \otimes v_j)}$. The answer to this question is "yes" if $(i, j) = (1, n), (1, 2)$ or $(n - 1, n)$. The answer is "no" if $j - i \geq 2$ and $(i, j) \neq (1, n)$

Question 1.1. Does every decomposable tensor in $V_{\omega_i+\omega_{i+1}}$ belong to $\overline{\mathrm{SL}_{n+1}(\mathbb{C}) \cdot (v_i \otimes v_{i+1})}$ when $i \notin \{1, n - 1\}$?

During the Kingston workshop Christian Ohn found the solution (for SL_n) to Question 1.1 which had been posed by Karin Baur. Later, Baur was able to generalize Ohn's result and classify all tensor products $V_\lambda \otimes V_\mu$ of irreducible representations of a reductive group G where the set of decomposable tensors in the Cartan component $V_{\lambda+\mu}$ is equal to the orbit closure $\overline{Gv_\lambda \otimes v_\mu}$ (orbit of the tensor of two highest weight vectors).

The general result ([B]) is the following:

Theorem 1.2. *Let G be a connected reductive complex algebraic group, let λ and μ be dominant integral weights (with respect to a fixed Borel subgroup).*

The set of decomposable tensors in the Cartan component $V_{\lambda+\mu}$ of $V_\lambda \otimes V_\mu$ is equal to the orbit closure $\overline{Gv_\lambda \otimes v_\mu}$ if and only if the following two conditions hold:

(i) $\overline{L_\lambda v_\mu} = \langle L_\lambda v_\mu \rangle$

(ii) $\overline{L_\mu v_\lambda} = \langle L_\mu v_\lambda \rangle$

(where $L_\mu \subset G$ is the Levi subgroup generated by the maximal torus and the root subgroups of the roots perpendicular to μ, and $\langle Gv \rangle$ is the linear span of the orbit Gv, i.e., the representation it generates).

References

[B] K. Baur, *Cartan components and decomposable tensors*, preprint.

2. Harm Derksen

Let V be an $\mathbb{F}$-vector space and let G be a linearly reductive subgroup of $\mathrm{GL}(V)$ such that G is a Zariski closed subset of $\mathrm{End}(V)$. Let W be a G-module.

Definition 2.1. The *complexity* of W is denoted $c(W)$ and defined by

$$c(W) := \min\{d \mid \text{every irreducible subrepresentation of } W \text{ appears in } \oplus_{i=0}^{d}\left(\otimes_{j=1}^{i} V\right)\}.$$

Example 2.2. If the characteristic of $\mathbb{F}$ is zero then $G = \mathrm{SL}(n, \mathbb{F})$ is linearly reductive. Let $\lambda = (\lambda_1, \ldots, \lambda_{n-1})$ be a highest weight and let $V(\lambda)$ be the corresponding irreducible $\mathrm{SL}(n, \mathbb{F})$-module. Then we have $c(V(\lambda)) = |\lambda| = \sum_{i=1}^{n-1} i\lambda_i$.

Definition 2.3. We also define $\mathcal{C}_d(W) :=$ smallest set of covariants, C, such that
(1) If Z is an irreducible G-module with $c(Z) \leq d$ and $\phi : W \to Z$ is a linear covariant, then $\phi \in C$;
(2) If $\phi_1, \phi_2 : W \to Z$ with $\phi_1, \phi_2 \in C$ then $\lambda_1\phi_1 + \lambda_2\phi_2 \in C$ for all scalars λ_1, λ_2; and
(3) If $\phi_1 : W \to Z_1$ and $\phi_2 : W \to Z_2$ are two covariants with $\phi_1, \phi_2 \in C$ and $\psi : Z_1 \otimes Z_2 \to Z_3$ with $\mathcal{C}(Z_3) \leq d$ then $\psi \circ (\phi_1 \otimes \phi_2) : W \to Z_3$ lies in C.

For a covariant $\phi : W \to Z$ we define the complexity $c(\phi)$ of ϕ as the smallest d such that $\phi \in \mathcal{C}_d(W)$. The constant $\mu(\mathbb{F}[W]^G) = \min\{d \mid c(f) \leq d \; \forall f \in \mathbb{F}[W]^G\} < \infty$ gives a measure for the complexity of the invariant ring $\mathbb{F}[W]^G$.

Problem 2.4. *Give bounds for* $\mu(\mathbb{F}[W]^G)$.

Example 2.5. View the algebraic group $\mathrm{GL}(V)$ as a Zariski closed subset of $\mathrm{End}(V \oplus V^*)$. Let $W = \mathrm{End}V \oplus \cdots \oplus \mathrm{End}V$. The algebra $\mathbb{F}[W]^{\mathrm{GL}(V)}$ is generated by traces of products, i.e., by elements of the form $\mathrm{Tr}(A_{i_1} \cdot \ldots \cdot A_{i_r})$. First observe that $c(\mathrm{End}(V)) = 2$ because $\mathrm{End}(V) \subset (V \oplus V^*) \otimes (V \oplus V^*)$. We can view A_{i_j} as a covariant $W \to \mathrm{End}(V)$ of complexity 2. It is now easy to see using the composition $\mathrm{End}(V) \times \mathrm{End}(V) \to \mathrm{End}(V)$ (which is also a covariant) that every product $A_{i_1} \cdots A_{i_r}$ defines a covariant of complexity 2. Finally the invariant $\mathrm{Tr}(A_{i_1} \cdots A_{i_r})$ is an invariant of complexity at most 2. This shows that $\mu(\mathbb{F}[W]^{\mathrm{GL}(V)}) \leq 2$.

A positive answer to the following question would imply that the isomorphism of two graphs can be tested in polynomial time (which is a famous still open problem).

Question 2.6. The symmetric group S_n can be viewed as the set of permutation matrices in $\mathrm{End}(\mathbb{F}^n)$. Does there exist a constant m such that for all n, $\mu(\mathbb{F}[\mathrm{S}^2(\mathbb{F}^n)]^{S_n}) \leq m$?

Question 2.7. (Popov) With G, V as before, find all G-modules W such that $\mu(\mathbb{F}[W]^G) = 1$.

3. Gene Freudenberg

Let $G_a = (\mathbb{C}, +)$. There exist non-linear actions of G_a on $\mathbb{C}^n$ for $n \geq 5$ with non-finitely generated rings of invariants. For linear actions of G_a the ring of invariants is always finitely generated by Weitzenböck's theorem.

Question 3.1. What happens in positive characteristic?

There exists a linear G_a^3 action on $\mathbb{C}^{18}$ with a non-finitely generated ring of invariants (Mukai).

Question 3.2. What about linear actions of G_a^2? Do these always have a finitely generated ring of invariants?

Question 3.3. (G. Kemper) For which groups G, is $\mathbb{F}[V]^G$ finitely generated for all G-modules V? Perhaps this class of groups consists only of those groups whose unipotent radical is trivial together with $(\mathbb{F}, +)$?

Problem 3.4. *(Nagata's Second Problem) Let K be a field with $\mathbb{C} \subset K \subset \mathbb{C}(x_1, x_2, \ldots, x_n)$ such that $K \cap \mathbb{C}[x_1, x_2, \ldots, x_n]$ is not finitely generated. Can K have transcendence degree 3?*

References

[DF] D. Daigle, G. Freudenburg, *A counterexample to Hilbert's Fourteenth Problem in dimension five*, J. Algebra **221** (1999) 353-371.

[N] M. Nagata, *On the 14-th problem of Hilbert*, Amer. J. Math. **81** (1959) 766-772.

4. Julia Hartmann

Let G be a finite subgroup of $GL(V)$ and let J denote the Hilbert ideal. The ring of coinvariants is $\mathbb{F}[V]_G := \mathbb{F}[V]/J$. Assume that $\mathbb{F}[V]^G$ is a polynomial ring. If V is a non-modular G-representation then the action of G on $\mathbb{F}[V]_G$ is equivalent to the regular representation. Conversely if the representation is modular, then $\mathbb{F}[V]_G$ is never the regular representation. What representation is it?

Example 4.1. (Jim Shank) Let V_2 be the 2 dimensional non-trivial representation of the cyclic group, C_p, of order p over the field $\mathbb{F}_p$ of order p. Then $\mathbb{F}_p[V_2]^{C_p} = \mathbb{F}_p[x, Ny]$ is a polynomial ring with generators x of degree 1 and Ny of degree p. Also $\mathbb{F}_p[V_2]_{C_p}$ is the trivial representation of dimension p.

5. Loek Helminck

Let G be a connected reductive group over $\mathbb{C}$ and let $\theta, \sigma : G \to G$ be two commuting involutions of G. Denote the fixed point groups of σ and θ by $H := G^\sigma$ and $K := G^\sigma$. Consider the double coset space $H\backslash G/K$. Let $\beta\colon G \to G$, $g \mapsto g\theta(g)^{-1}$ and let Q denote Image(β). Then β induces an isomorphism $G/K \xrightarrow{\sim} Q$. If $h \in H$ and $x \in G$ we let $h * x$ denote $hx\theta(h)^{-1}$. Then, via β, the natural left action of H on G/K becomes its $*$-action on Q. We want to compute the invariant theoretic quotient $Q/\!\!/H$ of Q by the $*$-action of H. In [HS] we established a version of the Chevalley-Luna-Richardson theorem [LR] showing that $Q/\!\!/H \simeq A/W_H^*(A)$ where A denotes a maximal θ and σ split torus in Q and $W_H^*(A) = N_H^*(A)/Z_H^*(A)$ is the natural finite group acting on A, which comes from the twisted action $*$ on A. Here $N_H^*(A) = \{h \in H \mid h * A = A\}$ and $Z_H^*(A) = \{h \in H \mid h * a = a \text{ for every } a \in A\}$. We also established the existence of slices, showed that the orbit type stratifications of $A/W_H^*(A)$ and $Q/\!\!/H$ agree on connected components and gave a characterization of the group $W_H^*(A)$, which is essentially a combination of a reflection group together with a set of translations.

Assume now that G, H and K are defined over $\mathbb{R}$ and denote their sets of $\mathbb{R}$-rational points by $G_{\mathbb{R}}$, $H_{\mathbb{R}}$ and $K_{\mathbb{R}}$. Can we find similar results in the real case?

Question 5.1. Does there exist a CHEVALLEY-LUNA-RICHARDSON type characterization for $H_{\mathbb{R}}\backslash G_{\mathbb{R}}/K_{\mathbb{R}}$, similar to the one above for all triples $(G_{\mathbb{R}}, H_{\mathbb{R}}, K_{\mathbb{R}})$? If $H_{\mathbb{R}}\backslash G_{\mathbb{R}}/K_{\mathbb{R}} \simeq \cup_{i\in I} E(A^i)_{\mathbb{R}}/W^*(A^i)$ with the $\{A^i \mid i \in I\}$ a set of maximal σ and θ-split $\mathbb{R}$-tori and $E(A^i) \subset A^i$, then can we find a characterization of $\{A^i \mid i \in I\}$ and of the finite groups $W^*(A^i)$? Do there exist slices and do we have an orbit type stratification? The answer to these questions is yes in the two extreme cases: If $G_{\mathbb{R}}$ is compact, then $H_{\mathbb{R}}\backslash G_{\mathbb{R}}/K_{\mathbb{R}} \simeq A_{\mathbb{R}}/W^*_{H_{\mathbb{R}}}(A)$ and if θ is a Cartan involution of $G_{\mathbb{R}}$, then $H_{\mathbb{R}}\backslash G_{\mathbb{R}}/K_{\mathbb{R}} \simeq A^2_{\mathbb{R}}/W_{(H\cap K)_{\mathbb{R}}}(A)$. Here $A^2_{\mathbb{R}} := \beta(A_{\mathbb{R}}) = \{a^2 \mid a \in A_{\mathbb{R}}\} \simeq \mathbb{R}^+)^l$ for some l.

References

[HS] A. G. Helminck and G. W. Schwarz, *Orbits and invariants associated with a pair of commuting involutions*, Duke Math. J. **106** (2001), no. 2, 237–279.

[LR] D. Luna and R. W. Richardson, *A generalization of the Chevalley restriction theorem*, Duke Math. J. **46** (1979), 487–496.

6. Wilberd van der Kallen

We would like to call this problem the Cohomological Mumford Problem.

Let G be a reductive linear algebraic group defined over a field $\mathbb{F}$ of finite characteristic. Let A be a commutative $\mathbb{F}$-algebra on which G acts rationally by $\mathbb{F}$-algebra automorphisms. Let I be a G invariant ideal of A. We ask if $H^{\text{even}}(G, A/I)$ is integral over the image of $H^{\text{even}}(G, A)$. Here we restrict to the even part of the rational cohomology [J, I Ch 4] for reasons of terminology: We do not want to explain the notion of integral extensions for noncommutative rings.

The case where G equals SL_2 follows, after reduction to the case where A is finitely generated, from [K, Thm 4.12] applied to the symmetric algebra $S^*_A(A/I)$.

Recall that the Mumford Conjecture for G is equivalent to the statement that $H^0(G, A/I)$ is integral over the image of $H^0(G, A)$ for all such A and I. In fact the Mumford Conjecture, proved by Haboush [J, II, Prop 10.7], concerns the following special case. Let V be a finite dimensional G-module with a nonzero invariant vector v. Let L be the line spanned by v. Take for A the ring $\mathbb{F}[V]$ of polynomial functions on V and take for I the ideal of the functions that vanish on L. Note that $\mathbb{F}[L]$ is just a polynomial ring in one variable. Therefore, to say that $\mathbb{F}[L] = H^0(G, A/I)$ is integral over the image of $H^0(G, A)$ is equivalent to saying that the image in $\mathbb{F}[L]$ is more than $\mathbb{F}$. Now Nagata has shown [N, Lemma 5.1B] that the case of general A, I follows from the special case.

References

[J] J.-C. Jantzen, Representations of Algebraic Groups, Pure and Applied Mathematics v. 131, Boston: Academic Press 1987.

[N] M. Nagata, Invariants of a group in an affine ring, J. Math. Kyoto Univ. 3 (1963/1964), 369–377.

[K] W. van der Kallen, Cohomology with Grosshans graded coefficients, these proceedings.

7. Dikran Karaguezian

The G-module structure of $S^*(V)$ should yield significant information about the invariant ring $S^*(V)^G$. Thus, one might hope to obtain results in invariant theory by proving results for the module structure of $S^*(V)$.

This general program has so far worked in practice in the following way. Given a G-module V of some special type, one proves a module-structure theorem for $S^*(V)$ with specific bounds on the degrees of a homogeneous system of parameters and bounds on the degrees in which various modules appear. Then consequences can be deduced for $S^*(V)^G$.

Examples of results whose proof follows this program can be found in [1], Theorem 16.4, and in [2], Section 8. These results suggest two questions. First, can module-structure theory be used to prove further results in invariant theory? Second, are there general theorems describing the relationship between the module-structure of $S^*(V)$ and the properties of the invariant ring $S^*(V)^G$?

References

[1] Karagueuzian, D. B., and Symonds, P. *The module structure of a group action on a polynomial ring: A finiteness theorem* preprint 2002.

[2] Karagueuzian, D. B., and Symonds, P. *The module structure of a group action on a polynomial ring: examples, generalizations, and applications*, these proceedings.

8. Gregor Kemper

Recall that Knop [K] showed that in characteristic zero, $\deg \mathcal{H}(\mathbb{F}[V]^G, t) \leq -\dim \mathbb{F}[V]^G$. Is this true in characteristic $p > 0$? Is it true for G a finite group in characteristic $p > 0$? It is shown in [HK] that this bound is valid if p^2 does not divide $|G|$.

Let G be a finite group whose order is divisible by the characteristic of $\mathbb{F}$. Let R denote the ring of invariants $R := \mathbb{F}[V]^G$. Take $f_1, f_2, \ldots, f_n$ a homogeneous system of parameters for R and put $A = \mathbb{F}[f_1, f_2, \ldots, f_n]$.. Then, R is a finitely generated A-module. Let $g_1, g_2, \ldots, g_m$ be homogeneous invariants which generate R as an A-module. Such a set of invariants is called a set of homogeneous *secondary invariants*. We order these secondary invariants such that $\deg g_1 \leq \deg g_2 \leq \cdots \leq \deg g_m$ and we assume that m is the minimum possible, i.e., $g_1, g_2, \ldots, g_m$ minimally generate R as an A-module.

Question 8.1. Is it always true that g_m does not occur in any of the generating A-linear relations among the g_i?

Note that this question is motivated by observation of a number of computations. Also, note that an affirmative answer implies that $A \cdot g_m$ is a free summand of R. Furthermore, an affirmative answer would also imply that $\deg g_m \leq (\sum_{i=1}^n \deg f_i) - n$ by an argument of Bram Broer.

References

[HK] Ian Hughes and Gregor Kemper, *Symmetric Powers of Modular Representations for Groups with a Sylow Subgroup of Prime Order*, J. of Algebra, **241**, (2001) 759–788.

[K] Friedrich Knop, *Der kanonische Modul eines Invariantenringes*, J. Algebra, **127**, (1989), 40–54.

9. Hanspeter Kraft

The group GL_n (or SL_n) acts on $M_{n\times n}$ the set of $n \times n$ matrices by conjugation. The invariants here are generated by the n functions $A \mapsto \mathrm{Tr}(A^k)$ for $k = 1, 2, \ldots, n$ where $\mathrm{Tr}(X)$ denotes the trace of the matrix X. The null cone for this representation is $\mathcal{N}(M_{n\times n})$ = the set of nilpotent matrices.

Next consider $2M_{n\times n} := M_{n\times n} \oplus M_{n\times n}$. The invariants here are generated by the functions $(A, B) \mapsto \mathrm{Tr}(A^{i_1}B^{j_1} \cdots A^{i_k}B^{j_k})$. Also, the null cone $\mathcal{N}(2M_{n\times n}) = \{(A, B) \mid \exists\, g \in \mathrm{GL}_n$ such that gAg^{-1} and gBg^{-1} are both strictly upper triangular matrices$\}$. Consider $\mathcal{M}_n := \{(A, B) \in 2M_{n\times n} \mid \lambda A + \mu B$ is nilpotent for all scalars $\lambda, \mu\}$.

For $n = 2$, $\mathcal{M}_2 = \mathcal{N}(2M_{2\times 2})$.

For $n = 3$, $\mathcal{M}_3 \supsetneq \mathcal{N}(2M_{3\times 3})$. To see this consider $A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$ and $B = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & -1 & 0 \end{pmatrix}$. Here $\lambda A + \mu B = \begin{pmatrix} 0 & \lambda & 0 \\ \mu & 0 & \lambda \\ 0 & -\mu & 0 \end{pmatrix}$ is nilpotent for all λ and μ. But $(A, B) \notin \mathcal{N}(2M_{3\times 3})$ since if $(X, Y) \in \mathcal{N}(2M_{3\times 3})$ then XY is nilpotent. But $AB = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$.

Definition 9.1. If we have a function $p(v)$ and we write $p(\lambda v_1 + \mu v_2) = \sum_i \lambda^i \mu^{d-i} p_i(v_1, v_2)$ then the functions p_i are *polarizations* of p.

Note that $\mathcal{M}_n$ = the common zero set of all the polarizations of all the functions $A \mapsto \mathrm{Tr}(A^k)$ for $k = 1, 2, \ldots, n$.

Consider again $n = 3$. The polarizations of $\mathrm{Tr}A$ are the two functions $\mathrm{Tr}A$ and $\mathrm{Tr}B$. The polarizations of $\mathrm{Tr}A^2$ are the three functions $\mathrm{Tr}A^2$, $\mathrm{Tr}AB$ and $\mathrm{Tr}B^2$ (since $\mathrm{Tr}AB = \mathrm{Tr}BA$). The polarizations of $\mathrm{Tr}A^3$ are the four functions $\mathrm{Tr}A^3$, $\mathrm{Tr}A^2B$, $\mathrm{Tr}AB^2$ and $\mathrm{Tr}B^3$. But there is another generating invariant, $\mathrm{Tr}(ABAB) = \mathrm{Tr}(BABA)$ which is not a polarization.

The representation $2M_{3\times 3}$ is 18 dimensional. The common zero set of all the polarizations, $\mathcal{M}_3$, has dimension 9. Thus these 9 functions arising as polarizations form a regular sequence in the ring of polynomial functions on $2M_{3\times 3}$. Also $\dim \mathcal{N}(2M_{3\times 3}) = 9$. Therefore, $\mathcal{N}(2M_{3\times 3})$ is a component of $\mathcal{M}_3$.

Question 9.2. There are $2 + 3 + \ldots + (n+1) = \binom{n+2}{2} - 1$ polarizations of the n functions $\mathrm{Tr}A^k$ for $k = 1, 2, \ldots, n$. Also $\operatorname{codim} \mathcal{N}(2M_{n\times n}) = \binom{n+2}{2} - 1$. Question: Does this set of polarizations always form a regular sequence in the ring of polynomial functions on $2M_{n\times n}$?

Note that this has been proved for $n = 4$. Also, note that $\mathcal{N}(2M_{n\times n})$ is always irreducible.

9.3. **Bialynicky-Birula Theory.**

Theorem 9.4. *(Bialynicky-Birula) Let X be a smooth projective variety on which $\mathbb{C}^*$ acts algebraically, i.e., via a morphism $\lambda : \mathbb{C}^* \to \mathrm{Aut}(X)$. Then the fixed points $F :=$*

$X^{\mathbb{C}^*}$ form a smooth variety with components $F = \cup_i F_i$. For the set $X_i := \{x \in X \mid \lim_{t\to 0} \lambda(t)\cdot x \in F_i\}$ we have the morphism $p_i : X_i \to F_i$ (by taking limits). Every fibre of p_i is a vector space. Also for all affine open subsets $U \subset F_i$, we have $p_i^{-1}(U) \cong U \times \mathbb{C}^d$.

Question 9.5. (George Lusztig) Is $p_i : X_i \to F_i$ an algebraic vector bundle? It is if $d = 1$.

10. Joel Segal

Let $G = S_d \times \mathrm{GL}(V)$ where V is an $\mathbb{F}$-vector space of dimension n. Consider the natural G-representation afforded by $\underbrace{V \otimes V \otimes \cdots \otimes V}_{d \text{ copies}} = V^{\otimes d}$. What is $\mathbb{F}[V \otimes \cdots \otimes V]^G$? Of particular interest is the case where $q = p^r$ and $\mathbb{F}$ is the field of order q.

A (hopefully) easier question is to determine the fixed subspace $(V \otimes \cdots \otimes V)^G$. For example, as a partial answer, it can be shown that this fixed subspace is non-zero only if $(q-1)$ divides d, using the fact that

$$D(V) := \bigoplus_{d=1}^{\infty} \left(V^{\otimes d}\right)^{S_d}$$

can be given a Hopf algebra structure.

The question can be generalized: what is $D(V)^H$, where $H \leq GL(V)$? It is not in general finitely generated as an algebra — can we find other finiteness properties that will make a 'finite description' of the invariants possible?

References

[AB] K. Akin, D. A. Buchsbaum, *Characteristic-free representation theory of the general linear group II. Homological considerations,* Adv. in Math. 72 (1988), 171-210.

[A] D. J. Anick, *On the homogeneous invariants of a tensor algebra,* Algebraic Topology, Proc. Int. Conf., Evanston 1988, Contemp. Math. 96 (1989), 15-17.

[N] M. Nicollerat, *Homology of symmetric algebras,* Comm. in Alg. 7 (1997), 657-676.

11. Nicolas Thiery

Problem 11.1. *Consider the $\binom{n}{2}$ dimensional vector space V_n whose basis is indexed by the pairs $\{i,j\}$ of $\{1,\ldots,n\}$ together with the natural permutation representation on V_n of the symmetric group S_n: $\sigma\cdot\{i,j\} = \{\sigma(i),\sigma(j)\}$. Find $\beta(\mathbb{F}[V_n]^{S_n})$.*

Here are some known values: $\beta(\mathbb{F}[V_3]^{S_3}) = 3$, $\beta(\mathbb{F}[V_4]^{S_4}) = 5$ and $\beta(\mathbb{F}[V_5]^{S_5}) = 9$. A partial computation shows that $\beta(\mathbb{F}[V_6]^{S_6}) \leq 11$, and equality is likely to hold.

Suppose V is a permutation representation of G. Since orbit sums always form a vector space basis for $\mathbb{F}[V]^G$, the Hilbert Series of $\mathbb{F}[V]^G$, $\mathcal{H}(\mathbb{F}[V]^G, t)$ is independent of the characteristic p of $\mathbb{F}$. Furthermore, algebraic properties like the degree bound or the size of a (homogeneous) minimal generating set only depends on the characteristic, not on the field $\mathbb{F}$ itself. The degree bound $\beta(\mathbb{F}[V]^G)$ may depend on the characteristic; take for example the cyclic group C_5: the degree bound is 5 in characteristic 0, and 7 in characteristic 5. Is this only a modular problem? More precisely, assume that the characteristic p does not divide the size of G (the non-modular case), and let respectively S_0 be a minimal generating set of $\mathbb{Q}[V]^G$ composed of orbit sums, and S_p be a minimal generating set of $\mathbb{F}[V]^G$. Here is a series of three conjectures concerning the non-modular case by order of likelihood:

Conjecture 11.2 (Folklore)**.** $\beta(\mathbb{F}[V]^G) = \beta(\mathbb{Q}[V]^G)$

Conjecture 11.3 (Folklore)**.** *The number of generators of each degree in S_p and S_0 coincide.*

Conjecture 11.4 (Folklore)**.** *The set of orbit sums S_0 is a minimal generating set of $\mathbb{F}[V]^G$.*

Positive answers would be of great help for practical computations!

For the next question, let $\mathbb{F}$ be a field of characteristic 0 and suppose V is a permutation representation of a finite group G. Let $\tau(d)$ denote the number of elements of degree d in a homogeneous minimal generating set for $\mathbb{F}[V]^G$.

Question 11.5. Characterize those permutation G-modules V such that τ is unimodular, i.e., such that there exits a positive integer m with $\tau(1) \leq \tau(2) \leq \cdots \leq \tau(m) \geq \tau(m+1) \geq \tau(m+2) \geq \cdots$.

This property has been tested on all transitive permutation groups for dimension $n \leq 6$. It seems to hold for most permutation groups. The alternating group, A_n is always a counter example for $n \geq 4$. Otherwise, the smallest counter example for a transitive group is for $n = 6$; in the computer algebra program GAP, this group is given by "TransitiveGroup(6,8)".

References

[Th1] Thiéry, Nicolas M., *Algebraic invariants of graphs: a study based on computer exploration*, SIGSAM Bulletin (ACM Special Interest Group on Symbolic and Algebraic Manipulation), **34**, No. 3, 9–20, September 2000.

[Th2] Thiéry, Nicolas M., *Invariants algébriques de graphes et reconstruction; une étude expérimentale*, Ph.D. Thesis, Université Lyon I, June 1999, N° d'ordre: 167-99.

[Th3] Thiéry, Nicolas M., *Computing minimal generating sets of invariant rings of permutation groups with SAGBI-Gröbner basis.* In Discrete models: combinatorics, computation, and geometry (Paris, 2001), Discrete Math. Theor. Comput. Sci. Proc., AA, 315–328 (electronic), Maison Inform. Math. Discrèt. (MIMD), (Paris, 2001).

12. David Wehlau

Let $\mathbb{F}$ be a field of characteristic p where $p \neq 0$. Let G be a finite group whose order is divisible by p.

Conjecture 12.1. *(Wehlau) If U is a G-submodule of the G-module V then $\beta(\mathbb{F}[U]^G) \leq \beta(\mathbb{F}[V]^G)$.*

This conjecture is easy to verify if U is a summand of V, i.e., if there exists a G-stable complement to U in V. As a consequence, the conjecture holds if V is a non-modular representation of G, (even if G is not finite). The main result in [SW2] is a proof that the conjecture holds when G is the cyclic group of order p.

One may also ask whether $\beta(\mathbb{F}[V/U]^G) \leq \beta(\mathbb{F}[V]^G)$. It would be very useful to develop relationships between $\beta(\mathbb{F}[V \otimes W]^G)$ and $\beta(\mathbb{F}[V]^G)$ and $\beta(\mathbb{F}[W]^G)$. Similarly we may ask about $\beta(\mathbb{F}[V \otimes W]^{G\times H})$ versus $\beta(\mathbb{F}[V]^G)$ and $\beta(\mathbb{F}[W]^H)$ where V is a G-module and W is an H-module.

We define an $\mathbb{F}[V]^G$-module homomorphism, called the *transfer* or *trace* map, $\mathrm{Tr}^G : \mathbb{F}[V] \to \mathbb{F}[V]^G$ by $\mathrm{Tr}^G(f) = \sum_{g\in G} g \cdot f$. We wish to consider the image of the transfer. We denote this image by $\mathrm{Im}\ \mathrm{Tr}^G$. Since Tr^G is an $\mathbb{F}[V]^G$-module homomorphism, $\mathrm{Im}\ \mathrm{Tr}^G$ is an ideal of $\mathbb{F}[V]^G$.

Similarly, if H is a subgroup of G we may define the relative transfer homomorphism $\mathrm{Tr}^G_H : \mathbb{F}[V]^H \to \mathbb{F}[V]^G$. This homomorphism is defined using a sum over a collection of left coset representatives for H in G: $\mathrm{Tr}^G_H(f) = \sum_{g\in G/H} g \cdot f$. Thus $\mathrm{Tr}^G = \mathrm{Tr}^G_{\{e\}}$.

False Conjecture 12.2. $\beta(\mathbb{F}[V]^G\ /\ \mathrm{Im}\ \mathrm{Tr}^G) \le |G|$.

This conjecture is now known to be false due to a counter example due to Fleischmann, Kemper and Shank (see [FKS, Remark 3.5]). A related conjecture which is still open is the following.

Conjecture 12.3. $\beta(\mathbb{F}[V]^G\ /\ \mathrm{Im}\ \mathrm{Tr}_{<G}) \le |G|$ *where* $\mathrm{Im}\ \mathrm{Tr}_{<G}$ *is the ideal generated by all* $\mathrm{Im}\ \mathrm{Tr}^G_H$ *as H varies over all proper subgroups of G.*

Here are two more open conjectures concerning Noether numbers.

Conjecture 12.4. *(Kemper) If* $\mathbb{F}[V]^G$ *is Cohen Macaulay then* $\beta(\mathbb{F}[V]^G) \le |G|$.

Conjecture 12.5. *(Derksen and Kemper* [DK, 3.8.6]*)* $\beta(J) \le |G|$ *where J is the Hilbert ideal in* $\mathbb{F}[V]^G$.

Conjecture 12.5 has been shown to be true if p does not divide $|G|$.

Finally, here is a conjecture which relates the structure of the ring of invariants to the transfer homomorphism.

Conjecture 12.6. *(Shank and Wehlau) Let G be a p-group. Then* $\mathbb{F}[V]^G$ *is a polynomial ring if and only if* $\mathrm{Im}\ \mathrm{Tr}^G$ *is a principal ideal.*

Let $\mathbb{F}_p$ denote the field of prime order p. Shank and Wehlau ([SW1]) proved that if $\mathbb{F}_p[V]^G$ is a polynomial ring then $\mathrm{Im}\ \mathrm{Tr}^G$ is principal. Broer ([Br]) later extended this result by showing that for any field $\mathbb{F}$ of characteristic p, if $\mathbb{F}[V]^G$ is a polynomial ring then $\mathrm{Im}\ \mathrm{Tr}^G$ is principal.

References

[Br] A. Broer, *Remarks on invariant theory of finite groups*, preprint, Université de Montréal, 1997.

[DK] H. Derksen and G. Kemper, Computational Invariant Theory, Encyclopaedia of Mathematical Sciences 130, Springer-Verlag, Berlin, Heidelberg, New York 2002, 268 pages.

[FKS] Peter Fleischmann, Gregor Kemper and R. James Shank , *On the depth of cohomology modules*, Report UKC/IMS/03/17, IMS, University of Kent, UK., 2003.

[SW1] R. James Shank and David L. Wehlau, *The transfer in modular invariant theory*, J. of Pure and Applied Algebra, **142** No. 1 (1999) 63-77.

[SW2] R. James Shank and David L. Wehlau, *Noether Numbers for Subrepresentations of Cyclic Groups of Prime Order*, Bulletin of the London Mathematical Society **34** No. 4 (2002) 438-450.

Department of Mathematics and Computer Science Royal Military College Kingston, Ontario, Canada K7K 7B4

E-mail address: `wehlau@rmc.ca`